Examples of Time Scale Models in Macroeconomics

(VERY PRELIMINARY)

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1 Introduction

The goal of this paper is to demonstrate how a new modelling technique – dynamic models on time scales – can be used in economics. Time scale calculus is a new and exciting mathematical theory[1] that unites two existing approaches to dynamic modelling – difference and differential equations – into a general framework called dynamic models on time scales. Because it is a more general approach to dynamic modelling, time scale calculus can be used to model dynamic processes whose time domains are more complex than the set of integers (difference equations) or real numbers (differential equations).

[1]German mathematician Stefan Hilger introduced time scale calculus in his Ph.D. dissertation in 1988. For comprehensive references on time scale calculus see Bohner and Peterson (2001 and 2003).

Since its inception, time scale calculus has found applications in entomology (Thomas and Urena (2005)), computer science (Atici and Atici (2005)), medical sciences (Thomas and Jones(2005)) and other areas but has virtually not been used in economics – which is surprising because economics is abundant with dynamic models that describe evolution of economic variables over time. Conventional dynamic models in economics are either discrete time or continuous time models. Although these two types of models generally produce similar conclusions, they require different techniques for solving them.

This paper aims to demonstrate the benefit of using time scale calculus: to unify discrete and continuous models into a single theory and to extend dynamic economic models beyond discrete and continuous models by allowing more complex time domains. Dynamic models on time scales have the potential to enrich economic models by providing a flexible and more capable way to model timing of events. Related work in economics have addressed statistical properties of the unevenly spaced data, also known as time-deformation models (Stock (1988), Meese and Rose (1991)). Engle and Russell (1988) study an autoregressive conditional duration model that explicitly recognizes that uneven trading intervals may cause irregularly spaced data.

The paper is structured as follows: The first section is this introduction. In the second section we provide several examples of dynamic models and present some theory behind the time scale calculus. In the third section, we provide a particular example of a dynamic economic model, which we set up in discrete time, continuous time and on a general time scale. The model is a familiar dynamic model of consumption in which consumer seeks the optimal consumption path given a certain income stream. Section four contains several examples of how the same model can be set up on particular time scales and demonstrate how the path of consumption may vary with different time scales. The third section is followed by conclusion.

2 Mathematics of Time Scale Calculus

2.1 Three Examples of Dynamic Models

We start our introduction to time scales with the following three examples to highlight the differences among differential equations, difference equations and dynamic equations on time scales.

Model I A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount currently present at time t, then

$$\frac{dQ}{dt} = -rQ, \tag{2.1}$$

where $r > 0$ is the decay rate.

Model II If $y(0)$ dollars are invested at an annual interest rate of 7 percent compounded quarterly, then $y(t)$, the value of the investment after t quarters of a year, is

$$y(t+1) - y(t) = .0175y(t) \tag{2.2}$$

for $t = 0, 1, 2, \ldots$.

Model III Let $N(t)$ be the number of plants of one particular species at time t in a certain area. By experiments we know that N grows exponentially according to $N' = N$ during the months of April until September. At the beginning of October, all plants die, but the seeds remain in the ground and start growing again at the beginning of April, with N now being doubled. So we have the following model[2]:

$$N'(t) = N(t) \tag{2.3}$$

for $t \in [2k, 2k+1)$ and

$$N(2k+2) - N(2k+1) = N(2k+1) \tag{2.4}$$

for $k = 0, 1, 2, \ldots$.

The domains of all these three models are different: $\mathbb{R}$ – the set of real numbers, $\mathbb{W}$ – the set of whole numbers and $\bigcup_{k=0}^{\infty}[2k, 2k+1]$. However, they all have at least one

[2] A similar model has been studied by Thomas and Urena [46] who modelled population of mosquitoes.

thing in common: They are closed subsets of $\mathbb{R}$. This observations demonstrates the premise for the time scale calculus: Time scales are defined as nonempty closed subsets of $\mathbb{R}$, and the aim of time scale calculus is to unify continuous and discrete analysis into a general theory of dynamic models. The motivation for such general theory is rooted in the fact that there is a disconnect between two methods of dynamic modelling: Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be different from continuous counterparts. Unification of these two types of dynamic equations in a general theory will help explain these similarities and discrepancies. In addition, dynamic models on time scales can be used to study problems that cannot be approached with differential and difference equations. So, unification and extension are the two main features of the time scales calculus (Bohner and Peterson (2001)).

2.2 Mathematics of Time Scale Calculus

In this section we give some basic definitions for time scales and discuss further the models introduced above.

We will denote a time scale by the symbol $\mathbb{T}$. We want to point out that each time scale differs from others in view of its point classification. To see this point classification scheme, first we define the forward and backward jump operators.

