1 Logic – Understanding the English language

Objective: To introduce the concept of symbolic logic as a means to simplifying complicated statements. Truth tables are introduced and used to show the truth values of “not,” “and,” and “or” statements. DeMorgan’s Laws are introduced as a means for switching “and/or” statements to equivalent “or/and” statements.

Hello and welcome back to Math Matters, a program where we try to take some of the math that’s being taught in the general education and teacher ed courses, and tie that to applications in the real world. We’ve got three fairly nice examples for you today. The first one comes from general math, the other two come from college algebra. I haven’t shown you much out of the teacher ed thing yet, but I will, be patient with me. There’s some good stuff coming up from out of there. Today’s first segment deals with logic. One of the first things we teach in general mathematics is typically a little segment on, a little bit on logic. We end up doing truth tables and a lot of things like that. The question would be “Why do I need to know this business?” It seems very abstract and not very useful. So one thing I would like to show you today is the usefulness for some of this business. It comes from the fact, the usefulness of this spawns from the fact that a lot of the English language comes from different places, and for that reason, it’s a little bit inconsistent, and I’ve got an example for you. Three words that end in the same four letters, okay? Now from a mathematician’s standpoint, I would expect those words to rhyme. Bough – when they bough breaks, the cradle will fall – cough, and dough. “Hey you, get down off the bough of that tree and come down here, and cough up the dough, before I break your legs.” Now, those words don’t rhyme. That’s just ridiculous from a mathematician’s standpoint. We would like those things to work consistently.

And also grammatically, and probably more importantly, grammatically, there’s so many different ways of saying exactly the same thing in English. And I’m not counting synonyms. I mean, there’s obviously different words for the same thing, but I mean just in sentence structure. “This town ain’t big enough for the two of us.” Well, I could have said “It is false that this town is big enough for the two of us.” Or I could have said “It is not true that this town is big enough for the two of us.” Or for the more sensitive galoot, “The two of us need a bigger town. Can’t we all just get along?” This is confusing and we have to sort out the meaning of these statements, and so, the logic we teach in general math is designed to help us get down to exactly what is being said in situations like this.

We do this in mathematics. I don’t want to steer you wrong. We do use different symbols for the same thing. For example, I could say zero or I could say five minus five. I could say square root of nine minus three, that’s also zero. One half minus five tenths is zero. Sine of zero degrees is zero. $e^{i\pi} + 1 \ldots$, okay, I probably got a little carried away on that one, but that does happen to be zero. But typically, these are things that pop up in our calculations.
and what we end up doing is simplifying all this to zero. So zero is the end, now that’s where we’re trying to get to in all of these things.

And so I’d like to show you how we do a similar thing in logic. We take a messy thing and we simplify it as much as possible. Okay, here’s a statement for you: “It is not the case that I don’t have the contract of I don’t get a commission but I have not set any records.” Well, how much more of a convoluted statement can you construct? What does that mean exactly? What is this person trying to say? And so what we like to do is break it down into symbols. And what I do here is I would take the basic kind of simple statements, things that are either true or false and allow them a letter, just like we would a variable or something like that. It’s just basically shorthand. I’ll use the letter $p$ for “I have a contract,” $q$ for “I get a commission,” and $r$ for “I have set a record.” (Figure 1)

“It is not the case that I don’t have the contract or I don’t get a commission, but I have not set any records.”

$p$: I have the contract. $\sim \Leftrightarrow$ not
$q$: I get a commission. $\land \Leftrightarrow$ and
$r$: I have set a record. $\lor \Leftrightarrow$ or

“... I have not set any records.”

$\sim (\sim p \lor \sim q) \land \sim r$

Figure 1, Segment 1

And then we take it even further to kind of – we could use the words “not,” obviously I’m negating some things, I’m saying “and,” sometimes I’m saying “or” – but we even go further and give symbols to those things as well. I’ll leave those up so you can kind of see what we’re doing. So when I say, “It’s not the case that,” what I’m doing is I’m negating something and I would put that negation symbol in front of the parentheses, and here’s all the stuff that I’m negating. “I don’t have the contract”: well that’s “not p.” I’ve negated the statement that I’m using for the letter $p$ for. “Or” – well that’s easy, that’s the symbol used for “or.” “I don’t get a commission”: well again, you’ve negated the statement “I get a commission” so that’s a “not q.” The comma means that we’re on a case to something else. “... but I haven’t set any records”: well, I didn’t get a symbol for “but,” but what that really means is this stuff has happened and now something else is going to happen. So basically,
even though it says “but,” what we’re saying here is “and.” So the symbol “and,” okay go back there, and then lastly “I have not set any records,” so you are negating the statement “I have set records.” And that’s the way we say that symbolically. (Figure 1)

