Dr. Neal, Spring 2008

MATH 540 The Random Exponential Growth Process

We now analyze the stochastic process of the form $Z_t = C \times A^t$, where $C \sim U[c, d]$, $A \sim U[r, s]$, and A is independent of C. For convenience, we shall assume $0 < c < d$ and $0 < r < s$. As an example throughout, we shall take $C \sim U[1000, 2000]$ and $A \sim U[0.90, 1.05]$, where Z_t then gives the value at time t.

The Average Value at Time *t*

By independence, we have $E[Z_t] = E[C \times A^t] = E[C] \times E[A^t]$. We know $E[C] = (c+d)/2$ is the average initial value; but we must determine $E[A^t]$. To do so, we use the fact that $E[g(A)] = \int_{RangeX} g(x)\, f_A(x)\, dx$, or in this case, $E[A^t] = \int_r^s x^t \frac{1}{s-r}\, dx$. Evaluating, we obtain

$$\begin{aligned} E[Z_t] &= E[C \times A^t] = E[C] \times E[A^t] \\ &= \frac{c+d}{2} \times \int_r^s x^t \frac{1}{s-r}\, dx = \frac{c+d}{2} \times \frac{x^{t+1}}{(t+1)(s-r)}\Bigg|_r^s \qquad (1) \\ &= \frac{c+d}{2} \times \frac{s^{t+1} - r^{t+1}}{(t+1)(s-r)}. \end{aligned}$$

In particular for $C \sim U[1000, 2000]$ and $A \sim U[0.90, 1.05]$ at $t = 4$, we have

$$E[Z_4] = E[C \times A^4] = 1500 \times \frac{1.05^5 - 0.9^5}{5 \times 0.15} \approx 1371.583.$$

The Variance at Time *t*

To compute the variance of Z_t, we first need to compute the expression

$$E[Z_t^2] = E[(C \times A^t)^2] = E[C^2] \times E[A^{2t}].$$

From (1), we can see that $E[A^t] = \dfrac{s^{t+1} - r^{t+1}}{(t+1)(s-r)}$. By replacing t with $2t$ we have

$$E[A^{2t}] = \frac{s^{2t+1} - r^{2t+1}}{(2t+1)(s-r)} \quad \text{(which is also found by } E[A^{2t}] = \int_r^s x^{2t} \frac{1}{s-r}\, dx\text{)}.$$

$$\text{Also, } E[A^2] = \frac{s^3 - r^3}{3(s-r)} = \frac{r^2 + rs + s^2}{3}, \text{ where } A \sim U[r, s].$$

But then for $C \sim U[c, d]$, we must have

$$E[C^2] = \frac{d^3 - c^3}{3(d-c)} = \frac{c^2 + cd + d^2}{3}\,.$$

Now by independence, we have

$$E[Z_t^2] = E[C^2 \times A^{2t}] = E[C^2] \times E[A^{2t}] = \frac{c^2 + cd + d^2}{3} \times \frac{s^{2t+1} - r^{2t+1}}{(2t+1)(s-r)}\,.$$

So the variance at time t is

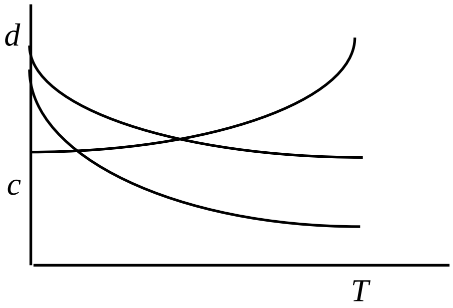
$$\begin{aligned} Var(Z_t) &= E[Z_t^2] - (E[Z_t])^2 \\ &= \frac{(c^2 + cd + d^2)(s^{2t+1} - r^{2t+1})}{3(2t+1)(s-r)} - \left(\frac{(c+d)(s^{t+1} - r^{t+1})}{2(t+1)(s-r)}\right)^2 \end{aligned}$$

(which does not simplify significantly.)

