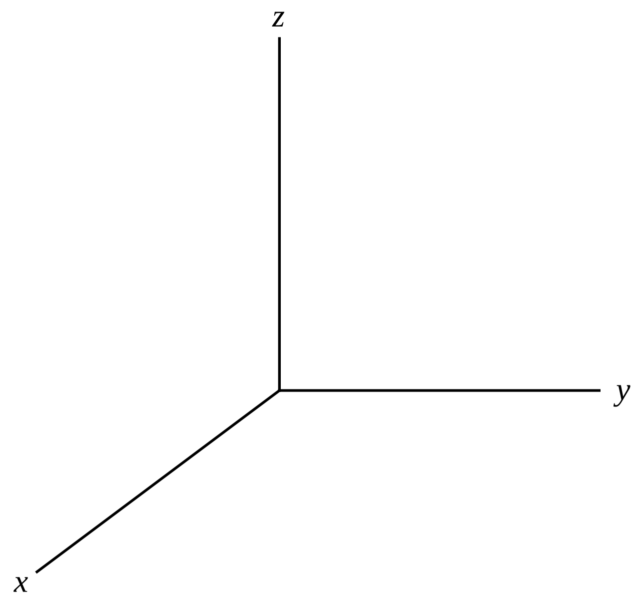
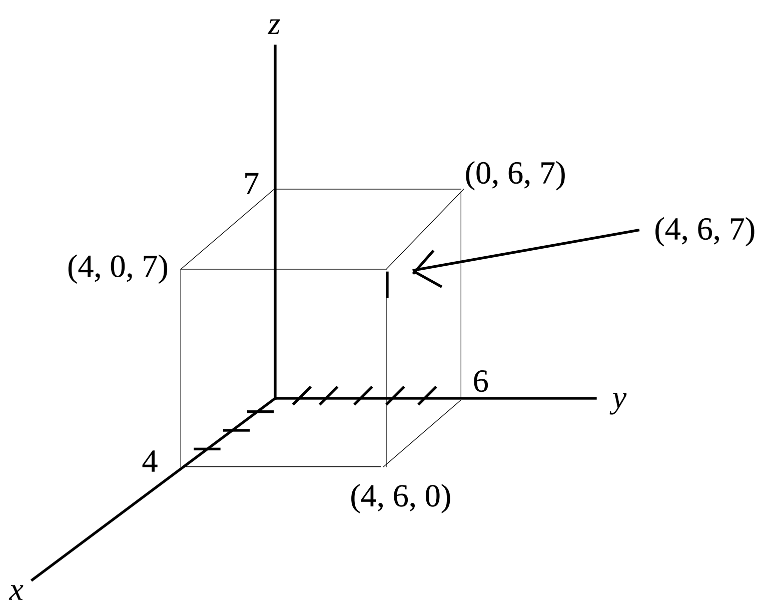
MATH 117 Spherical Coordinates

Points in three-dimensional space can be labeled in rectangular coordinates (x, y, z) and plotted with the positive axes oriented as shown below:

