MATH 540 Poisson Arrival Process

One of the most basic stochastic processes is the *Poisson Process* which is measured over continuous time but has a discrete range. This process is based upon a Poisson distribution $Y \sim Poi(\lambda)$ that counts the number of occurrences during a fixed period of time, where $\lambda > 0$ is the average number of occurrences per unit time. In this setting, an occurrence will be an arrival to a group that has an initial amount of J members. The Poisson process A_t counts the *number of arrivals* to the group through time t. We also can count the *number in the group* at time t and denote this amount by X_t. We then have

$$X_t = J + A_t, \text{ for } t \geq 0. \tag{1}$$

General Assumptions

We assume that we have an initial group of J members so that $X_0 = J$. No members leave the group. Arrivals occur according to a $Poi(\lambda)$ distribution. That is,

(i) Arrivals occur independently.

(ii) The rate of arrival is constant with the average time between arrivals being $L = \dfrac{1}{\lambda}$.

(iii) The probability of exactly k arrivals per unit time is

$$P(A_1 = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

for $k = 0,1,2,...$, and the average number of arrivals per *unit* time is λ, with $\lambda > 0$.

(iv) Because the rate of arrival is constant, the average number of arrivals through time t is proportional to the interval length $[0, t]$. Thus, there are an average of λt arrivals through time t, and the number of arrivals per time t is a $Poi(\lambda t)$ distribution.

Range and Probability Distributions

Each process A_t and X_t has a range consisting of its possible values. In this case, A_t counts the number of arrivals through time t while X_t counts the total number in the group at time t. Thus,

$$Range\ A_t = \{0,1,2,...\} \qquad \text{and} \qquad Range\ X_t = \{J, J+1, J+2, ...\} .$$

The *probability distribution function* (pdf) of a discrete-valued process Z gives the probability, denoted by $P(Z_t = k)$, of Z equaling k at time t, for all k in $Range Z_t$.

The number of arrivals through time t is still a Poisson distribution, but the average number of arrivals is λt. Thus, the probability of exactly k arrivals through time t is

$$P(A_t = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \text{ for } k = 0,1,2,... \tag{2}$$

On the other hand, there are k members in the group if and only if there have been $k-J$ arrivals. So the probability of having exactly k members in the group at time t is given by

$$P(X_t = k) = P(A_t = k-J) = \frac{(\lambda t)^{k-J} e^{-\lambda t}}{(k-J)!} \text{ for } k = J, J+1, J+2, \ldots \tag{3}$$

Mean, Variance, and Standard Deviation

A $Poi(\lambda)$ distribution already has it specified that the average value is λ. And it is also the case that its standard deviation is $\sqrt{\lambda}$. Because $A_t \sim Poi(\lambda\, t)$, we then know that the average number of arrivals per time is t is $\lambda\, t$ with a standard deviation of $\sqrt{\lambda\, t}$. But these facts also can be quickly derived using the pdf of A_t along with the fact that $\sum_{k=0}^{\infty} \frac{x^k}{k\,!} = e^x$ for all x. So the average number of arrivals per time t (also called the *mean* or *expected value*) can be derived by

$$\begin{aligned}
E[A_t] &= \sum_{k \quad RangeA_t} k\, P(A_t = k) = \sum_{k=0}^{\infty} k \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\
&= \sum_{k=1}^{\infty} k \frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad \text{(because first term cancels)} \\
&= \sum_{k=1}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{(\lambda t)^{k+1} e^{-\lambda t}}{k\,!} \qquad \text{(re - index)} \\
&= (\lambda t) e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k\,!} \\
&= (\lambda t) e^{-\lambda t} e^{\lambda t} \\
&= \lambda\, t\,.
\end{aligned} \tag{4}$$

The *variance* of a random variable Y is found by $Var(Y) = E[Y^2] - (E[Y])^2$, and then its standard deviation is given by $\sigma(Y) = \sqrt{Var(Y)}$. The standard deviation gives a way of measuring the *average spread from the mean*.