Specifically for $C \sim U[1000, 2000]$ and $A \sim U[0.90, 1.05]$ at time $t = 4$, we have $Var(Z_4) \approx 130{,}452.0988$ which gives a standard deviation of about 361.182.

The Average Area Up to Time *T*

Each trajectory of the stochastic process $Z_t = C\,A^t$ creates a non-negative function that has positive area under the curve over the interval $[0, T]$. We now wish to determine the average area created by all trajectories over $[0, T]$.



Upon choosing a random initial value C from $[c, d]$, then independently choosing a random exponential base A from $[r, s]$, the area under a single path is given by

$$\int_0^T Z_t\,dt = \int_0^T C \times A^t\,dt = C \int_0^T A^t\,dt$$

Because the value C is independent of A, it will also be independent of the created area $\int_0^T A^t\,dt$. And because the average of an independent product equals the product of the averages, we obtain the average area under all trajectories through time T as

$$\begin{aligned}
E\left[\int_0^T Z_t\,dt\right] &= E\left[C\int_0^T A^t\,dt\right] = E[C]\times E\left[\int_0^T A^t\,dt\right] \\
&= E[C]\times E\left[\frac{A^t}{\ln A}\bigg|_0^T\right] = E[C]\times E\left[\frac{A^T-1}{\ln A}\right] \\
&= \frac{c+d}{2}\times\int_r^s \frac{x^T-1}{\ln x}\times\frac{1}{s-r}\,dx \qquad (2)\\
&= \frac{c+d}{2(s-r)}\int_r^s \frac{x^T-1}{\ln x}\,dx\,.
\end{aligned}$$

Unfortunately, this last integral has no antiderivative; thus, this average area cannot be simplified further.

The Volume of a Solid

To understand this integral some more, let us consider the volume of the solid S under the surface $z=x^y$ over the rectangular region $[r,s]\times[0,\ T]$ in the xy plane. Because $z=x^y$ is continuous and bounded over the region $[r,s]\times[0,\ T]$, S must have a finite volume which can be computed as

$$Vol(S)=\int_r^s\int_0^T x^y\,dy\,dx=\int_r^s\left(\frac{x^y}{\ln x}\bigg|_0^T\right)dx=\int_r^s\left(\frac{x^T-1}{\ln x}\right)dx.$$

This is the same integral obtained in (2). Thus, we can say

$$E\left[\int_0^T Z_t\,dt\right]=\frac{c+d}{2(s-r)}\,Vol(S). \qquad (3)$$

Will we have better luck evaluating $Vol(S)$ if we re-arrange the order of integration? The volume also can be written as

$$Vol(S)=\int_0^T\int_r^s x^y\,dx\,dy=\int_0^T\left(\frac{x^{y+1}}{y+1}\bigg|_r^s\right)dy=\int_0^T\left(\frac{s^{y+1}-r^{y+1}}{y+1}\right)dy\,. \qquad (4)$$

Again, no closed-form antiderivative exists for this integral.

The Area Under the Average Curve

Next, consider a function g that is defined to be the average value at time t of all trajectories of the original process $Z_t = C\,A^t$. From Equation (1) we have

$$g(t) = E[Z_t] = \frac{c+d}{2} \times \frac{s^{t+1} - r^{t+1}}{(1+t)(s-r)}.$$

Then a single curve is created by g over the interval $[0, T]$. What is the area under this curve? Integrating we obtain

$$\begin{aligned}\int_0^T g(t)\,dt &= \int_0^T E[Z_t]\,dt = \int_0^T \frac{c+d}{2} \times \frac{s^{t+1} - r^{t+1}}{(t+1)(s-r)}\,dt \\ &= \frac{c+d}{2(s-r)} \int_0^T \frac{s^{t+1} - r^{t+1}}{(t+1)}\,dt.\end{aligned}$$

But this integral is the same as that obtained in Equation (4), which combined with Equation (3) gives

$$\int_0^T E[Z_t]\,dt = \frac{c+d}{2(s-r)} \int_0^T \frac{s^{t+1} - r^{t+1}}{(t+1)}\,dt = \frac{c+d}{2(s-r)}\,Vol(S) = E\left[\int_0^T Z_t\,dt\right].$$

Thus, $E\left[\int_0^T Z_t\,dt\right] = \int_0^T E[Z_t]\,dt$; that is, the average of the area under all curves equals the area under the single average curve.

