

Consider again the scenario of a random walk X that starts at positive height j , and on each independent step, moves upward a units with probability p , moves downward b units with probability q , or remains at the same height with probability $r = 1 - p - q$. It stops upon reaching any height greater than or equal to n or dropping to any height less than or equal to m . (We again assume that all height units are integers.)

We now will use a system of linear equations that gives an *exact simultaneous solution* for the probability of reaching the upper boundary before the lower boundary for *all* starting heights j . In addition, we can easily adjust the system to find the average number of steps needed to hit a boundary. However, the solutions will still be numerical rather than closed-form algebraic formulas.

The System of Equations

Let x_j be the probability of reaching the upper boundary first when starting at height j . Then we know that $x_m = 0$ and $x_n = 1$ (because if we start at height m then we can't reach n first, and if we start at height n then we are certain to reach n first.) In fact, in because we may surpass n or drop below m , then $x_j = 0$ for all $j < m$, and $x_j = 1$ for all $j > n$.

Now consider the values of x_j for $m < j < n$. Assuming we go up a units at a time or down b units at a time, then x_j can be written in terms of x_{j-b} and x_{j+a} by

$$x_j = q x_{j-b} + r x_j + p x_{j+a}.$$

We can re-write this set of equations, for $m < j < n$, as

$$q x_{j-b} + (r - 1) x_j + p x_{j+a} = 0$$

The system of equations is completed with the equations $x_j = 0$ for $j < m$, and $x_j = 1$ for $j > n$.

Example 1. Suppose we go up or down 1 unit at a time, where $p = 0.4$, $q = 0.5$, and $r = 0.1$. We quit if we reach height 8 or height 0. (Here we do not specify an initial starting height.) What are the probabilities of reaching 8 before 0 when starting at each height of 0, 1, 2, ..., 8?

Solution. We let x_j be the probability of reaching height 8 before height 0 when starting at height j . Then we have the system of equations:

$$\begin{array}{rcl}
 x_0 = 0 & & x_0 = 0 \\
 qx_0 + (r-1)x_1 + px_2 = 0 & & 0.5x_0 - 0.9x_1 + 0.4x_2 = 0 \\
 qx_1 + (r-1)x_2 + px_3 = 0 & & 0.5x_1 - 0.9x_2 + 0.4x_3 = 0 \\
 qx_2 + (r-1)x_3 + px_4 = 0 & \text{or} & 0.5x_2 - 0.9x_3 + 0.4x_4 = 0 \\
 \cdot & & \cdot \\
 \cdot & & \cdot \\
 x_8 = 1 & & x_8 = 1
 \end{array}$$

Note that the matrix of coefficients matrix is nearly identical to the matrix of transition probabilities used in the Markov Chain method. The only difference is that now there is a string of $r - 1$ down the inner portion of the main diagonal rather than r . If $r = 0$, then these terms are -1 .

We shall call this matrix of coefficients T . Let F be the 9×1 column of constants. Then we are solving the system $TX = F$.

$$\begin{array}{cccccccccc}
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & 0 \\
 0.5 & -0.9 & 0.40 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\
 0 & 0.5 & -0.9 & 0.40 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 \\
 0 & 0 & 0.5 & -0.9 & 0.40 & 0 & 0 & 0 & 0 & x_3 & 0 \\
 0 & 0 & 0 & 0.5 & -0.9 & 0.40 & 0 & 0 & 0 & x_4 & 0 \\
 0 & 0 & 0 & 0 & 0.5 & -0.9 & 0.40 & 0 & 0 & x_5 & = 0 \\
 0 & 0 & 0 & 0 & 0 & 0.5 & -0.9 & 0.40 & 0 & x_6 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.9 & 0.40 & x_7 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_8 & 1
 \end{array}$$

The matrix of coefficients T will always be invertible. Here $\det(T) = -0.0325 \neq 0$. Hence we can solve by $X = T^{-1}F$.

$$\begin{array}{r}
 x_0 \\
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
 x_6 \\
 x_7 \\
 x_8
 \end{array}
 = X = T^{-1}F =
 \begin{array}{r}
 0 \\
 0.0503985 \\
 0.113397 \\
 0.192144 \\
 0.290579 \\
 0.413622 \\
 0.567426 \\
 0.759681 \\
 1
 \end{array}
 .$$

Thus when starting at height 7, there is almost a 76% chance of reaching 8 before dropping to 0, but only a 29% chance when starting at height 4.

Because the up/down movements are both 1 unit each, these values can be found much more quickly with the known closed-form formula $\frac{1-(q/p)^j}{1-(q/p)^8}$ for $0 \leq j \leq 8$.

However, this system of equations method is very useful when $a \neq b$. We also know a closed-form formula for the average number of steps to hit a boundary in this case:

$$E[S_0^n] = \frac{n}{p-q} \times \frac{1-(q/p)^j}{1-(q/p)^n} - \frac{j}{p-q}$$

Example 2. Suppose now we go up 3 units with probability $p = 0.4$, go down 2 units with probability $q = 0.5$, or stay constant with $r = 0.1$. We wish to reach a goal of 10 (or more) before dropping to 0 (or below). (a) What are the probabilities of doing so when starting at height j , for $1 \leq j \leq 9$? (b) What is the average number of steps needed to hit a boundary?

