Let $A_t \sim Poi(\lambda t)$ count the number of arrivals to a group per time t, where λ is the average numbers of arrivals per unit time, and let $X_t = J + A_t$, for $t \geq 0$. Then the probability of i arrivals through time t is

$$P(A_t = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \text{ for } k = 0, 1, 2, \ldots$$

Let T_i measure the total amount of time until the ith occurrence. Then $T_i \sim \Gamma[i, \lambda]$ with pdf $f_i(t) = \frac{\lambda^i}{(i-1)!} t^{i-1} e^{-\lambda t}$ for $t \geq 0$. Moreover, the average time of the ith occurrence is $E[T_i] = i/\lambda$.

For example, suppose there is an average $\lambda = 12$ arrivals every hour. Then the average time of an arrival is $L = 1/12$ hours, or 5 minutes. That is, the first arrival, on average, comes at 5 minutes. The second arrival comes on average at 10 minutes; the third arrival comes on average at 15 minutes, etc. So, on average, the arrivals are equally spaced out:

$$E[T_1] = 5 \text{ min} \quad E[T_2] = 10 \text{ min} \quad E[T_3] = 15 \text{ min} \quad \ldots. \quad E[T_i] = \frac{i}{12} \text{ hrs } = 5i \text{ min}$$

But suppose we have exactly n arrivals through time t. When can we expect these arrivals to occur on average? We claim that the average times of the n arrivals under this condition will divide the time interval into $n+1$ equal segments.

Example 1. Consider the occasions for which we only have *one* arrival in an hour. Among all the occasions when this happens, the average time of this one arrival is 30 minutes which divides the time interval [0, 1 hr] into two equal pieces.

Example 2. Consider the occasions for which we have *two* arrivals in 30 minutes. Among all the occasions when this happens, the average times of the two arrivals must divide the interval [0, 30] into 3 equal pieces of length 30/3 = 10. Thus, under this condition, T_1 averages 10 minutes and T_2 averages 20 minutes.

Example 3. Consider the occasions when we have 4 arrivals in 45 minutes. Then [0, 45] must be divided into 5 pieces of length 9. So under this condition, T_1 averages 9 minutes, T_2 averages 18 minutes, T_3 averages 27 minutes, and T_4 averages 36 minutes.

We state the result formally as:

Theorem: Let $A_t \sim Poi(\lambda t)$. Given that there are exactly n arrivals through time t, the average time of the ith arrival, for $1 \leq i \leq n$, is given by

$$E[T_i \mid A_t = n] = \frac{it}{n+1}.$$

Proof. To prove the result, we will need a triple integral over the region that describes the following conditions:

(i) i arrivals occur within time t
(ii) $n-i$ more arrivals occur after the time of the first i arrivals, but still within time t
(iii) The time of the $(n+1)$st arrival is beyond t.

We first let $X = T_i \sim \Gamma[i,\lambda]$ be the time of the ith arrival. By (i), we need $X \le t$. Next, we let $Y \sim \Gamma[n-i,\lambda]$ be the time for $n-i$ more arrivals to occur, starting at time X. By (ii), these arrivals must occur in the *remaining* time of $t-X$. So we need $Y \le t-X$. Now $X+Y$ is the time of all n arrivals. Finally, we let $Z \sim \exp(1/\lambda)$ measure the time of one more arrival. But this arrival *cannot* be within the remaining time of $t-X-Y$ or else we would have $n+1$ arrivals. Thus, we need $Z > t-X-Y$.

Because X, Y, and Z are independent, their joint pdf is the product of their individual pdf's. So by conditioning over the event $\{A_t = n\}$ and integrating over the above described region, we obtain the conditional average of the time of the ith arrival as

