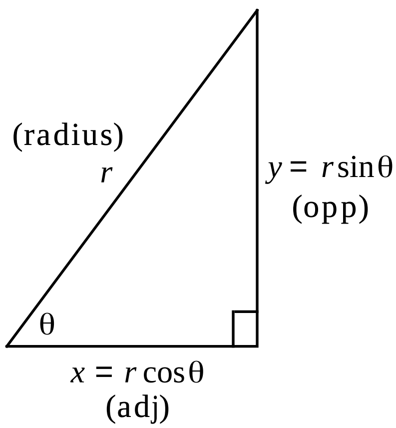
Dr. Neal, Spring 2009

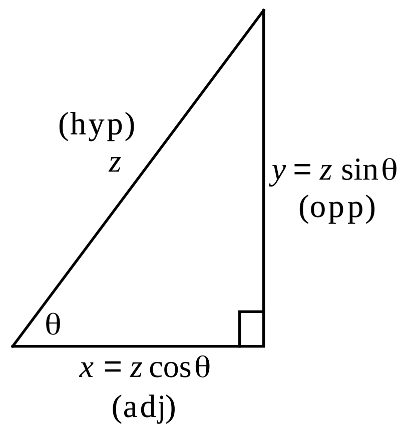
MATH 117 Right Triangle Trig

Previously, we have seen the right triangle formulas

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

where the hypotenuse r comes from the radius of a circle, and x is "adjacent" to θ and y is "opposite" of θ. But these formulas hold in any right triangle with hypotenuse z, even outside the context of a circle:





Solving for $\cos\theta$ and $\sin\theta$, we have

$$\cos\theta = \frac{x}{z} = \frac{\text{“Adjacent”}}{\text{“Hypotenuse”}} \qquad \sin\theta = \frac{y}{z} = \frac{\text{“Opposite”}}{\text{“Hypotenuse”}}$$

The Other Trig Functions

We define four other trigonometric functions the "tangent", the "cotangent", the "secant", and the "cosecant" as follows:

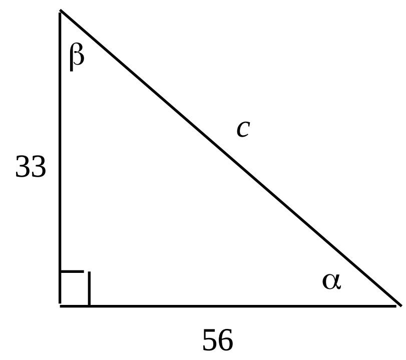
Tangent: $\tan\theta = \dfrac{y}{x} = \dfrac{\text{“Opposite”}}{\text{“Adjacent”}}$ **Cotangent**: $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{x}{y} = \dfrac{\text{“Adjacent”}}{\text{“Opposite”}}$

Secant: $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{z}{x} = \dfrac{\text{“Hypotenuse”}}{\text{“Adjacent”}}$ **Cosecant**: $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{z}{y} = \dfrac{\text{“Hypotenuse”}}{\text{“Opposite”}}$

Finding the Trig Function Values and Angles Given Two Sides

Given any two sides of a right triangle, we can use the Pythagorean Theorem to find the third side. Then we can compute the six trig function values of the acute angles in the triangle by using the above ratios. We also can use a calculator to approximate the measure of the angles.

Example 1. In the triangle below: (a) Find the remaining side. (b) Find the six trig. function values of the acute angles. (c) Find the approximate degree measures of the acute angles.



