Dr. Neal, Spring 2008

MATH 540 **The Boundary Problem: Markov Chain Solution**

Consider a random walk X that starts at positive height j, and on each independent step, moves upward a units with probability p, moves downward b units with probability q, or remains at the same height with probability $r = 1 - p - q$. It stops upon reaching any height greater than or equal to n or dropping to any height less than or equal to 0. (We assume all height units are integers.)

Rather than just analyzing the end state upon reaching a boundary, how do we analyze the state after each step? In particular, after a fixed number of steps, what are the probabilities of being at each possible height 0, 1, . . ., j, . . ., n? What is the average height after this number of steps? Can we still find the "long-term" probability of reaching n or above before dropping to 0 or below? Can we find the average number of steps needed to reach (or surpass) a boundary, and can we find the average ending height when stopping upon reaching a boundary?

There are no closed-form algebraic formulas for this general *non-symmetric* boundary problem for which the upward jumps a do not equal the downward jumps b. However, by using *Markov Chains* we can find numerical solutions.

Example Scenario

Suppose we begin at height $j = 6$, go up $a = 3$ units with probability $p = 0.40$, go down $b = 2$ units with probability $q = 0.50$, or remain constant with probability $r = 0.10$. We wish to reach a goal of $n = 10$ (or above) before dropping to 0 (or below).

(a) After 5 steps, what are the probabilities of each possible height? What is the average height after these 5 steps?

(b) What happens as the number of steps gets very large?

Range of Heights

Throughout, we will use boundaries of 0 and n and initial height j, where $0 < j < n$. As with the symmetric random walk, we can always use vertical translation to consider a different lower boundary. That is, we can translate $m < j < n$ to $0 < (j - m) < (n - m)$.

Now if we are at height $n - 1$ and we move upward on the next step, then we reach height $(n - 1 + a) \geq n$ and we stop. And if we are at height 1 and move downward on the next step, then we drop to height $(1 - b) \leq 0$ and we stop. These give the extreme values. So the range of possible heights is

$$1 - b,\ -b,\ \ldots,\ 0,\ 1,\ \ldots,\ j,\ \ldots,\ n - 1,\ n,\ \ldots,\ n - 1 + a$$

In all, there are $(n - 1 + a) - (1 - b - 1) = n + a + b - 1$ possible heights. For example, if our boundaries are 0 to 10 but we move up 3 units or down 2 units at a time, then we can reach height 12 or drop down to height –1. So there are 14 possible heights from –1 to 12.

Initial State Matrix

We let B be the $1\times(n+a+b-1)$ *initial state matrix* that gives the initial probabilities of being at each possible height from $1-b$, $-b$, . . . , 0, 1, . . . , j, . . . , $n-1$, n, . . . , $n-1+a$.

If we always start at height j, then every entry in B is 0 except for a 1 in the j spot which is actually the $(j+b)th$ column. In our example, we initially have probability 1 of starting at height $j=6$; so B is given by

	–1	0	1	2	3	4	5	6	7	8	9	10	11	12
***B* =**	**(0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0)**

Note 1: The matrix B only consists of the row of (0 , 0 , . . . , 1, . . . ,0, 0). The values above are only used to identify the height corresponding to the column position.

Note 2: If we were to start at uniformly random heights strictly between 0 and 10, then B would look like:

	–1	0	1	2	3	4	5	6	7	8	9	10	11	12
***B* =**	**(0**	**0**	**1/9**	**1/9**	**1/9**	**1/9**	**1/9**	**1/9**	**1/9**	**1/9**	**1/9**	**0**	**0**	**0)**

Transition Matrix

We next let A be the $(n+a+b-1)\times(n+a+b-1)$ *matrix of transition probabilities* from each height to every height. The values $a_{i\,j}$ are the probabilities of being at height j after a step given that the walk was at height i on the previous step.

In our example, suppose at some point we are at height 3. Then after the next step, it is only possible to be at heights 1, 3, or 6. We drop 2 units to height 1 with probability q, stay at height 3 with probability r, or go up 3 units to height 6 with probability p.