To find the variance of A_t, we first find the average square of A_t by

$$
\begin{aligned}
E[A_t^2] &= \sum_{k \quad RangeA_t} k^2 P(A_t = k) = \sum_{k=0}^{\infty} k^2 \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\
&= \sum_{k=1}^{\infty} k^2 \frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad \text{(because first term cancels)} \\
&= \sum_{k=1}^{\infty} k \frac{(\lambda t)^k e^{-\lambda t}}{(k-1)!} = \sum_{k=0}^{\infty} (k+1) \frac{(\lambda t)^{k+1} e^{-\lambda t}}{k\,!} \qquad \text{(re - index)} \\
&= (\lambda t) \left(\sum_{k=0}^{\infty} k \frac{(\lambda t)^k e^{-\lambda t}}{k\,!} + \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k\,!} \right) \\
&= (\lambda t) \big(E[A_t] + 1 \big) = (\lambda t)^2 + \lambda t \,.
\end{aligned}
$$

So the variance of A_t is $Var(A_t) = E[A_t^2] - (E[A_t])^2 = (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t$, and its standard deviation is $\sigma(A_t) = \sqrt{Var(A_t)} = \sqrt{\lambda t}$.

We also can use the pdf of X_t to derive its average and standard deviation (an exercise), or we can derive it more quickly using some basic facts about expected value and variance:

(i) For any constant c, $E[c] = c$. (The average value of a constant is that constant.)
(ii) For all random variables Y and Z, $E[Y+Z] = E[Y] + E[Z]$. (The average of a sum is the sum of the averages.)
(iii) $Var(Y+c) = Var(Y)$. (Adding a constant does not change the variance or spread.)
(iv) If Y and Z are independent, then $Var(Y+Z) = VarY) + Var(Z)$.

Now because the total number in the group at time t is $X_t = J + A_t$ (Eq. 1), we have

$$
\begin{aligned}
E[X_t] &= E[J + A_t] \\
&= E[J] + E[A_t] = J + E[A_t] \qquad (5) \\
&= J + \lambda t
\end{aligned}
$$

Moreover, $Var(X_t) = Var(J + A_t) = Var(A_t) = \lambda t$. Thus, the standard deviation of the number in the group at time t is $\sigma(X_t) = \sqrt{\lambda t} = \sigma(A_t)$.

Note: If the initial group number X_0 were *random* but independent of arrivals, then we would have $E[X_t] = E[X_0] + \lambda t$, $Var(X_t) = Var(X_0) + \lambda t$, and $\sigma(X_t) = \sqrt{Var(X_0) + \lambda t}$.

Probability Generating Function

For a discrete-valued stochastic process Z, we can define a probability generating function (pgf) by

$$G_Z(s,t) = E[s^{Z_t}] = \sum_{k \in RangeZ_t} s^k P(Z_t = k)$$

The pgf completely determines the process due to the following properties:

(i) $E[Z_t] = \left.\frac{\partial G_Z(s,t)}{\partial s}\right|_{s=1}$ (the first derivative of G_Z with respect to s, evaluated at $s=1$)

(ii) $P(Z_t = k) = \frac{1}{k!} \times \left.\frac{\partial^k G_Z(s,t)}{\partial s^k}\right|_{s=0}$ (the kth derivative of G_Z wrt s, evaluated at $s=0$)

(iii) If Y and Z are independent, then $G_{(Y+Z)}(s,t) = G_Y(s,t) \times G_Z(s,t)$.