This interesting interchange of average also occurs with trajectories created by one-dimensional random walks (to be seen later). It also applies to a stochastic process of the form $Z_t = Mt + B$, where M and B are any type of distribution (an easy exercise).

In general, whenever X and Y are independent distributions, then the interchange of average will apply to a process of the form $Z_t = g(X, Y, t)$, where g is a continuous function. Indeed, if X has probability density $f_X(x)$ and Y has probability density $f_Y(y)$, then

$$\begin{aligned}E\left[\int_0^T Z_t\,dt\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^T g(x,y,t)\,dt\right) f_X(x)\,f_Y(y)\,dy\,dx \\ &= \int_0^T \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y,t)\,f_X(x)\,f_Y(y)\,dy\,dx\right) dt = \int_0^T E[Z_t]\,dt.\end{aligned}$$

But it remains a nice exercise to compute these values, as well to determine the average and variance at time t, for various other processes such as $Z_t = C \times t^A$, $C\sin(At)$, or $\cos(At + C)$, where $C \sim U[c, d]$ and $A \sim U[r, s]$.

Example. Let $Z_t = C \times A^t$ where $C \sim U[1000, 2000]$ and $A \sim U[0.90, 1.05]$. Compute $E\left[Z_t \,\middle|\, C \le 1500 \cap A \ge 1\right]$.

Solution. Assuming that $1000 \le C \le 1500$ and $1 \le A \le 1.05$, we obtain

$$\begin{aligned}
E[Z_t \mid C \le 1500 \cap A \ge 1] &= E\left[C \times A^t \,\middle|\, C \le 1500 \cap A \ge 1\right] \\
&= E[C \mid C \le 1500] \times E\left[A^t \,\middle|\, A \ge 1\right] \\
&= E[C \mid 1000 \le C \le 1500] \times E\left[A^t \,\middle|\, 1 \le A \le 1.05\right] \\
&= 1250 \times \left(\frac{1}{P(1 \le A \le 1.05)} \int_1^{1.05} x^t \, \frac{1}{(1.05 - 0.90)} \, dx \right) \\
&= 1250 \times \left(\frac{1}{0.05/\ 0.15} \right) \frac{x^{t+1}}{0.15(t+1)} \Bigg|_1^{1.05} \\
&= \frac{25000(1.05^{t+1} - 1)}{t+1}\,.
\end{aligned}$$

In particular at $t = 4$, we have $E\left[Z_4 \,\middle|\, C \le 1500 \cap A \ge 1\right] \approx 1381.4078$ compared to $E[Z_4] \approx 1371.583$. Also, note that $E\left[Z_t \,\middle|\, C \le 1500 \cap A \ge 1\right]$ can be obtained with the formula $E[Z_t] = \dfrac{c+d}{2} \times \dfrac{s^{t+1} - r^{t+1}}{(t+1)(s-r)}$ using $[c, d] = [1000, 1500]$ and $[r, s] = [1, 1.05]$.

The Distribution of a Specific Time T

Assume now that the exponential base A is always greater than 1. That is, assume $C \sim U[c, d]$, $A \sim U[r, s]$, with A independent of C, but also with $r > 1$. Now let T denote the time needed for path $Z_t = C \times A^t$ to double its initial amount of C. We now shall find the mean and standard deviation of T as well as its cdf and pdf.

The time T needed to double is given by the solution to $C \times A^T = 2C$, which is

$$T = \frac{\ln 2}{\ln A}.$$

Conveniently, T is simply a function of the single random variable A and does not involve C. Thus, the expected value of T is given by

$$\begin{aligned}
E[T] &= E\left[\frac{\ln 2}{\ln A}\right] = \ln 2\, E\left[\frac{1}{\ln A}\right] \\
&= \ln 2 \int_r^s \frac{1}{\ln x} f_A(x)\, dx = \frac{\ln 2}{s - r} \int_r^s \frac{1}{\ln x}\, dx\,.
\end{aligned}$$

Unfortunately, there is no closed-form antiderivative of $\int 1/\ln x\,dx$; thus, $E[T]$ does not simplify further.