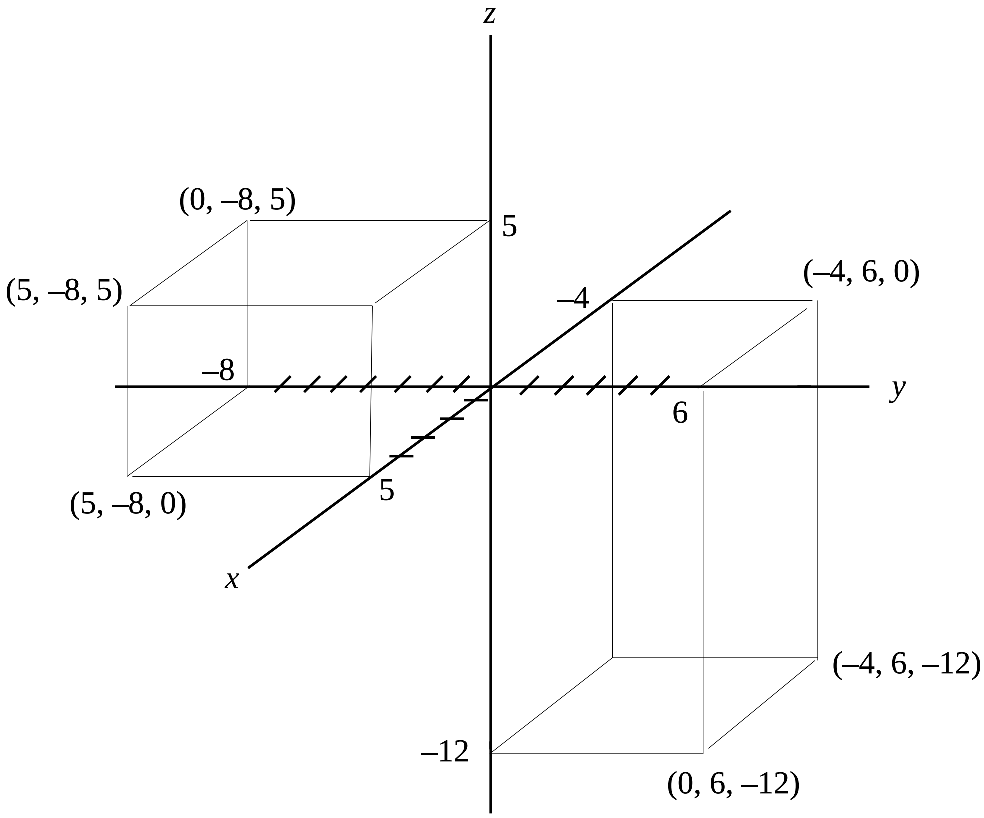


Example 1. Plot the points $u = (4, 6, 7)$, $v = (5, -8, 5)$, and $w = (-4, 6, -12)$.

Solution. The key to plotting points is drawing a box with edges that are parallel to the axes. The point (4, 6, 7) is shown below along with three other corner points on the xy, xz, and yz planes:

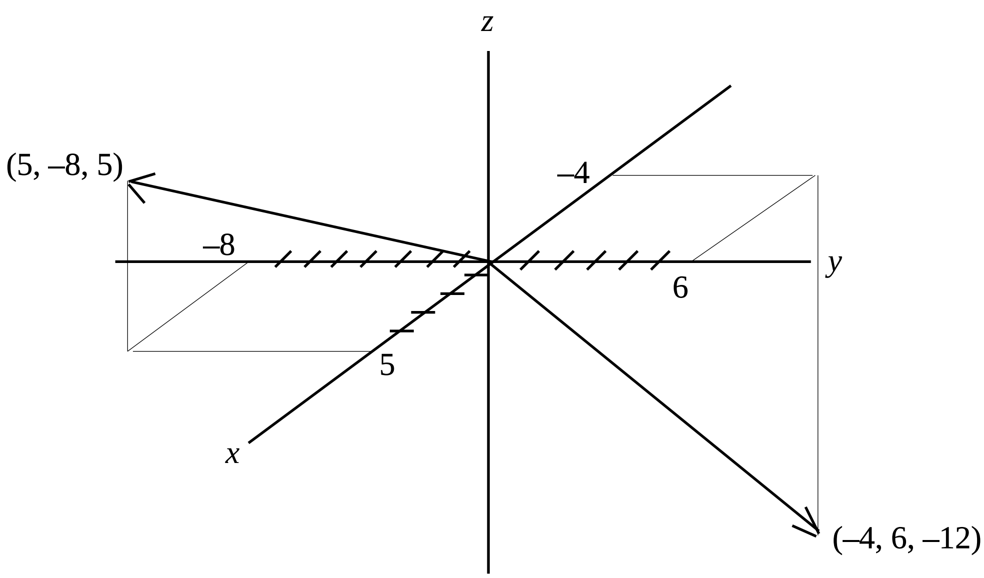


Below are the points $v = (5, -8, 5)$ and $w = (-4, 6, -12)$ and some other vertices.