Thus, the transition row in A that represents the height 3, will look like

–1	0	1	2	3	4	5	6	7	8	9	10	11	12
(0	**0**	***q***	**0**	***r***	**0**	**0**	***p***	**0**	**0**	**0**	**0**	**0**	**0)**

Also, if we reach any of the boundary heights of –1, 0, 10, 11, or 12, then we remain at that height with probability 1.

Here is the complete matrix A for our example. (Only the terms in bold are part of the matrix. The other terms on the outside are simply place holders that tell the possible height values.)

	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
-1	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
0	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	***q***	**0**	***r***	**0**	**0**	***p***	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
2	**0**	***q***	**0**	***r***	**0**	**0**	***p***	**0**	**0**	**0**	**0**	**0**	**0**	**0**
3	**0**	**0**	***q***	**0**	***r***	**0**	**0**	***p***	**0**	**0**	**0**	**0**	**0**	**0**
4	**0**	**0**	**0**	***q***	**0**	***r***	**0**	**0**	***p***	**0**	**0**	**0**	**0**	**0**
5	**0**	**0**	**0**	**0**	***q***	**0**	***r***	**0**	**0**	***p***	**0**	**0**	**0**	**0**
6	**0**	**0**	**0**	**0**	**0**	***q***	**0**	***r***	**0**	**0**	***p***	**0**	**0**	**0**
7	**0**	**0**	**0**	**0**	**0**	**0**	***q***	**0**	***r***	**0**	**0**	***p***	**0**	**0**
8	**0**	**0**	**0**	**0**	**0**	**0**	**0**	***q***	**0**	***r***	**0**	**0**	***p***	**0**
9	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	***q***	**0**	***r***	**0**	**0**	***p***
10	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**
11	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**
12	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**

The place holders on the left represent the *previous* state, and the place holders on the top represent the possible heights after another step is taken.

Height Range Matrix

Lastly, let H be the $1 \times (n + a + b - 1)$ matrix of all possible heights:

$$H = \mathbf{(-1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12)}$$

Markov Chain Solution

(a) To find the probabilities of having each possible height after k steps, we multiply $B \times A^k$.

(b) To find the average height after k steps, multiply $B \times A^k \times H^T$.

(c) To find the “long-term” probabilities of reaching the boundaries, multiply $B \times A^k$ for a “large” k such as $k = 200$. All of the probabilities for the inner heights 1 to $(n - 1)$ will converge to 0. Then multiply $B \times A^k \times H^T$ with this large k to approximate the final average height when quitting upon reaching a boundary.

Numerical Example

We now complete the example with $p = 0.40$, $q = 0.50$, $r = 0.10$, $j = 6$, $n = 10$, $a = 3$, and $b = 2$. First, let us determine the probabilities after $k = 5$ steps. The matrix $B \times A^5$ is

-1	0	1	2	3	4	5	6	7	8	9	10	11	12
(0	**0.17**	**0.075**	**0.0025**	**0.075**	**0.00025**	**0.03**	**0.12**	**0.004**	**0.06**	**0.0002**	**0.2176**	**0.048**	**0.19744)**

Thus, there is a 46.3% chance of reaching the goal of 10 or more within 5 steps and a 17% chance of having bottomed out.

The average height after 5 steps is given by the product $B \times A^5 \times H^T \approx 6.75914$. Thus after 5 steps, the average height will be *more* than the initial height of 6. Why?

After 200 steps, the matrix $B \times A^{200}$ is

-1	0	1	2	3	4	5	6	7	8	9	10	11	12
(0.096789	**0.243163**	**≈ 0**	**≈ 0**	**≈ 0**	**≈ 0**	**≈ 0**	**≈ 0**	**≈ 0**	**≈ 0**	**≈ 0**	**0.2959**	**0.1115**	**0.2526**

And the average height is 7.12038.

This technique also applies when the payoff is 1:1.