Solution. By the Pythagorean Theorem, $c^2 = 33^2 + 56^2$; so $c = \sqrt{33^2 + 56^2} = \sqrt{4225} = \mathbf{65}$

With respect to angle α, adj = 56 opp = 33 hyp = 65	With respect to angle β, adj = 33 opp = 56 hyp = 65
$\cos\alpha = \dfrac{adj}{hyp} = \mathbf{\dfrac{56}{65}}$	$\cos\beta = \dfrac{adj}{hyp} = \mathbf{\dfrac{33}{65}}$
$\sin\alpha = \dfrac{opp}{hyp} = \mathbf{\dfrac{33}{65}}$	$\sin\beta = \dfrac{opp}{hyp} = \mathbf{\dfrac{56}{65}}$
$\tan\alpha = \dfrac{opp}{adj} = \mathbf{\dfrac{33}{56}}$	$\tan\beta = \dfrac{opp}{adj} = \mathbf{\dfrac{56}{33}}$
$\cot\alpha = \dfrac{1}{\tan\alpha} = \mathbf{\dfrac{56}{33}}$	$\cot\beta = \dfrac{1}{\tan\beta} = \mathbf{\dfrac{33}{56}}$
$\sec\alpha = \dfrac{1}{\cos\alpha} = \mathbf{\dfrac{65}{56}}$	$\sec\beta = \dfrac{1}{\cos\beta} = \mathbf{\dfrac{65}{33}}$
$\csc\alpha = \dfrac{1}{\sin\alpha} = \mathbf{\dfrac{65}{33}}$	$\csc\beta = \dfrac{1}{\sin\beta} = \mathbf{\dfrac{65}{56}}$

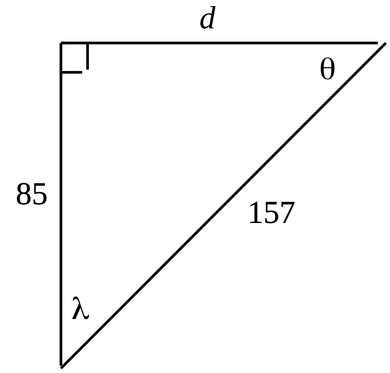
To find the measures of α and β, put your calculator in *degree* mode and evaluate

COS^{-1}(56/65) and **COS^{-1}(33/65)**

The **COS^{-1}** (inverse cosine) button is **2ND COS**. Then,

$$\alpha = \cos^{-1}\left(\frac{56}{65}\right) \approx \mathbf{30.51^\circ} \quad \text{and} \quad \beta = \cos^{-1}\left(\frac{33}{65}\right) \approx \mathbf{59.49^\circ}$$

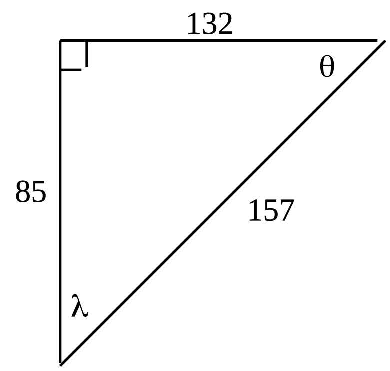
Example 2. In the triangle below: (a) Find the remaining side. (b) Find the six trig. function values of the acute angles. (c) Find the approximate degree measures of the acute angles.



Solution. Because $85^2 + d^2 = 157^2$;