$$
\begin{aligned}
E\left[X \mid A_t = n\right] &= \frac{1}{P(A_t = n)} \int_0^t \int_0^{t-x} \int_{t-x-y}^{\infty} x \, f_X(x)\, f_Y(y)\, f_Z(x)\, dz\, dy\, dx \\
&= \frac{n!}{(\lambda t)^n e^{-\lambda t}} \int_0^t \int_0^{t-x} \int_{t-x-y}^{\infty} x \left(\frac{\lambda^i}{(i-1)!} x^{i-1} e^{-\lambda x} \right) \left(\frac{\lambda^{n-i}}{(n-i-1)!} y^{n-i-1} e^{-\lambda y} \right) \left(\lambda e^{-\lambda z} \right) dz\, dy\, dx \\
&= \frac{n!}{(i-1)!(n-i-1)!\, t^n e^{-\lambda t}} \int_0^t \int_0^{t-x} \int_{t-x-y}^{\infty} \left(x^i e^{-\lambda x} \right) \left(y^{n-i-1} e^{-\lambda y} \right) \left(\lambda e^{-\lambda z} \right) dz\, dy\, dx \\
&= \frac{n!}{(i-1)!(n-i-1)!\, t^n e^{-\lambda t}} \int_0^t \int_0^{t-x} \left(x^i e^{-\lambda x} \right) \left(y^{n-i-1} e^{-\lambda y} \right) \left(-e^{-\lambda z} \Big|_{t-x-y}^{\infty} \right) dy\, dx \\
&= \frac{n!}{(i-1)!(n-i-1)!\, t^n e^{-\lambda t}} \int_0^t \int_0^{t-x} \left(x^i e^{-\lambda x} \right) \left(y^{n-i-1} e^{-\lambda y} \right) \left(e^{-\lambda (t-x-y)} \right) dy\, dx \\
&= \frac{n!}{(i-1)!(n-i-1)!\, t^n} \int_0^t \int_0^{t-x} \left(x^i \right) \left(y^{n-i-1} \right) dy\, dx \\
&= \frac{n!}{(i-1)!(n-i-1)!\, t^n} \int_0^t \left(x^i \right) \left(\frac{y^{n-i}}{n-i} \Big|_0^{t-x} \right) dx \\
&= \frac{n!}{(i-1)!(n-i)!\, t^n} \int_0^t x^i (t-x)^{n-i}\, dx
\end{aligned}
$$

To evaluate $\int_0^t x^i (t-x)^{n-i}\, dx$, we shall use a tabular form of repeated integration by parts with $u = x^i$ and $v' = (t-x)^{n-i}$.

In the table below, we take successive derivatives down the first column, take successive anti-derivatives down the second column, and alternate signs down the third column. The terms in the fourth column are the products of the terms in the first three columns, evaluated from 0 to t, and the sum of the fourth column is $\int_0^t x^i (t-x)^{n-i}\, dx$.

	$v' = (t-x)^{n-i}$		
$u = x^i$	$v = -\dfrac{(t-x)^{n-i+1}}{n-i+1}$	$+$	$-\dfrac{(t-x)^{n-i+1}}{n-i+1} x^i \Big\vert_0^t$
$i\, x^{i-1}$	$+\dfrac{(t-x)^{n-i+2}}{(n-i+1)(n-i+2)}$	$-$	$-\dfrac{(t-x)^{n-i+2}}{(n-i+1)(n-i+2)} i\, x^{i-1} \Big\vert_0^t$
$P(i,2)\, x^{i-2}$	$-\dfrac{(t-x)^{n-i+3}}{P(n-i+3,3)}$	$+$	$-\dfrac{(t-x)^{n-i+3}}{P(n-i+3,3)} P(i,2)\, x^{i-2} \Big\vert_0^t$
.	.	.	.
.	.	.	.
$i!x^0$	$(-1)^{i+1} \dfrac{(t-x)^{n+1}}{P(n+1,i+1)}$	$(-1)^i$	$-\dfrac{i!(t-x)^{n+1}}{P(n+1,i+1)} \Big\vert_0^t$

We see that each term in the fourth column is 0, except for the last term which is $i!\, t^{n+1}/P(n+1,i+1)$. Thus, we obtain our result with

$$
\begin{aligned}
E[X \mid A_t = n] &= \frac{n!}{(i-1)!(n-i)!\, t^n} \times \int_0^t x^i (t-x)^{n-i}\, dx \\
&= \frac{n!}{(i-1)!(n-i)!\, t^n} \times \frac{i!\, t^{n+1}}{P(n+1,\, i+1)} \\
&= \frac{n!}{(i-1)!(n-i)!\, t^n} \times \frac{i!\, t^{n+1}(n-i)!}{(n+1)!} \\
&= \frac{it}{n+1}.
\end{aligned}
$$