We now shall derive the pgf of the Poisson arrival process A_t:

$$\begin{aligned} G_A(s,t) = E[s^{A_t}] &= \sum_{k \in RangeA_t} s^k P(A_t = k) \\ &= \sum_{k=0}^{\infty} s^k \frac{(\lambda t)^k e^{-\lambda t}}{k!} \\ &= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda st)^k}{k!} \qquad (6) \\ &= e^{-\lambda t} e^{\lambda st} \\ &= e^{\lambda t(s-1)}, \quad \text{for all } s. \end{aligned}$$

We note that $\frac{\partial G_A(s,t)}{\partial s} = (\lambda t)e^{\lambda t(s-1)}$ and $\frac{\partial^k G_A(s,t)}{\partial s^k} = (\lambda t)^k e^{\lambda t(s-1)}$; thus,

$$E[A_t] = \left.\frac{\partial G_A(s,t)}{\partial s}\right|_{s=1} = \left.(\lambda t)e^{\lambda t(s-1)}\right|_{s=1} = \lambda t$$

and

$$P(A_t = k) = \left.\frac{1}{k!} \times (\lambda t)^k e^{\lambda t(s-1)}\right|_{s=0} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$

The Average Curve

We define the *average curve* of a stochastic process Z_t to be the single function

$$a_Z(t) = E[Z_t], \text{ for } t \geq 0.$$

The average curve simply gives the average value at time t. For the process $X_t = J + A_t$, we know that the average value at time t is $J + \lambda t$ (Eq. 5); thus, the average curve in this case is the single continuous linear function

$$a_X(t) = E[X_t] = J + \lambda t, \text{ for } t \geq 0. \tag{7}$$

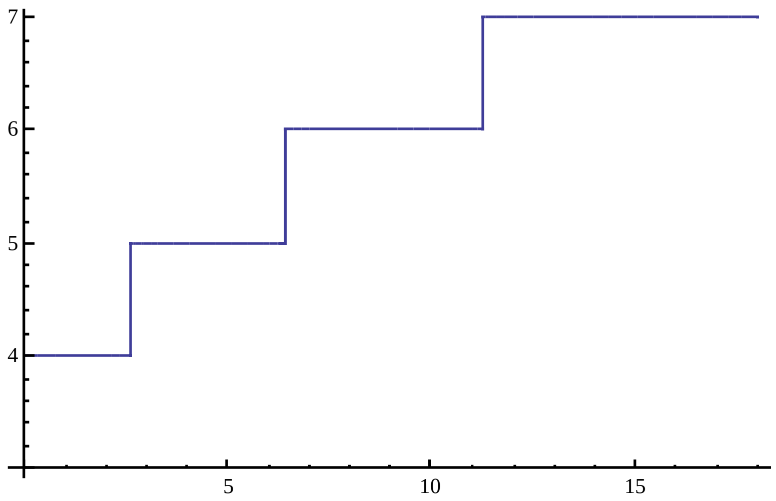
Area Under the Average Curve Up to Deterministic Time *s*

Given the function $a_X(t) = E[X_t]$, we can integrate it from 0 to s to find the area under this average curve. In the case of our process $X_t = J + A_t$, we obtain

$$\begin{aligned}\int_0^s a_X(t)\,dt = \int_0^s E[X_t]\,dt = \int_0^s (J + \lambda t)\,dt = Jt + \frac{\lambda t^2}{2}\Bigg|_0^s \\ = Js + \frac{\lambda s^2}{2}.\end{aligned} \tag{8}$$

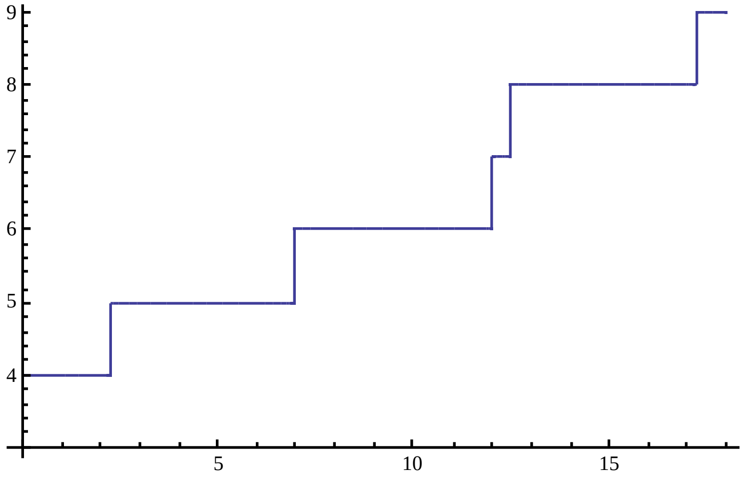
Example. Suppose we begin with $J = 4$ members initially in the group and a new arrival occurs on average every 5 minutes. Let X_t denote the number in the group after t minutes.

