

The Boundary Problem formulas for ${}_jP_m^n$, $E[{}_jS_m^n]$, and $E[{}_jF_m^n]$ provide opportunities to seek some interesting interchange of limit results. We now shall find and interpret the limits as $n \rightarrow \infty$ and as $m \rightarrow \infty$. We shall need the following proposition on the interchange of limits for the probabilities of events.

Proposition 1. (a) Let $A_k \subset A_{k+1} \subset \dots \subset A_n \subset \dots$ be a nested increasing sequence of events.

Then $P\left(\bigcup_{i=k}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} P(A_n)$.

(b) Let $A_k \supset A_{k+1} \supset \dots \supset A_n \supset \dots$ be a nested decreasing sequence of events.

Then $P\left(\bigcap_{i=k}^{\infty} A_i\right) = \lim_{n \rightarrow \infty} P(A_n)$.

If a sequence of events is such that $A_k \subset A_{k+1} \subset \dots \subset A_n \subset \dots$, then we could interpret the union as the “limit” of these events. If the events decrease, then the intersection would be the “limit.” The proposition gives important interchange of limit results in these cases and states that the probability of the limit equals the limit of the probabilities.

1. The Probabilities of Reaching a Single Boundary

We first seek to find the probability that simple random walks will ever drop to height m when starting at height $j > m$. To do so, we again let ${}_jU_m^n$ denote the set of paths that cause simple random walks to reach height n before height m when starting at height j where $m < j < n$. These sets form a nested, decreasing sequence as n increases: ${}_jU_m^i \supset {}_jU_m^{i+1}$ for all $i < n$. Indeed, if a walk reaches height $i+1$ before height m , then it also must have reached height i before height m . Thus by Proposition 1,

$$\begin{aligned}
 P\left(\bigcap_{i=n}^{\infty} {}_jU_m^i\right) &= \lim_{n \rightarrow \infty} P\left({}_jU_m^n\right) = \lim_{n \rightarrow \infty} {}_jP_m^n \\
 &= \lim_{n \rightarrow \infty} \frac{j-m}{n-m} && \text{if } p = q \\
 &= \lim_{n \rightarrow \infty} \frac{1-(q/p)^{j-m}}{1-(q/p)^{n-m}} && \text{if } p < q \\
 &= 0 && \text{if } p < q \\
 &= 1-(q/p)^{j-m} && \text{if } p > q.
 \end{aligned} \tag{1.1}$$

How can we interpret this limit? We know that a walk will reach a boundary of n or m with probability 1. So we may let ${}_jV_m^n$ be the set of probability 0 consisting of the paths along which walks do not reach either boundary of n or m . Then the countable union $V = \bigcup_{i=n} {}_jV_m^i$ still has probability 0. We then exclude those paths and are left with V , those all paths that do reach either n or m , where $P(V) = 1$. Within V , paths never drop to height m if and only if they belong to $\bigcup_{i=n} {}_jU_m^i$. Then $V \setminus \bigcup_{i=n} {}_jU_m^i$ are all paths that never drop to height m . Because $P(V) = 1$, then $P(\bigcup_{i=n} {}_jU_m^i)$ by itself gives the probability that walk will not drop to height m . By subtracting this value from 1, we also have the probability ${}_jP_m$ that a simple random walk will drop to height m .

Theorem 1. For simple random walks beginning at height j , the probability of dropping to height m , for $m \geq j$, is given by

$${}_jP_m = \begin{cases} 1 & \text{if } p \leq q \\ (q/p)^{j-m} & \text{if } p > q \end{cases}.$$

The probability of not dropping to height m is given by

$${}_jQ_m = 1 - {}_jP_m = \begin{cases} 0 & \text{if } p \leq q \\ 1 - (q/p)^{j-m} & \text{if } p > q \end{cases}.$$

The formula for ${}_jP_m$ for $p \leq q$ and $m = 0$ gives us the famous ‘‘Gambler’s Ruin.’’ Suppose p is the probability of winning a bet in which we win or lose \$1 at a time. No matter how much money j we initially start with, we eventually will drop to \$0 with certainty. That is, the persistent gambler goes broke with probability 1!

In a similar fashion, we also can find the probability that simple random walks will eventually increase to height n . Here we observe that the sets ${}_jU_m^n$ form a nested, increasing sequence as m decreases to i : ${}_jU_i^n \supseteq {}_jU_{i-1}^n$ for all $i \geq m$. Indeed, if a walk reaches height n before height i , then it also reaches height n before height $i - 1$. Thus,

$$P\left(\bigcup_{m=i} {}_jU_m^n\right) = \lim_{m \rightarrow \infty} P({}_jU_m^n) = \lim_{m \rightarrow \infty} {}_jP_m^n = \begin{cases} 1 & \text{if } p \leq q \\ (p/q)^{n-j} & \text{if } p > q \end{cases},$$

which gives the probability of eventually reaching height n (because the paths actually reach n before some particular lower boundary $i \geq j$). So we have