Average Area Under the Curves

We previously have considered the stochastic process $X_t = J + A_t$ that gives the total number at time t in a group with J initial members, where $A_t \sim Poi(\lambda t)$. We have seen that the average at time t is $a_X(t) = E[X_t] = J + \lambda t$, which defines the single average curve. Moreover, the area under the average curve over the interval [0, s] is

$$\int_0^s E[X_t]\,dt = Js + \frac{\lambda s^2}{2}.$$

On the other hand, we can consider the area $\int_0^s X_t(\omega)\,dt$ under each individual curve for all possible outcomes of arrivals ω. We then may compute the average area under all possible curves, denoted by $E\left[\int_0^s X_t\,dt\right]$. To simplify our work some, we note that

$$\int_0^s X_t(\omega)\,dt = \int_0^s \big(J + A_t(\omega)\big)\,dt = Js + \int_0^s A_t(\omega)\,dt;$$

thus,

$$E\left[\int_0^s X_t\,dt\right] = J\,s + E\left[\int_0^s A_t\,dt\right].$$

To compute the average of a random variable W, we may partition our sample space of outcomes into events $\{B_n\}$. We then compute the conditional average of W on each B_n, and then find the total average by summing the weighted conditional averages as

$$E[W] = \sum_n E[W \mid B_n] \times P(B_n)$$

In our case, we shall use the partition of events $B_n = \{A_s = n\}$, for $n \geq 0$, which are the events of having exactly n arrivals through time s. We then must compute $E\left[\int_0^s A_t\,dt \,\middle|\, A_s = n\right]$, the average area under the arrival process given exactly n arrivals, for each of these events.

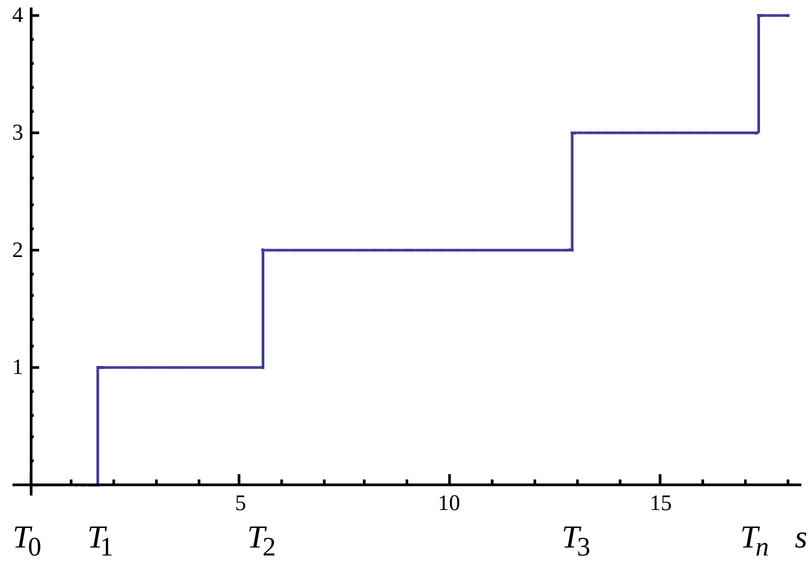
So consider an outcome ω with exactly n arrivals through time s, and let $T_i(\omega)$ be the time of the ith arrival, where $T_0(\omega) = 0$. The area under such a curve having initial height 0 is given by

$$\int_0^s A_t(\omega)\,dt = 0(T_1(\omega) - T_0(\omega)) + 1(T_2(\omega) - T_1(\omega)) + \ldots + (n-1)(T_n(\omega) - T_{n-1}(\omega)) + n(T - T_n(\omega))$$