Solution. Here we can actually reach a height of 12 (if going up while at height 9) and can actually drop to -1 (if going down while at height 1). Thus, $x_j = 0$ for $-1 \leq j \leq 0$, and $x_j = 1$ for $10 \leq j \leq 12$. For $1 \leq j \leq 9$, we have $qx_{j-2} + (r-1)x_j + px_{j+3} = 0$. We thereby have a system $TX = F$ of 14 equations and 14 unknowns with the following augmented matrix $T|F$:

													T	F
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
q	0	$r-1$	0	0	p	0	0	0	0	0	0	0	0	0
0	q	0	$r-1$	0	0	p	0	0	0	0	0	0	0	0
0	0	q	0	$r-1$	0	0	p	0	0	0	0	0	0	0
0	0	0	q	0	$r-1$	0	0	p	0	0	0	0	0	0
0	0	0	0	q	0	$r-1$	0	0	p	0	0	0	0	0
0	0	0	0	0	q	0	$r-1$	0	0	p	0	0	0	0
0	0	0	0	0	0	q	0	$r-1$	0	0	p	0	0	0
0	0	0	0	0	0	0	q	0	$r-1$	0	0	p	0	0
0	0	0	0	0	0	0	0	q	0	$r-1$	0	0	p	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

The solution is given by $X = T^{-1}F$. Below is the result from a *Mathematica* program that solves the system.

12	1
11	1
10	1
9	0.873637
8	0.811138
7	0.772546
6	0.660048
5	0.590583
4	0.489177
3	0.414139
2	0.262481
1	0.217412
0	0
-1	0

For each possible initial height j , we have the probability of reaching 10 or more before dropping to 0 or less.

Average Number of Steps

Suppose we want to find the *average number of steps* needed to hit either boundary of n or m . To do so, we can create a similar system of equations. Now let x_j be the average number of steps when starting at height j . Then we know $x_j = 0$ for all $j = m$ and for all $j = n$.

Now consider the values of x_j for $m < j < n$. Assuming we go up a units at a time or down b units at a time, then x_j can be written in terms of x_{j-b} and x_{j+a} . First one step must be taken, then we start over at either $(j-b)$, or at $(j+a)$, or still at j if we have remained constant. Thus,

$$x_j = 1 + q x_{j-b} + r x_j + p x_{j+a}.$$

The middle set of equations, for $m < j < n$, becomes

$$q x_{j-b} + (r-1)x_j + p x_{j+a} = -1$$

The system of equations is completed with the equations $x_j = 0$ for $j = m$, and $x_j = 0$ for $j = n$. So the only changes required are in generating the matrix F with -1 in the middle terms and 0 for $j = n$. The matrix of coefficients T remains the same.

So in Example 2, the solution is $X = T^{-1}F$ where

	0	12	0
	0	11	0
	-1	10	0
	-1	9	3.59441
	-1	8	4.22327
	-1	7	4.46995
	-1	6	5.60188
$F =$	-1	5	6.0459
	-1	4	5.20785
	-1	3	5.50401
	-1	2	3.79818
	0	1	3.42571
	0	0	0
	0	-1	0

With *Mathematica*,
we obtain the following
table for the average
number of steps:

Note: A drawback of this system of equations method is that we do not obtain the average final height. In Example 2, if we start at height $j = 6$, then about 66% of the time we will reach the top boundary first and about 34% of the time we hit the lower boundary first. But these boundaries could be $-1, 0, 10, 11$, or 12 . We do not have the probabilities of ending at each specific height, so we cannot find the average final height. But in this case, we can use the Markov Chain Method with a large number of steps to find the average final height for a specific starting height j .

Mathematica Exercises (See system.nb file)

1. In a gambling game, you bet \$30 at a time with a 4 : 3 payoff and a probability of $p = 0.42$ of winning. (There are no ties.) You want at least a 75% chance of reaching \$300 (or more) before going bust.

- What is the minimum dollar amount with which you must start?
- If you start with this amount, then what is the exact probability of attaining your goal, and what is the average number of bets you can make before attaining your goal or going bust?
- In order to maximize the average number of bets you can make, with what dollar amount should you begin, and what would be the probability of attaining your goal when starting with this amount?

2. In a gambling game, let $p = 0.10$ and let the payoff be 8 : 1. (Again, no ties.) You bet \$10 at a time. You wish to quit when you “go ahead” or go bust (i.e., start with \$10 and have a goal of \$20 or more; or start with \$30 and have a goal of \$40 or more, etc.). In general, you start with \$10 k and have a top goal of \$10 $(k + 1)$ or more.

You want at least an 80% chance of attaining your goal. What is the minimum dollar amount with which you must start? With this initial amount, what is the average number of bets that would be made?