Next, we shall use the cdf technique to find the distribution of $T = \ln 2/\ln A$. First, because $r \le A \le s$ with $r > 1$, we have $\ln r \le \ln A \le \ln s$, then $\dfrac{\ln r}{\ln 2} \le \dfrac{\ln A}{\ln 2} \le \dfrac{\ln s}{\ln 2}$, then $\dfrac{\ln 2}{\ln r} \ge \dfrac{\ln 2}{\ln A} \ge \dfrac{\ln 2}{\ln s}$. Thus, $\dfrac{\ln 2}{\ln s} \le T \le \dfrac{\ln 2}{\ln r}$.

Now suppose $\dfrac{\ln 2}{\ln s} \le z \le \dfrac{\ln 2}{\ln r}$ and consider the cdf of T at z:

$$\begin{aligned}
F_T(z) &= P(T \le z) = P\left(\frac{\ln 2}{\ln A} \le z\right) = P\left(\frac{1}{z}\ln 2 \le \ln A\right) = P\left(\ln 2^{1/z} \le \ln A\right) \\
&= P(2^{1/z} \le A) = 1 - P(A \le 2^{1/z}) \\
&= 1 - F_A(2^{1/z}) \\
&= 1 - \begin{cases} 0 & \text{if } 2^{1/z} < r \\ \dfrac{2^{1/z} - r}{s - r} & \text{if } r \le 2^{1/z} \le s \\ 1 & \text{if } 2^{1/z} > s \end{cases} \\
&= \begin{cases} 1 & \text{if } \dfrac{\ln 2}{\ln r} < z \\ \dfrac{s - 2^{1/z}}{s - r} & \text{if } \dfrac{\ln 2}{\ln s} \le z \le \dfrac{\ln 2}{\ln r} \\ 0 & \text{if } z < \dfrac{\ln 2}{\ln s} \end{cases} .
\end{aligned}$$

Taking the derivative with respect to z, we obtain the pdf of T as

$$f_T(z) = \frac{\ln 2 \times 2^{1/z}}{(s - r)z^2} \quad \text{for } \frac{\ln 2}{\ln s} \le z < \frac{\ln 2}{\ln r} .$$

Note that $E[T^2] = \displaystyle\int_{\ln 2/\ln s}^{\ln 2/\ln r} z^2\, f_T(z)\,dz = \frac{\ln 2}{(s - r)} \int_{\ln 2/\ln s}^{\ln 2/\ln r} 2^{1/z}\,dz$, which does not simplify, and then $\sigma(T) = \sqrt{E[T^2] - (E[T])^2}$.

Exercise

Let $Z_t = C \times t^A$, where $C \sim U[c, d]$, $A \sim U[r, s]$, with $0 \le r < s$, and A, C independent.

(a) Evaluate $E[Z_t]$. (Recall: In general, $\int a^{bx}\, dx = \dfrac{a^{bx}}{b \ln a}$).

(b) Evaluate $E[Z_t^2]$ and $Var(Z_t)$.

(c) Simplify the average area under the curve $E\left[\int_0^T Z_t\, dt\right]$ down to a single integral.

(d) Simplify the area under the average curve $\int_0^T E[Z_t]\,dt$ down to a single integral.

(e) Write the expressions obtained in (c) and (d) in terms of the volume of a three-dimensional solid. Explain the solid you are using and show how (c) and (d) can be written in terms of this volume. Then what can you conclude about the terms computed in (c) and (d)?

(f) For $C \sim U[2, 5]$ and $A \sim U[1, 4]$, compute the mean of Z_3. Then compute $E\left[Z_3 \mid C \ge 3 \cap A \le 2\right]$.

(g) Let T be the total time needed for the process to triple its amount attained at time 1. (i) Give an expression for $E[T]$. (ii) Derive the cdf and pdf of T.