Vectors in 3D

Every point (x, y, z) forms a *vector* from the origin (0, 0, 0) to (x, y, z). Below are the vectors $v = (5, -8, 5)$ and $w = (-4, 6, -12)$.



As with vectors in the xy plane, a 3-dimensional vector u has a length, now labeled ρ. But in three-dimensional space, a vector has *two* directional angles, labeled θ and ϕ. Collectively, (ρ, θ, ϕ) are the *spherical* coordinates of the vector $u = (x, y, z)$.

Length and Angles

The *length* of vector $u = (x, y, z)$ is the distance from (0, 0, 0) to (x, y, z) given by

$$\rho = \|u\| = \sqrt{x^2 + y^2 + z^2}$$

Length (or norm)

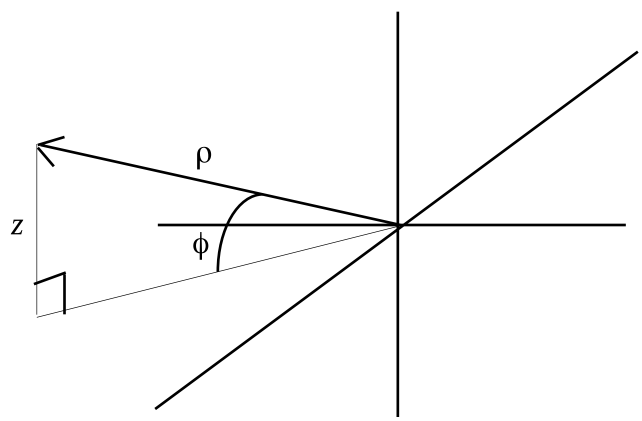
The angle θ is determined solely by the (x, y) coordinates. As always, θ is measured from the positive x-axis and $\tan\theta = y/x$. Then is θ is given by

$$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } \theta \text{ is in Quad I} \\ \tan^{-1}(y/x) + 180^\circ & \text{if } \theta \text{ is in II or III} \\ \tan^{-1}(y/x) + 360^\circ & \text{if } \theta \text{ is in IV} \end{cases}$$

Sideways Angle

The angle ϕ is measured from the xy plane up (or down) to the point (x, y, z). By convention, we let ϕ be positive if $z > 0$ and let ϕ be negative if $z < 0$. Then we always have $-90^\circ \le \phi \le 90^\circ$.

From the illustration to the right, we see that



$$\sin\phi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

So the angle ϕ is given by

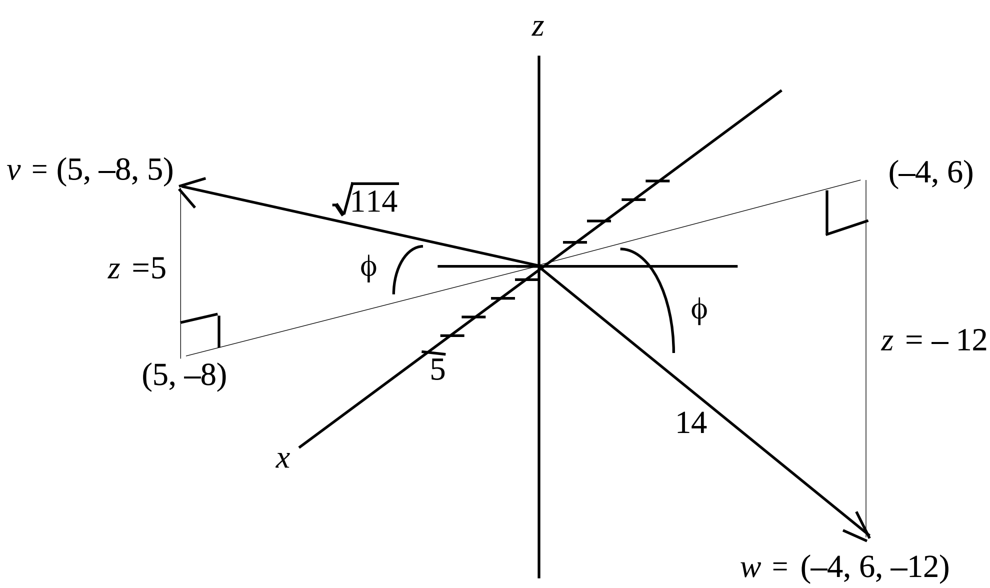
$$\phi = \sin^{-1}\left(\frac{z}{\rho}\right) = \sin^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Vertical Angle