we have

$$d = \sqrt{157^2 - 85^2} = \sqrt{17424} = \mathbf{132}$$



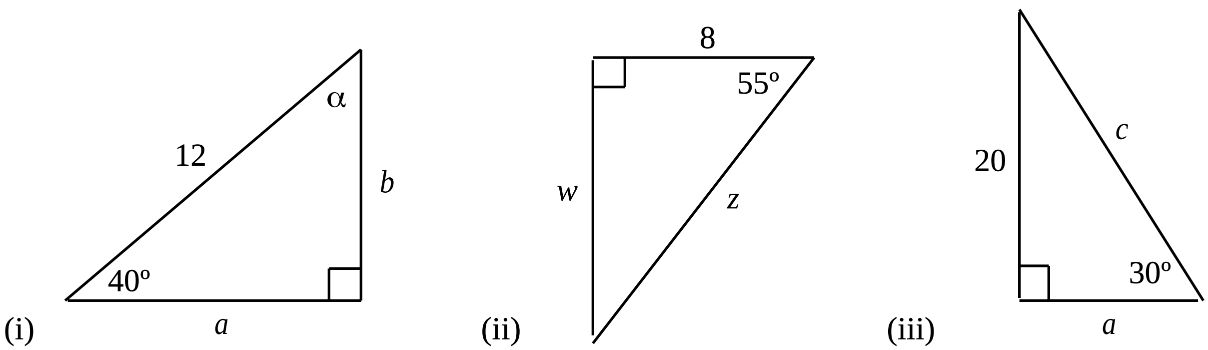
With respect to angle λ, adj = 85 opp = 132 hyp = 157	With respect to angle θ, adj = 132 opp = 85 hyp = 157
$\cos\lambda = \frac{adj}{hyp} = \mathbf{\frac{85}{157}}$	$\cos\theta = \frac{adj}{hyp} = \mathbf{\frac{132}{157}}$
$\sin\lambda = \frac{opp}{hyp} = \mathbf{\frac{132}{157}}$	$\sin\theta = \frac{opp}{hyp} = \mathbf{\frac{85}{157}}$
$\tan\lambda = \frac{opp}{adj} = \mathbf{\frac{132}{85}}$	$\tan\theta = \frac{opp}{adj} = \mathbf{\frac{85}{132}}$
$\cot\lambda = \frac{1}{\tan\lambda} = \mathbf{\frac{85}{132}}$	$\cot\theta = \frac{1}{\tan\theta} = \mathbf{\frac{132}{85}}$
$\sec\lambda = \frac{1}{\cos\lambda} = \mathbf{\frac{157}{85}}$	$\sec\theta = \frac{1}{\cos\theta} = \mathbf{\frac{157}{132}}$
$\csc\lambda = \frac{1}{\sin\lambda} = \mathbf{\frac{157}{132}}$	$\csc\theta = \frac{1}{\sin\theta} = \mathbf{\frac{157}{85}}$

The measures of λ and θ are $\lambda = \cos^{-1}\left(\frac{85}{157}\right) \approx \mathbf{57.22^\circ}$ and $\theta = \cos^{-1}\left(\frac{132}{157}\right) \approx \mathbf{32.78^\circ}$.

Finding the Remaining Sides Given a Side and an Angle

Given a side and one acute angle in a right triangle, then the other angle is simply the complement of the first (i.e., they add up to 90º). The other two sides can be found by using the appropriate right triangle ratio.

Example 3. Find the remaining pieces of the following right triangles using only the given labeled information on each:



Solution. (i) First, $\alpha = 50^\circ$, which is the complement of the given 40º angle. But α is not needed to find the other sides. We need to apply the appropriate formula below:

$$\cos\theta = \frac{Adj}{Hyp} \qquad \sin\theta = \frac{Opp}{Hyp} \qquad \tan\theta = \frac{Opp}{Adj}$$

Part (i) – With respect to $\theta = 40^\circ$, we have: adj = a opp = b hyp = 12

$$\text{So, } \cos 40^\circ = \frac{Adj}{Hyp} = \frac{a}{12} \quad \to \quad a = 12\cos 40^\circ \approx \mathbf{9.19} \text{ (via calculator)}$$

$$\text{and } \sin 40^\circ = \frac{Opp}{Hyp} = \frac{b}{12} \quad \to \quad b = 12\sin 40^\circ \approx \mathbf{7.71} \text{ (via calculator)}$$

Part (ii) – With respect to $\theta = 55^\circ$, we have: adj = 8 opp = w hyp = z

$$\text{So, } \cos 55^\circ = \frac{Adj}{Hyp} = \frac{8}{z} \quad \to \quad z = \frac{8}{\cos 55^\circ} \approx \mathbf{13.94757} \text{ (via calculator)}$$

$$\text{and } \tan 55^\circ = \frac{Opp}{Adj} = \frac{w}{8} \quad \to \quad w = 8\tan 55^\circ \approx \mathbf{11.425} \text{ (via calculator)}$$