In this case, $L = 5$ minutes, which is the average time of an arrival, and $\lambda = 1/L = 1/5$ which gives the average number of arrivals per minute. We also can say that there are, on average, 1 arrival every 5 minutes, 2 arrivals every 10 minutes, etc. Below is a *single trial* ω_1 that shows the additions to the group during the first 18 minutes.

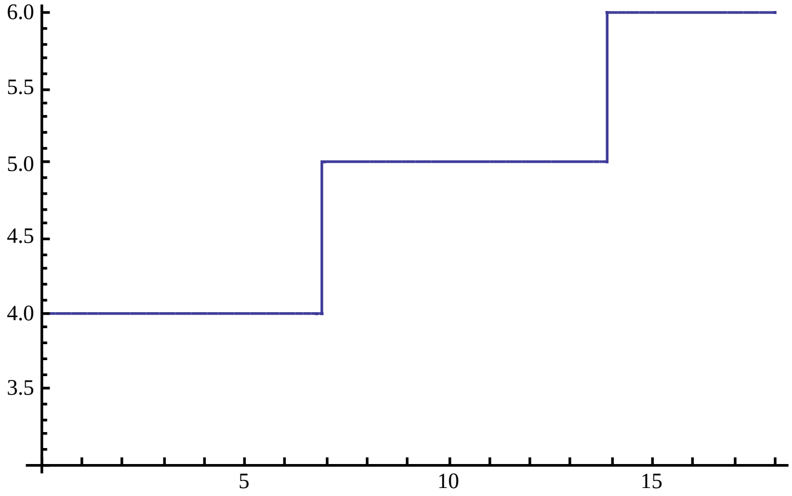


Times of Arrivals for ω_1: 2.60611, 6.40263, 11.2933

A trial ω is only one possible outcome; but there are uncountable possibilities of what could possibly happen over the course of time. Below are two other curves created by different outcomes over the time interval $0 \leq t \leq 18$:



Times of Arrivals for ω_2: 2.27315, 6.94523, 12.0358, 12.4466, 17.1812



Times of Arrivals for ω_3: 6.90239, 13.8854

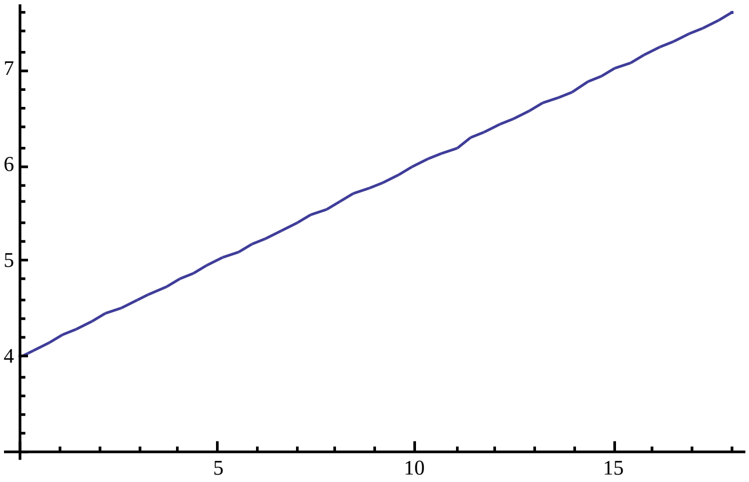
Given a single trial ω, $X_t(\omega)$ represents the total number in the group at time t for that individual trial (or generically, the height at time t for that trial). So $X_t(\omega)$ represents only one single curve (or path or trajectory). To represent all possible curves, we write X_t which is the process as a whole. In particular, the average amount at time t over all possible curves is denoted by $E[X_t]$, and $P(X_t = k)$ denotes the probability of having exactly k in the group at time t when considering all possible curves.