Example 2. Suppose we start with \$450 and win or lose \$50 at a time with $p = 0.49$. We wish to quit when we "go ahead" to \$500 or drop to \$100. Initially, we shall make a maximum of 7 bets. Rather than create a huge 501×501 matrix A, we can divide the values by \$50 and then note that this scenario is equivalent to starting with \$9 and winning or losing \$1 at a time while trying to reach \$10 before dropping to \$2. Now the matrices B, H, and A are

\$ (× 50)	2	3	4	5	6	7	8	9	10
B **:**	**(0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0)**

H :	**(100**	**150**	**200**	**250**	**300**	**350**	**400**	**450**	**500)**

(× 50)	2	3	4	5	6	7	8	9	10
2	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
3	***q***	***r***	***p***	**0**	**0**	**0**	**0**	**0**	**0**
4	**0**	***q***	***r***	***p***	**0**	**0**	**0**	**0**	**0**
5	**0**	**0**	***q***	***r***	***p***	**0**	**0**	**0**	**0**
6	**0**	**0**	**0**	***q***	***r***	***p***	**0**	**0**	**0**
7	**0**	**0**	**0**	**0**	***q***	***r***	***p***	**0**	**0**
8	**0**	**0**	**0**	**0**	**0**	***q***	***r***	***p***	**0**
9	**0**	**0**	**0**	**0**	**0**	**0**	***q***	***r***	***p***
10	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**

(Here $p = 0.49$, $q = 0.51$, and $r = 0$.)

If we compute $B \times A^7$, we find that there is a 0.7119 probability of having reached \$500 within 7 bets, and only 0.00897411 probability of having lost each bet and dropped to \$100. However on average after 7 bets, we will have $B \times A^7 \times H^T =$ \$446.55, which is less than the starting amount of \$450.

Using $k = 500$, we find that there is about 0.856815 probability of reaching \$500 before dropping to \$200, with an average final fortune of \$442.73. In this case, the probability of reaching the top first is also given by

$$\frac{1-(q/p)^{j-m}}{1-(q/p)^{n-m}} = \frac{1-(0.51/0.49)^7}{1-(0.51/0.49)^8} \approx 0.856815369$$

It is quite tedious to compute the matrix products by hand, or even to enter the matrices into a calculator. However, a simple *Mathematica* program can be used to create the matrices and the necessary products after one enters the variables n, j, p, q, a, b, and k. We note though that this method has not given us the average number of steps needed to hit a boundary.

Mathematica **Exercises (See markov.nb file)**

1. When betting on "Red" in roulette, the probability of winning is $p = 18/38$. Suppose you start with $5000 and bet $100 at a time. The payoff is also $100. Your goal is to reach $6000. You will quit if you reach $6000 or drop to $2000.

(a) (i) After a maximum of 10 bets, what are your 4 most likely remaining dollar values and what are the probabilities of having these dollar amounts?

(ii) What is the average fortune after these 10 possible bets?

(iii) What is the probability of having at least as much money as you started with after at most 10 bets?

(b) (i) What is the long-term probability of reaching $6000 before dropping to $2000?

(ii) What is the average fortune when quitting upon reaching $6000 or dropping to $2000?

(iii) What is the average number of bets that would be made when stopping upon reaching one of these boundaries?

2. When betting on a "column" in roulette, the probability of winning is $p = 12/38$. Suppose you start with $200 and bet $5 at a time. The payoff is $10. Your goal is to reach $225. You will quit if you reach $225 (or higher) or drop to $50.

(a) (i) After a maximum of 5 bets, what is the probability of having hit your goal of $225 or higher?

(ii) What is the average fortune after these 5 possible bets?

(iii) What is the probability that you are still playing after 5 bets?

(b) (i) What is the long-term probability of reaching $225 or higher before dropping to $50?

(ii) What is the average fortune when quitting upon reaching $225 (or higher) or dropping to $50?

(iii) Given that you do in fact reach $225 or higher, what proportion of the time is it actually $225?