Now the question is, can I simplify that to make more sense of this crazy piece of English? And this is kind of where truth tables comes in. Truth tables tell us what happens when a certain statement starts out as true or false, and then we apply one of these things “not” or “and” or “or.” So, back to the screen here, if the statement is originally true and I say “not” the statement, that makes the statement false. If it’s false to begin with and I say “not” the statement, that would make the statement true, for example. So the truth table just shows us the operation of that symbol. It changes the truth value. With “or,” okay, obviously with “or” one of the two things has to be true: we deal with an inclusive “or,” which means actually both of the two things could be true and that’s okay, it’s not an “either/or.” So true in that case, true in that case, true in that case and false in that one. Okay, and then with “and,” both things have to be true, so true, false, false, and false. (Figure 2)

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Figure 2, Segment 1

Now the reason I bring this up is, if I look at this complicated kind of beginning of the statement right here, and I do a truth table for the four situations that can arise, and I apply these rules: it’s true in the first case, false in the other three. That matches my “and” truth table. So, with the same kind of beginnings and endings, these mean exactly the same thing, they are logically equivalent. This is called DeMorgan’s Laws, by the way, that I can take that messy thing and replace it with a simple “and” statement. Also, and I didn’t prove this one to you, if I start with an “and” statement, I can turn it into an “or.” (Figure 3) So this messy thing that I had translates and I can replace all of this mess with just “p and q,” and
then go back and restate it: “I have a contract, I get a commission, but I have not yet set a record,” and I added the yet because I’m an optimist. (Figure 4)

That’s just a simple example and I’m going to come back next week hopefully and show you some other ways that the logic is useful to us, okay? And I’m going to flash up some of the work that I did in a summary page.
“It is not the case that I don’t have the contract or I don’t get a commission, but I have not set any records.”

\[ \sim (\sim p \lor \sim q) \land \sim r = p \land q \land \sim r \]

“I have a contract, I get a commission, but I have not set a record, yet.”

**Summary**

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Summary

DeMorgan’s Laws:
\[ \sim (\sim p \lor \sim q) = p \land q \]
\[ \sim (\sim p \land \sim q) = p \lor q \]
2 Factoring – Solving for unknowns in area problems

Objective: To show how factoring can be used to solve real-world problems by looking at a realistic area problem. It also illustrates the weaknesses of factoring as a method of solving quadratic equations.

The second little trick I’d like to show you today involve factoring. We do this in our college algebra courses and the kids get to be pretty good at it, but the general question we get is “Why do I need to know this stuff? How is this actually useful to me?” and I’d like to show you an example today of how factoring, what it does, what it means, and how it’s useful in solving things that perhaps, maybe involve calculating area.

Factoring, anything that has a power of two, we call a quadratic, and above that, polynomials. It seems kind of academic, you’ve seen this kind of work before where you have $2x$ times $x$ gives me $2x^2$. You have to arrange your middle terms, the outer and inner kind of terms so that the difference in this case, because this is a minus, you’re looking for the difference and the difference needs to be a positive $x$, one really. And if I arrange things like this I get $5x$. This is $1x$ and that’s $6x$, so the difference is $5x$. (Figure 1) If I switch them, though, the difference is $x$. That’s good, that’s what I was looking for and I want it to be positive so I just have to put the plus and the minus in the right place. (Figure 2) Now we do that kind of stuff in college algebra all the time, the question is “Why?” and what I’d like to do is kind of show you and go one step further and factor the two out of there, but I’d like to talk a little about what this actually means, and to do that I would like to look at two formulas.

Factoring quadratics and other polynomials may seem like just an academic exercise.

$$2x^2 + x - 3 = (2x - 1)(x + 3)$$

Difference $= 6x - x = 5x$

Figure 1, Segment 2
The first, I’ll have “a,” “b,” and “c” as kind of parameters. The second I’ll have “a,” it’ll be the same “a” really, and then “m” and “n” as different parameters. So now to do this I’ve got to jump out of this business for just a second and I’ll pull up something else.