Theorem 2. For simple random walks beginning at height j , the probability of reaching height n , for $n \geq j$, is given by

$${}_jP^n = \begin{cases} 1 & \text{if } p \geq q \\ (p/q)^{n-j} & \text{if } p < q \end{cases}.$$

The probability of not reaching height n is given by

$${}_jQ^n = 1 - {}_jP^n = \begin{cases} 0 & \text{if } p \geq q \\ 1 - (p/q)^{n-j} & \text{if } p < q \end{cases}.$$

2. Average Number of Steps to Reach a Single Boundary

We now seek to find the average number of steps needed to decrease to height m . To do so, we again let ${}_jS_m^n$ be the number of steps needed for simple random walks to reach a boundary of height n or m when starting at height j . Then the times $\{{}_jS_m^n\}_{n=j}$ form an increasing sequence. Indeed, the number of steps needed to reach n or m must be less than or equal to the number of steps needed to reach $n+1$ or m .

Now let ${}_jS_m$ be the number of steps needed to decrease to height m . Because $P({}_jS_m < \infty) = {}_jP_m$, then by Theorem 1, ${}_jS_m$ is finite on a set of probability 1 when $p \geq q$. It is also clear that ${}_jS_m^n \leq {}_jS_m$ for all $n \geq j$, for if a walk reaches m before n , then the times are the same; but if the walk reaches n first, then ${}_jS_m^n < {}_jS_m$. We now assert that ${}_jS_m^n(\omega)$ converges to ${}_jS_m(\omega)$ whenever ${}_jS_m(\omega)$ is finite.

Lemma 1. If ${}_jS_m(\omega) < \infty$, then $\lim_n {}_jS_m^n(\omega) = {}_jS_m(\omega)$.

Proof. Consider a path ω such that ${}_jS_m(\omega) < \infty$ and let $N = {}_jS_m(\omega)$. Then for all n such that $n - j > N$, a simple walk along path ω cannot reach n within N steps. Thus ${}_jS_m^n(\omega) = N$ for all $n > j + N$ and therefore $\lim_n {}_jS_m^n(\omega) = {}_jS_m(\omega)$. 🍏

Because ${}_jS_m < \infty$ with certainty for $p \geq q$, we have:

Corollary 1. Let $p \geq q$. On a set of probability 1, ${}_jS_m^n$ increases to ${}_jS_m$ as $n \rightarrow \infty$.

We now can assert that, for $p < q$, the averages of ${}_j S_m^n$ converge to the average of ${}_j S_m$ as $n \rightarrow \infty$. That is, the limit of the averages is the average of the limit. To obtain our result, we simply apply the following theorem from analysis:

The Monotone Convergence Theorem. Let $\{X_n\}_{n=1}^{\infty}$ be sequence of non-negative, increasing random variables that converge on a set of probability 1 to the random variable X . Then $\lim_{n \rightarrow \infty} E[X_n] = E[X]$.

Hence we have:

Theorem 3. For $p < q$, $\lim_{n \rightarrow \infty} E[{}_j S_m^n] = E[{}_j S_m]$.

Now we simply evaluate the limit:

$$\begin{aligned} \lim_{n \rightarrow \infty} E[{}_j S_m^n] &= \begin{cases} \lim_{n \rightarrow \infty} \frac{(n-j)(j-m)}{2p} & \text{if } p = q \\ \lim_{n \rightarrow \infty} \frac{n-m}{p-q} \times \frac{1-(q/p)^{j-m}}{1-(q/p)^{n-m}} - \frac{j-m}{p-q} & \text{if } p < q \end{cases} \\ &= \begin{cases} \frac{j-m}{q-p} & \text{if } p < q \end{cases} \end{aligned}$$

And we now have the following result:

Theorem 4. For simple random walks beginning at height j , the average of the number of steps ${}_j S_m$ needed to decrease to height m , for $m < j$, is given by

$$E[{}_j S_m] = \begin{cases} \frac{j-m}{q-p} & \text{if } p < q \end{cases}$$

Similarly, for a fixed n the times $\{{}_j S_m^n\}$ increase, as $m \rightarrow \infty$, to the number of steps ${}_j S^n$ needed to reach height n . Therefore we again can apply a monotone convergence argument to obtain the following results:

(i) If $j S^n(\omega) < \infty$, then $\lim_{m \rightarrow \infty} j S_m^n(\omega) = j S^n(\omega)$.

(ii) Let $p < q$. On a set of probability 1, $j S_m^n$ increases to $j S^n$ as $m \rightarrow \infty$.

(iii) $\lim_{m \rightarrow \infty} E[j S_m^n] = E[j S^n]$.

Theorem 5. For simple random walks beginning at height j , the average of the number of steps $j S^n$ needed to increase to height n , for $n > j$, is given by

$$E[j S^n] = \begin{cases} \frac{n-j}{p-q} & \text{if } p < q \\ \infty & \text{if } p > q \end{cases}$$

Theorem 5 shows the fallacy in the “quit when you get ahead” gambling philosophy for $p < q$. If p is the probability of winning a bet in which you gain or lose \$1 at a time (or \$ a at a time), then on average it will take an infinite number of plays to go ahead just \$1 even if $p = 0.5$.