Part (iii) – With respect to $\theta = 30^\circ$, we have: adj = a opp = 20 hyp = c

$$\text{So, } \sin 30^\circ = \frac{Opp}{Hyp} = \frac{20}{c} \quad \to \quad c = \frac{20}{\sin 30^\circ} = \frac{20}{\frac{1}{2}} = \mathbf{40}$$

$$\text{and } \tan 30^\circ = \frac{Opp}{Adj} = \frac{20}{a} \quad \to \quad a = \frac{20}{\tan 30^\circ} = \frac{20}{\frac{1}{\sqrt{3}}} = \mathbf{20\sqrt{3}}$$

Right Triangle Trig on the xy Plane

Any set of (x, y) coordinates determines an angle θ by considering the segment from (0, 0) to (x, y). The trig function values can be quickly found by using

$$\cos\theta = \frac{x}{z} \qquad \sin\theta = \frac{y}{z} \qquad \tan\theta = \frac{y}{x}$$

where we always take $z = +\sqrt{x^2 + y^2}$.

Example 4. Find the six trig function values of the *positive* angle θ determined by the following points. Then find the approximate measure of the angle.

(i) (–9, 40) (ii) (–80, –39) (iii) (88, –105)

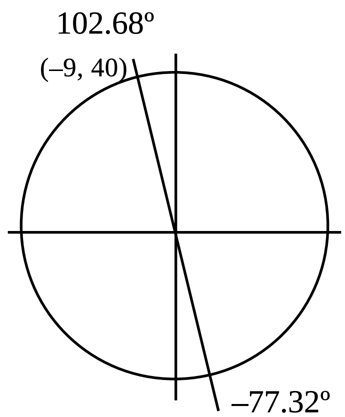
Solution. In each case, we first need to find the hypotenuse z. Then we apply the right triangle trig formulas to find the six trig function values. Finally, we'll use **tan^{-1} (y/x)** to find the angle θ. In each case though, we will have to add either 180º or 360º to adjust for the correct quadrant of θ.

Part (i): First, $x = -9$, $y = 40$, and $z = \sqrt{9^2 + 40^2} = \sqrt{1681} = 41$. Then,

$$\cos\theta = \frac{-9}{41} \quad \sin\theta = \frac{40}{41} \quad \tan\theta = \frac{40}{-9} \quad \cot\theta = \frac{-9}{40} \quad \sec\theta = \frac{41}{-9} \quad \csc\theta = \frac{41}{40}$$

In degree mode: **tan^{-1} (40/–9)** ≈ –77.32º which is in the 4th Quadrant.

But (–9, 40) is in the 2nd Quadrant, so add 180º to adjust: $\theta \approx -77.32^\circ + 180^\circ \approx$ **102.68º**.

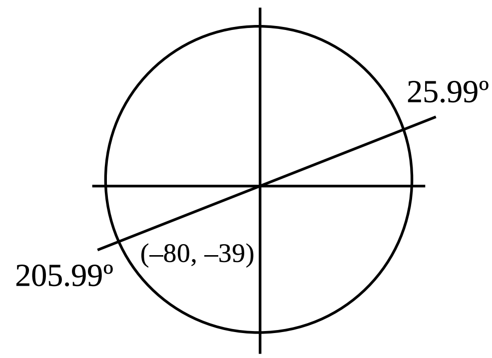


Part (ii): First, $x = -80$, $y = -39$, and $z = \sqrt{80^2 + 39^2} = \sqrt{7921} = 89$. So,

$$\cos\theta = -\frac{80}{89} \quad \sin\theta = -\frac{39}{89} \quad \tan\theta = \frac{39}{80} \quad \cot\theta = \frac{80}{39} \quad \sec\theta = -\frac{89}{80} \quad \csc\theta = -\frac{89}{39}$$

In degree mode: **tan^{-1} (–39/–80)** ≈ 25.99º which is in the 1st Quadrant. But the given point (–80, –39) is in the 3rd Quadrant, so add 180º to adjust.

Then θ ≈ 25.99º + 180º ≈ **205.99º**.