Okay, what you see here is just a piece of software called Geometric Sketchpad, Geometer’s Sketchpad, and it allows me to get some information in here. Here’s the basic kind “ax^2” plus “bx” plus “c”, but I’m able to change the values. For example here with “a”, I am able to kind of make “a” bigger and I’ve got “b” and “c” set to zero, so this is just “ax^2” right now. And you see the effect of the “a” – it makes it fatter or skinnier, that’s kind of the idea. And then the “c” out here, so the “bx” is still out of there, moves it up and down, that’s kind of the effect, and then the “b” moves it around. In fact that “c” becomes my y-intercept and then this moves it around. The point is that sometimes this thing will cross the x-axis and sometimes it won’t. What that x-axis tells us is that when is this thing equal to zero. So if I’m solving that equation, this would have two answers here and here wherever those x-coordinates are, whereas this one may not or does not actually have any real solutions. If I set that equal to zero, there is no answer for the “x”, at least not in the real numbers.

Now let me flash out of this again real quick. Here we go. The “a”, “x” minus “m,” and “x” minus “n”. Now in this case, the “a” I’m going to leave alone, actually the “a” does exactly the same thing, it makes it fatter or skinnier, it’s the same “a” we were dealing with. But here’s what I wanted to show you: “m” and “n”, “m” is zero, “n” is basically zero, let me put it right back over there. The only solutions to this thing, the only place it crosses the x-axis are at zero. Watch what happens when I slide the “m” around. When “m” is for example four, set it right down on four, there, I’ve got the solution here at “n” equals...
zero, I’ve got the solution here at “m” equals four. The “m” and the “n”, and the same thing with “n”, let me do the same thing with the “n” real quick to convince you. And if I put that on a negative two, those are solutions to the equation this thing equals zero. Those are the x-intercepts. So that is what it’s really telling you is it’s telling you what are the answers to this equation. The hard part though, you can see that everything I’m dealing with here crosses the x-axis. Everything I dealt with in the first case, not everything did cross the x-axis.

So, let me just, I’ll flash back to my show now. Excuse me while I get used to the machinery here. Some of the things in the form “ax^2” plus “bx” plus “c” don’t cross the x-axis. Everything in the other form does. So the real trick is, can I get things in the first form into the second form. If so, I can find solutions. Something like this doesn’t cross the x-axis. I cannot find that “m” and “n”. Something like this does. I may be able to find “m” and “n” but they may not be rational numbers and it’s going to be hard for me to factor it if they’re not. So factoring is good, but it’s inadequate for a lot of the solving we will eventually need to do. But if it works, then it’s the easiest way to do it. (Figures 3 and 4)

It’s clear from our experimentation that there are more curves of the form

\[ y = ax^2 + bx + c, \quad a,b,c \text{ real numbers} \]

than of the form

\[ y = a(x-m)(x-n), \quad m,n \text{ real.} \]

\[ y = ax^2 + bx + c = 0 \]

has no real solutions, so no \( m,n \) exist.

Figure 3, Segment 2

So let me show you a little example of where it kind of would pop up. And I have to kind of cook the numbers on this in order to make it nice so that I could factor it. But let’s say you’ve got 100 feet of fencing and you want to enclose a rectangular area with that fencing. And I’m going to keep the width actually less than the length, I’m just going to kind of impose that. Good question would be, “What’s the minimum width I need in order to get an area of 600 square feet?” (Figure 5) Now, what you do is you have formulas actually for perimeter and area and I’m going to fix the perimeter. It’s two times the width here and then
the length and then I’m going to set that equal to 100. That’s the fencing I’m going to use. And the area is width times length. That is up in the air. Now what you can do, you’ve got two equations with two unknowns, so you can actually solve for one of the unknowns in one of the equations. I’m going to solve for “l” in the first one and then you shoot that into the second one. And that does give you a constraint. If you’re going to let “w” be less than “l” then 25 is as high as it can go. (Figure 6)

Now if you graph that, over the range zero to 25, that’s the curve you get, you get the left hand part of a parabola. Can I get 600 square feet with this? I sure can. Let me show you where the two, there’s the horizontal line 600 and they cross. So to solve that, I could just use my calculator to get the intersection or I could solve this equation. There’s the standard kind of thing that you’re learning or talking about factoring now in class. I need two factors of 600 that when they add up, they give me 50. Well 20 and 30 right? And 20 would be the good answer. It’s the small answer, it’s the answer below 25. That is actually the x-coordinate or the w-coordinate, excuse me, where these two things cross. That’s exactly where factoring comes in. That’s not an unrealistic example of something you might want to pull off. We don’t use it all the time, because we just slap something together, but doing things the optimum way, that’s the hard part. (Figure 6)

Oh, and I should say this, make sure you put your units in there and then we’ll go to the summary page and kind of talk about some of the things we’ve spoken about.
So, factoring is an inadequate way of solving quadratic equations, but it is also one of the easiest ways, if it works.