When $p = q = 0.5$, then a walk will go ahead of the initial starting height with probability 1, but the average time is ∞ which simply means that the infinite series $E[j S^n] = \sum_{k=0}^{\infty} k P(j S^n = k)$ diverges. On the other hand, the series converges to $(n-j)/(p-q)$ for $p > q$. In this case, paths tend to drift upward which causes them to reach the upper boundary in finite average time.

Note that when $p < q$, then there is a set of positive probability on which the paths do not reach height n . Thus $j S^n = \infty$ on a set of positive probability which clearly makes $E[j S^n] = \infty$.

3. Limits of the Average Ending Heights

Lastly, we shall analyze the limits of the averages of the final ending heights $j F_m^n$. We begin by showing pointwise limit for paths that reach a single boundary.

Lemma 2. If a simple random walk path decreases to height m along path ω , then $\lim_n j F_m^n(\omega) = m$.

Proof. If a walk decreases to height m along ω , then $j S_m(\omega) < \infty$. Consider such a path and let $N = j S_m(\omega)$. Then for all n such that $n - j > N$, a simple walk along path ω cannot reach n within N steps. Thus $j F_m^n(\omega) = m$ for all $n > j + N$ and $\lim_n j F_m^n(\omega) = m$.

Because ${}_j S_m < \infty$ on a set of probability 1 for $p < q$, we obtain:

Corollary 2. For $p < q$, the final ending heights ${}_j F_m^n$ converge to m on a set of probability 1 as $n \rightarrow \infty$.

Next using the formula for $E[{}_j F_m^n]$, we obtain the limit of the average final ending height as $n \rightarrow \infty$:

$$\lim_n E[{}_j F_m^n] = \begin{cases} j & \text{if } p = q \\ + & \text{if } p > q \\ m & \text{if } p < q \end{cases}.$$

How can this limit be interpreted? First for $p < q$, the walks tend to drift downward and will reach the lower boundary of m with certainty in finite average time. Moreover, ${}_j F_m^n$ converges to m pointwise and

$$\lim_n E[{}_j F_m^n] = m = E[m] = E \lim_n {}_j F_m^n.$$

That is, just like with the times needed to hit a boundary, the limit of the average equals the average of the limit.

However for $p = q$, we still have that ${}_j F_m^n$ converges pointwise to m on a set of probability 1, but the averages of ${}_j F_m^n$ do not converge to the average of the limit. In this case,

$$\lim_n E[{}_j F_m^n] = j \neq E \lim_n {}_j F_m^n = m.$$

Although $\{{}_j F_m^n\}_{n=j}^{\infty}$ converges pointwise to m for $p = q$, the sequence is not bounded by another random variable with finite expectation since ${}_j F_m^n$ always equals either n or m and n is increasing to infinity. Had this sequence been bounded, then the limit of the averages would have to equal the average of the limits by Lebesgue's Dominated Convergence Theorem. The analogous results for the limit as $m \rightarrow \infty$ are stated below:

- (i) If a simple random walk path increases to height n along path ω , then $\lim_{m \rightarrow \infty} {}_j F_m^n(\omega) = n$.
- (ii) For $p < q$, the final ending heights ${}_j F_m^n$ converge to n on a set of probability 1 as $m \rightarrow \infty$.

$$\text{(iii)} \quad \lim_{m \rightarrow \infty} E[{}_j F_m^n] = \begin{cases} j & \text{if } p = q \\ - & \text{if } p < q \\ n & \text{if } p > q \end{cases}$$

Exercises

1. You have a 48% chance of winning a bet. You start with \$100. You bet \$5 at a time with a \$5 payoff for winning.

(a) You will quit if you reach \$150 or if you go drop to \$10. What is the probability of reaching \$150 first? What is the average number of bets needed to quit? What is your average final fortune?

(b) What is the probability of getting ahead to \$105 before going broke? What is the average number of bets needed to hit one of these boundaries? What is your average final fortune?

(c) You wish to gain \$50 before going broke while starting with an amount $5k$. What is the maximum probability of doing so regardless of starting amount?

(d) Assume you can play with unlimited debt. What is the probability of ever “getting ahead” to \$105?

(e) What is the probability of ever falling behind to \$50? What would be the average number of bets needed to do so?

Assume $q > p$. For simple random walks that begins at height j , and move up or down 1 unit at a time, the average maximum height is given by

$$E[jM] = j + \frac{p}{q-p}.$$

The average maximum attained before dropping to height m is

$$E[jM_m] = j + \left(1 - (q/p)^{j-m}\right) \sum_{i=1}^{\infty} \frac{1}{1 - (q/p)^{j-m+i}}.$$

(f) Under the conditions stated above, find the average maximum fortune attained before dropping to \$50.

(g) If you could play with unlimited debt, find the average maximum fortune you would attain.