Definition 2.1 *Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator*

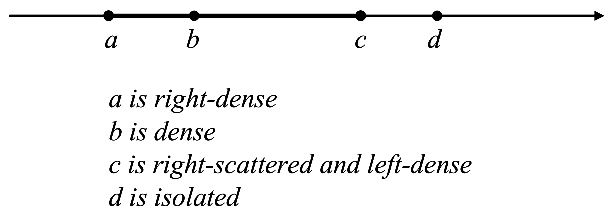

Figure 1: Point Classification in Time Scale Calculus

$\sigma : \mathbb{T} \to \mathbb{T}$ *by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by*

$$\rho(t) := \sup\{s \in T : s < t\}.$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$, we say t is *left-scattered*. Points that are both right-scattered and left-scattered are called *isolated*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. Points that are right-dense and left-dense at the same time are called *dense*. Figure 1 demonstrates this point classification.

The set $\mathbb{T}^\kappa$ which is derived from $\mathbb{T}$ is defined as follows: If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, we define the function $f^\sigma : \mathbb{T}^\kappa \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}^\kappa$.

Definition 2.2 *If $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$, then the delta-derivative of f at a point t is defined to be the number $f^\Delta(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood of U of t such that*

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$.

We note that if $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f'(t)$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$.

Definition 2.3 *A function $F : \mathbb{T} \to \mathbb{R}$ we call a delta-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. We then define the Cauchy Δ-integral from a to t of f by*

$$\int_a^t f(s)\Delta s = F(t) - F(a)$$

for all $t \in \mathbb{T}$.

Note that in the case $\mathbb{T} = \mathbb{R}$ we have

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

and in the case $\mathbb{T} = \mathbb{Z}$ we have

$$\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k),$$

where $a, b \in \mathbb{T}$ with $a \leq b$.

Now if we look back to the three models, we can restate all of them in the same way:

$$x^{\Delta}(t) = \alpha x(t), \tag{2.5}$$

where $t \in \mathbb{T}$ and α is a constant.

This equation is called a first order dynamic equation on a time scale $\mathbb{T}$. If $\alpha = -r$ and $\mathbb{T} = \mathbb{R}$, then it corresponds to the first order differential equation (2.1). If $\alpha = .0175$ and $\mathbb{T} = \mathbb{Z}$, then it corresponds to the first order difference equation (2.2). Finally if $\alpha = 1$ and $\mathbb{T} = P_{1,1}$, where $P_{1,1} = \bigcup_{k=0}^{\infty}[2k, 2k+1]$, then it corresponds to the system of equations (2.3)-(2.4).

This is a new representation for all three models. Moreover, because we have a developed theory in time scales, we can search for the solution that covers all three cases. On the other hand, this is not only a unification but also an extension of the given three equations to other problems with different domains. Next we present some of the definitions and a theorem required for solving equation (2.5).

Definition 2.4 *A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous or rd-continuous provided it is continuous at right dense points in $\mathbb{T}$ and its left-sided limits exist at left dense points of $\mathbb{T}$.*

If $\mathbb{T} = \mathbb{R}$, then f is rd-continuous if and only if f is continuous.

Definition 2.5 *The function $p : \mathbb{T} \to \mathbb{R}$ is μ-regressive if*

$$1 + \mu(t)p(t) \neq 0$$

for all $t \in \mathbb{T}^{\kappa}$, where $\mu(t) = \sigma(t) - t$.

Define the μ-regressive class of functions on $\mathbb{T}^{\kappa}$ to be

$$\mathcal{R}_{\mu} = \{p : \mathbb{T} \to \mathbb{R} : p \text{ is } rd - continuous \text{ and } \mu - regressive\}.$$

Definition 2.6 *If $p \in \mathcal{R}_{\mu}$, then the delta exponential function is defined by*

$$e_p(t, s) := \exp(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau)$$

for $s, t \in \mathbb{T}$, *where the* μ*-cylinder transformation* ξ_μ *is as in* [20, *page* 57].

Note that in the case $\mathbb{T} = \mathbb{R}$, then $e_\alpha(t, s) = e^{\alpha(t-s)}$, and if $\mathbb{T} = \mathbb{Z}$, then $e_\alpha(t, s) = (1 + \alpha)^{t-s}$, where $\alpha \in \mathbb{R}$.

Definition 2.7 *If* $p \in \mathcal{R}_\mu$, *then the first order linear dynamic equation*

$$y^\Delta = p(t)y$$

is called regressive.

Theorem 2.8 S*uppose* $y^\Delta = p(t)y$ *is regressive. Let* $t_0 \in \mathbb{T}$ *and* $y_0 \in \mathbb{R}$. *The unique solution of the initial value problem*

$$y^\Delta = p(t)y, \qquad y(t_0) = y_0$$

is given by

$$y(t) = e_p(t, t_0)y_0.$$

We have presented only a small part of the theory and applications of time scales. A time scales theory has been developed for nonlinear and higher order dynamic equations (Atici et al (2000)), boundary value problems (Atici et al. (2002, 2004, 2005) and calculus of variations (Atici et al (2005)). We also note that an analogous theory was developed later for the "nabla derivative," denoted y^{∇}, which is a generalization of the backward difference operator from discrete calculus (see Atici and Guseinov (2002)).

Definition 2.9 *If $f : T \to R$ is a function and $t \in T_{\kappa}$, then we define the nabla derivative of f at a point t to be the number $f^{\nabla}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood of U of t such that*

$$|[f(\rho(t)) - f(s)] - f^{\nabla}(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|$$

for all $s \in U$.