Example: A farmer wishes to enclose a rectangular area with 100 feet of electric fencing.

![Figure 5, Segment 2](image1)

What is the minimum width $w$ needed to guarantee an area of $600 \text{ ft}^2$?

![Figure 6, Segment 2](image2)
Summary

The curve $y = a(x - m)(x - n)$ has zeros at $x = m$ and $x = n$.

If we are able to factor $ax^2 + bx + c$, then we can solve $ax^2 + bx + c = 0$.

Area calculations frequently require us to solve these types of equations.
3 Rational expressions – Solving for unknowns in area problems

Objective: To show how rational functions can appear in real-world problems by looking at a realistic area problem. We take this opportunity to explain why we are not allowed to divide by zero. The example given also illustrates how algebra is inadequate for finding the extrema of a function, and how we bypass this inadequacy by using graphing calculators to find numerical answers.

We’re back with another example of out of algebra and that is rational expressions. We typically kind of go, and this is going to go along with what I just did, frequently I hear “How would I ever use this?” and this is going to come down to helping us with things that involve for example area. Now, what I mean by a rational expression is, let’s go to the board here, something in this form. You’ve got some kind of polynomial on the top and some kind of polynomial on the bottom. We call the top part the numerator and this grade school stuff. The bottom is the denominator, but when we go to these things, there is a complication, we have to be sure that we don’t plug in values of x into the denominator that will make it zero. If we do, if we look at the graph of things like this or we think about this expression, and we look at the graph of it, we say that “y” equals this thing, that’s going to be the location of a vertical asymptote if it also makes the numerator something other than zero. If they’re both zero, if the denominator is zero and the numerator is zero, then the graph is going to have a hole in it and that’s a hole that won’t show up in most cases when you graph things. (Figure 1)

Let me just quickly jump out of here, I just want to show you very quickly some of the kind of things that can pop up with this. Let’s see, is that the one I really want? Sorry, I’m fumbling around with this, the stuff. Tell you what, before I show you this one, let me show you, right here I’ll get it. Save it to here we go, sorry for the fumbling around. Something like this, this is a perfect example of a polynomial kind in the top, polynomial kind in the bottom and I want to point out to you what these kind of things mean. Now right now I’ve got everything zeroed out. This is zero, that’s this big purple line across the bottom. But as I perhaps let this “b” be a positive number, watch what happens. You can see here, we’ve got a vertical asymptote right here when “x” is equal to zero and then I can move that around when I change “c”, that asymptote moves left and right. That’s really what the “c” controls, and then the “a” would control up and down kind of changes and that little dotted line you see is the horizontal asymptote that’s going on. The other thing is what if I go to a square, sorry, let me do this a little better, if this is an “ax” squared up here. The change is about the same, the “c” is going to again, oh wait, let me make this, you get that same kind of shape and the “c” moves it left and right, we talked about this, but because the power of this is higher than the power of that, you don’t end up with a horizontal asymptote, you end up with a kind of a slant. Watch what happens when I tilt this “a” around. Now how that’s useful to us is, it has the way things are drawn now, just look at the positive things. I actually have a minimum here and this is the kind of thing your calculator can find for you. It’s also the kind of thing we can find with calculus, but now calculus is a little bit above what we’re going to be talking about. If I want exact answers, I have to use calculus. If I’m willing to
settle for approximate answers, I can just go with the calculator and just let it tell me the answer. So, let’s go back then to this kind of stuff.