Part (iii): We have $x = 88$, $y = -105$, and $z = \sqrt{88^2 + 105^2} = \sqrt{18769} = 137$. Thus,

$$\cos\theta = \frac{88}{137} \quad \sin\theta = -\frac{105}{137} \quad \tan\theta = -\frac{105}{88} \quad \cot\theta = -\frac{88}{105} \quad \sec\theta = \frac{137}{88} \quad \csc\theta = -\frac{137}{105}$$

In degree mode: **tan^{-1} (–105/88)** ≈ –50º which is in the 4th Quadrant as is (88, –105). But to get a *positive* angle, add 360º. So we actually have θ ≈ **310º**.

Finding Other Trig Function Values of θ

Given one trig function value of an angle θ and its quadrant, we can make an (x, y, z) system to find the other trig function values. Always keep the hypotenuse z *positive*, and use the appropriate +/– signs for x and y depending on the quadrant.

Example 5. Find the remaining trig function values of the described angle θ.

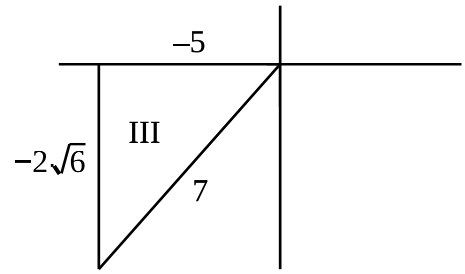
(i) $\sec\theta = -\frac{7}{5}$ and θ is in III (ii) $\csc\theta = \frac{9}{4}$ and θ is in II

(iii) $\cot\theta = -\frac{5}{12}$ and θ is in IV

Solution. (i) For $\sec\theta = -\frac{7}{5}$, then $\cos\theta = \frac{-5}{7} = \frac{x}{z}$. And y is also *negative*, because θ is in Quadrant III. Then $y = -\sqrt{z^2 - x^2} = -\sqrt{49 - (-5)^2} = -\sqrt{24} = -2\sqrt{6}$. The other four trig function values are:

$\sin\theta = \dfrac{y}{z} = -\dfrac{2\sqrt{6}}{7}$ $\csc\theta = \dfrac{z}{y} = -\dfrac{7}{2\sqrt{6}}$

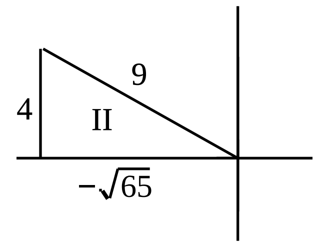
$\tan\theta = \dfrac{y}{x} = \dfrac{2\sqrt{6}}{5}$ $\cot\theta = \dfrac{x}{y} = \dfrac{5}{2\sqrt{6}}$



(ii) For $\csc\theta = \dfrac{9}{4}$, then $\sin\theta = \dfrac{4}{9} = \dfrac{y}{z}$. But x is *negative*, because θ is in Quadrant II. So $x = -\sqrt{z^2 - y^2} = -\sqrt{81-16} = -\sqrt{65}$. The other four trig function values are:

$\cos\theta = \dfrac{x}{z} = -\dfrac{\sqrt{65}}{9}$ $\sec\theta = \dfrac{z}{x} = -\dfrac{9}{\sqrt{65}}$

$\tan\theta = \dfrac{y}{x} = -\dfrac{4}{\sqrt{65}}$ $\cot\theta = \dfrac{x}{y} = -\dfrac{\sqrt{65}}{4}$



(iii) $\cot\theta = -\dfrac{5}{12}$ $\tan\theta = \dfrac{-12}{5} = \dfrac{y}{x}$

(θ is in IV, so x is pos. and y is neg.)

$$z = \sqrt{5^2 + (-12)^2} = 13$$

$\cos\theta = \dfrac{x}{z} = \dfrac{5}{13}$ $\sec\theta = \dfrac{z}{x} = \dfrac{13}{5}$

$\sin\theta = \dfrac{y}{z} = -\dfrac{12}{13}$ $\csc\theta = \dfrac{z}{y} = -\dfrac{13}{12}$