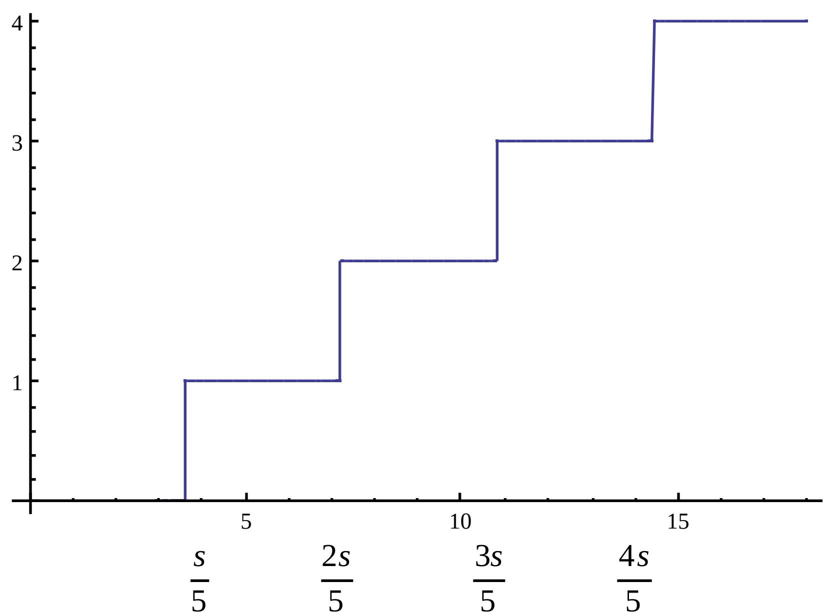
$$= \sum_{i=1}^{n-1} i\,(T_{i+1}(\omega) - T_i(\omega)) + n(s - T_i(\omega))\,.$$

In general, the area under the Poisson Arrival process can be expressed as

$$\int_0^s A_t\,dt = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n-1} i(T_{i+1} - T_i) + n(s - T_i) \right) 1_{\{A_s = n\}}$$



A typical graph with $n = 4$ arrivals through time $s = 18$



The average graph with 4 arrivals in time $s = 18$: $E[T_i \mid A_s = 4] = \frac{is}{5}$

So the average area under such curves having exactly n arrivals through time s is

$$E\left[\int_0^s A_t \, dt \,\middle|\, A_s = n\right] = \sum_{i=1}^{n-1} i\,(E[T_{i+1} \mid A_s = n] - E[T_i \mid A_s = n]) + n(s - E[T_n \mid A_s = n])$$

$$= \sum_{i=1}^{n-1} i \left(\frac{(i+1)s}{n+1} - \frac{is}{n+1}\right) + n\left(s - \frac{n\,s}{n+1}\right)$$

$$= \sum_{i=1}^{n} i \left(\frac{s}{n+1}\right) = \left(\frac{s}{n+1}\right) \times \frac{n(n+1)}{2} = \frac{n\,s}{2}.$$

The total average area under the Poisson arrivals curves is then

$$E\left[\int_0^s A_t \, dt\right] = \sum_{n=0}^{\infty} E\left[\int_0^s A_t(\omega)\, dt \,\middle|\, A_s = n\right] P(A_s = n)$$

$$= \frac{s}{2} \sum_{n=0}^{\infty} n\, P(A_s = n)$$

$$= \frac{s}{2} E[A_s] = \frac{s}{2}(\lambda s) = \frac{\lambda\, s^2}{2}.$$

Finally, we have

$$E\left[\int_0^s X_t \, dt\right] = J\,s + E\left[\int_0^s A_t \, dt\right] = J\,s + \frac{\lambda s^2}{2} = \int_0^s E[X_t]\,dt.$$

That is, the average area under all curves $E\left[\int_0^s X_t \, dt\right]$ equals the area under the single average curve $\int_0^s E[X_t]\,dt$.

Exercise

1. Let A_t be a standard Poisson arrival process with an average of 30 arrivals every 12 hours, and let s = 3 hours.

(a) What is $E\left[\int_0^s A_t\,dt\right]$ in this case?

(b) Compute $E\left[\int_0^s A_t\,dt \mid A_s = 0\right]$, $E\left[\int_0^s A_t\,dt \mid A_s = 1\right]$, $E\left[\int_0^s A_t\,dt \mid A_s = 2\right]$, and $E\left[\int_0^s A_t\,dt \mid A_s = 3\right]$. For each, show a graph that illustrates the result.

(c) Weight the results in (b) to compute $E\left[\int_0^s A_t\,dt \mid 0 \le A_s \le 3\right]$.

Bonus Challenge: Consider a modified Poisson arrival process A_t, with initial height $A_0 = 0$, that moves to height 2^i (rather than i) upon the ith arrival.

(a) Derive $E[A_t]$, the average height at time t, for this process.

(b) Derive $\int_0^s E[A_t]dt$, the area under the average curve, for this process.

(c) Derive $E\left[\int_0^s A_t\,dt \mid A_s = n\right]$ for this process.

(d) Derive $E\left[\int_0^s A_t\,dt\right]$ for this process.

(e) Based on (b) and (d), what conclusion can you draw?