Example 2. Find the spherical coordinates of $v = (5, -8, 5)$ and $w = (-4, 6, -12)$.

Solution. The length of vector v is $\rho = \sqrt{5^2 + 8^2 + 5^2} = \sqrt{114}$. For v, the angle θ in the xy plane is in the 4th Quadrant and is determined by the first two coordinates (5, –8). So $\theta = \tan^{-1}(-8/5) + 360^\circ \approx 302^\circ$. The vertical angle ϕ satisfies $\sin\phi = \dfrac{z}{\rho} = \dfrac{5}{\sqrt{114}}$; so $\phi = \sin^{-1}\left(\dfrac{5}{\sqrt{114}}\right) \approx 27.92^\circ$.

Thus the spherical coordinates of vector v are $\rho = \sqrt{114}$, $\theta \approx 302^\circ$, and $\phi \approx 27.92^\circ$.



The length of vector w is $\rho = \sqrt{4^2 + 6^2 + 12^2} = 14$. For w, the angle θ is in the 2nd Quadrant and is determined by the first two coordinates (–4, 6). So $\theta = \tan^{-1}(6/-4) + 180^\circ \approx 123.69^\circ$. The vertical angle ϕ satisfies $\sin\phi = \dfrac{z}{\rho} = \dfrac{-12}{14}$; so $\phi = \sin^{-1}\left(\dfrac{-12}{14}\right) \approx -59^\circ$.

Thus the spherical coordinates of vector w are

$$\rho = 14,\ \theta \approx 123.69^\circ,\ \text{and}\ \phi \approx -59^\circ.$$

Distance and Angle In Between

Given two vectors $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$, we can find the vector from u to v, the direct distance from u to v, and the angle in between u and v. The formulas for doing so are the same as for two-dimensional vectors, but with an added z term:

Vector from *u* to *v*: $v - u = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Distance from *u* to *v*: $\|v - u\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Dot Product: $u \cdot v = x_1 x_2 + y_1 y_2 + z_1 z_2$

Angle in Between: $\alpha = \cos^{-1}\left(\dfrac{u \cdot v}{\|u\|\,\|v\|}\right)$

Example 3. Let $v = (5, -8, 5)$ and $w = (-4, 6, -12)$. Find the vector from v to w, the direct distance from v to w, and the angle in between v and w.

Solution. The vector from v to w is $w - v = (-9, 14, -17)$. The distance from v to w is simply the length of $w - v$ given by $\|w - v\| = \sqrt{9^2 + 14^2 + 17^2} = \sqrt{566} \approx 23.79$.

As found in the previous example, $\|v\| = \sqrt{114}$ and $\|w\| = 14$. The dot product is $v \cdot w = (5)(-4) + (-8)(6) + (5)(-12) = -128$. So the angle in between vectors v and w is

$$\alpha = \cos^{-1}\left(\frac{-128}{\left(\sqrt{114} \times 14\right)}\right) \approx 148.9^\circ.$$