What are $E[X_{10}]$, and $P(X_{10} = 6)$ in this case? Again, $\lambda = 1/5$ is the average number of arrivals per minute; so $E[X_{10}] = J + \lambda \times 10 = 4 + \frac{10}{5} = 6$. The standard deviation of the number in the group at time 10 minutes is $\sigma(X_{10}) = \sqrt{\lambda \times 10} = \sqrt{2} \approx$ 1.414.

The probability of having exactly 6 members at time $t = 10$ equals the probability of having exactly 2 arrivals through time t, which is

$$P(A_{10} = 2) = \frac{(\lambda \times 10)^2 e^{-\lambda \times 10}}{2!} = \frac{2^2 e^{-2}}{2!} \approx 0.27$$

The average curve is again one single function of time t given in Equation (7) as $a_X(t) = E[X_t] = J + \lambda t$. In this example, $J = 4$ and $\lambda = 1/5$; thus, the average of all possible curves is the linear function $a_X(t) = 4 + \frac{t}{5}$.



The average curve $a_X(t) = E[X_t] = 4 + \frac{t}{5}$, for $t \geq 0$.

Interchange of Average

We have noted that we can find the area under the single average curve up to time s as given in Equation (8) by $Js + \frac{\lambda s^2}{2}$. But we also can find the areas $\int_0^s X_t(\omega)\,dt$ under all the individual curves and then compute the average area. It turns out that the average area under all possible curves equals the area under the average curve. That is,

$$E\left[\int_0^s X_t\,dt\right] = \int_0^s E[X_t]\,dt. \qquad (9)$$

(This property holds with many different types of stochastic processes,
and we shall eventually prove it for this one.)

Exercises

Throughout, let $X_t = J + A_t$, where $A_t \sim Poi(\lambda\, t)$.

1. Use the pdf of X_t to derive $E[X_t]$ and $E[X_t^2]$. (You may use the known facts/sums already derived for A_t.) Then use your results to derive $\sigma(X_t)$.

2. Derive the pgf of X_t. Then use this pgf to compute $E[X_t]$.

3. (a) Write $P(A_t \geq n)$, the probability of having n or more arrivals by time t, as a finite sum.
(b) Using your result in (a), compute $\lim_{t \to \infty} P(A_t \geq n)$ and explain the result.
(c) Using your result in (a), compute $\lim_{n \to \infty} P(A_t \geq n)$ and explain the result.
(d) Use the result in (a) to derive an expression for $P(X_t \geq K)$.

4. (Stopped Poisson Process). For a fixed integer $n \geq 1$, let

$$S_{n,t} = \begin{cases} A_t & \text{if } 0 \leq A_t < n \\ n & \text{otherwise} \end{cases}$$

That is, arrivals continue until the nth arrival. When we obtain n or more arrivals, then we group them all as the nth arrival. (Essentially, we bound the maximum number of arrivals as n.)

(a) What is the range of $S_{n,t}$?
(b) Use the pdf of A_t and the result in 3(a) to write the pdf of $S_{n,t}$.
(c) Give an expression for $E[S_{n,t}]$. Then compute $\lim_{t \to \infty} E[S_{n,t}]$ and explain the result.
(d) For a fixed t, what would you conjecture $\lim_{n \to \infty} E[S_{n,t}]$ to be. Explain.
(e) Let there be an average of $\lambda = 2.5$ arrivals per hour, let $n = 4$, and let $t = 2$ hours. Compute $E[A_2]$ and $E[S_{4,2}]$.