Now I do want to say just very quickly, this whole business of dividing by zero, why can’t you do it? You know, this is something we all take for granted but probably nobody has ever explained it to you. What we mean when we say “a” divided by “b” is “c”, it means that “a” is equal to “b” times “c” and that “c” is a unique value. So if “b” is zero and that’s what we’re talking about, dividing by zero, that turns into “a” equals zero times “c” and if “a” is not zero, then we’ve got a problem. It has to be zero. If we let “a” be zero however, then I get zero equals zero times “c” and the “c” can be anything, it’s not unique. So because it doesn’t fit the way we defined division, we say it is undefined. I don’t know if anyone ever explained that or not, I felt like I owed it to you to explain why we leave that zero out of the denominator. (Figure 2)

Alright, let’s go to an example. I want to go back and look at that problem I worked before actually with the farmer wanting to enclose the rectangular area. And we made the assumption when we worked that problem that we were going to use all 100, all the fencing we had, the 100 feet of fencing. But I was only trying to enclose 600 square feet. So the question, a good question would be, “Could I get by with less fencing material? In fact, what would be the minimal amount of fencing I would need to enclose an area of 600 square feet? I know I can do it with a hundred, can I do it with less? And how does this involve rational things in the first place?” (Figure 3) Well, now I’ve kind of switched things out. I’m fixing the area, so the width times the length would be 600 and now the perimeter can change. The only thing, the only constraint I really have is that I have to use less than 100 feet. I mean

A rational expression has the form

\[
\frac{P(x)}{Q(x)} \quad \text{numerator} \\
\text{denominator}
\]

The expression will be undefined at any value \(x = a\) such that \(Q(a)=0\).

- If \(P(a) \neq 0\), then it has a vertical asymptote at \(x = a\).
- If \(P(a) = 0\), then the graph has a hole in it at \(x = a\).
I know that’s less than 100 feet because I’ve already worked that problem. That does mean still that the width, it’s going to range between zero and 25. Twenty-five would be a square right? Twenty-five, 25, 25, 25 gives me 100. But now I’ve got the two equations, the two unknowns, I want to solve for one of the unknowns. The one I will solve for is the length. (Figure 4)

I’m dividing by the width, now that means I can no longer use zero as an answer. It’s a subtlety, but it’s a fact that I now want to avoid that answer zero. And I’ll do what I did last time and that is jam it into the other equation which you can clean up a little bit. This is an equation now that involves rational expressions and that’s what we’re talking about and this is how I would use it. To solve this, you can clean it up a little bit, fine. To solve this without calculus, I would go to my graphing calculator, I would draw that curve. Can I do this in less than 100 feet perimeter? Yes, I can and I can tell you exactly that those lines cross again at 20, okay? But if I want the minimum, I use the functions on my calculator and find that to be about 24 and a half feet. Which is roughly, it turns out that is roughly the square root of 600. That’s actually the exact answer. If you use calculus, that’s the exact answer which means that the length would also be the square root of 600 and you end up with a square. (Figure 4)

Cute little example of how rational expressions . . ., you know right now, you’re just playing with them, you’re not solving equations, but here’s exactly the way they’re going to be used later on when you actually start getting down to the nitty gritty of solving for distances and stuff involving area. And I’ll just quickly flash up some of the things I talked about for you to kind of look at.
Example: A farmer wishes to enclose a rectangular area with 100 feet of electric fencing.

What is the minimum width $w$ needed to guarantee an area of 600 ft$^2$?

We found the answer to be $w = 20$ ft. But what if you only wanted a region with area 600 ft$^2$? What is the minimal amount of fencing needed?

According to my calculator, the curve has a minimum at $w \approx 24.4949$ ft $\approx \sqrt{600}$ ft.

The rectangle is square!
Summary

\[
\frac{P(x)}{Q(x)} \quad \text{numerator}
\]
\[
\frac{Q(x)}{Q(x)} \quad \text{denominator}
\]

The expression will be undefined at any value \( x = a \) such that \( Q(a) = 0 \).
- If \( P(a) \neq 0 \), then it has a vertical asymptote at \( x = a \).
- If \( P(a) = 0 \), then the graph has a hole in it at \( x = a \).

Summary page 1, Segment 3

Summary

Division by 0 is bad!

We can use the “maximum” and “minimum” feature of our calculators to find approximate max’s and min’s when the exact values are not known.

Summary page 2, Segment 3
Closing

We're at the end of another episode of “Math Matters.” I hope you enjoyed the examples that I showed you today. I realize I moved a little fast through some of those so let me remind you that all the stuff you’ve seen here today, the segments we’ve produced are located on the “Math Matters” webpage and we’ll flash up that address at the end of the program. Take advantage of these things, I think you’ll, hopefully you’ll find it helpful as you proceed through your classes. Also, let me remind you that I’m always looking for great examples or suggestions for improving the show so make sure and use that email address and contact me if I can be of any help to you. We’ll call it a day then. Thanks a lot for tuning in, see you next week.