3 Discrete, Continuous and Time Scale Models of Utility Maximization

In this section we include three examples of how a simple utility maximization problem can be set up and solved in discrete, continuous and time scale settings. The main purpose of these examples to bridge familiar continuous and discrete models with time scale models and to demonstrate the fact that the time scale calculus model is a general framework for dynamic models. All three versions of the model assume perfect foresight.

3.1 Discrete Time Model

A representative consumer seeks to maximize the lifetime utility U:

$$U = \sum_{t=0}^{T} \left(\frac{1}{1+\delta} \right)^{t} u\left(C_t\right), \tag{3.1}$$

where $0 < \delta < 1$ is the (constant) discount rate and $u\left(C_t\right)$ is the utility the consumer derives from consuming C_t units of consumption in periods $t = 0, 1, .., \infty$. Utility is assumed to be concave: $u(C_t)$ has $u(C_t)' > 0$ and $u(C_t)'' < 0$. The consumer is limited by the *budget constraint*:

$$A_{t+1} = (1+r)A_t + Y_t - C_t, \tag{3.2}$$

where A_{t+1} is the amount of assets held at the beginning of period $t+1$, Y_t is the income (determined exogenously) received in period t and r is the constant interest rate. Thus, saving and consumption decisions are assumed to be made simultaneously during the same time period. Ponzi schemes are not allowed.

This problem can be solved using a Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \left\{ \left(\frac{1}{1+\delta} \right)^{t} u\left(C_t\right) + \lambda_t \left(A_{t+1} - (1+r)A_t - Y_t + C_t\right) \right\}.$$

Differentiating with respect to C_t and A_{t+1} and combining these two first order conditions yields the familiar Euler equation that relates current and future consumption:

$$u'(C_t) = \frac{1+r}{1+\delta} u'(C_{t+1}). \tag{3.3}$$

Equation (2.8) describes the optimal behavior for the consumer at any period. It shows how the consumer will schedule the consumption path given the impatience level δ and the interest rate r. Because $u'(C_t) > 0$ and $u''(C_t) < 0$, if $u'(C_{t+1}) < u'(C_t)$, then $C_{t+1} > C_t$. Therefore, when the interest rate r is higher than internal rate of preference δ, the consumer will wait to consume until later periods. If $\frac{1+r}{1+\delta} < 1$, the consumer is impatient and will consume more in the earlier periods and less in the future periods.

3.2 Continuous Time Model

The same problem can be solved in a continuous time setting, where lifetime utility is the sum of discounted instantaneous utilities:

$$U = \int_0^T u(C_t) e^{-\delta t} dt. \tag{3.4}$$

This is the equivalent of the utility function in the discrete case (3.1). The consumer's goal is to maximize lifetime utility with respect to the path $\{C_t\}_{t=0}^{\infty}$ subject to the budget constraint

$$A_t' = A_t r + Y_t - C_t.$$

So, consumption, and asset holdings are continuous functions of time.

Using calculus of variation, the problem can be set up as

$$\mathcal{H}(C_t, A_t, t) = u\left(C_t\right) e^{-\delta t} + \lambda_t \left(A_t' - rA_t - Y_t + C_t\right).$$

To solve the model, first we derive Euler equations

$$\begin{aligned} e^{-\delta t} u'\left(C_t\right) &= -\lambda_t, \\ -r\lambda_t &= \lambda_t{}'. \end{aligned}$$

Substituting and solving the system of equations, we obtain the solution for optimal consumption

$$C_t{}' = (\delta - r) \frac{u'\left(C_t\right)}{u''\left(C_t\right)}, \tag{3.5}$$

which states that the growth rate of consumption is positive if $\delta - r < 0$ and negative when $\delta - r > 0$. Therefore we obtain a result similar to the discrete case.

3.3 Time Scale Calculus Model

The time scale calculus version of this model retains a form similar to the continuous case:

$$U = \int_0^{\sigma(T)} u(C(t))\hat{e}_{-\delta}(t,0)\nabla t$$

subject to the budget constraint

$$A^{\nabla}(t) = rA(t) + Y(t) - C(t), t \in [\sigma(0), T].$$

This problem can be set up and solved using calculus of variation on time scales:

$$\mathcal{G}(C(t), A(t), t) = u(C(t))\hat{e}_{-\delta}(t,0) + \mu(t)[A^{\nabla}(t) - rA(t) - Y(t) + C(t)].$$

Just as in the continuous case, the first step is to find Euler equations:

$$\begin{aligned} u'(C(t))\hat{e}_{-\delta}(t,0) &= -\mu(t), \\ -r\mu(t) &= \mu^{\nabla}(t). \end{aligned}$$

Substitution gives us the following dynamic equation:

$$\hat{e}_{-\delta}(\rho(t),0)[u'(C(t))]^{\nabla} = [\delta - r]\hat{e}_{-\delta}(t,0)u'(C(t)).$$

Using a property of the ∇-exponential function [21], namely

$$\hat{e}_{-\delta}(\rho(t),0) = (1+\delta\nu(t))\hat{e}_{-\delta}(t,0),$$

we obtain the following expression:

$$[u'(C(t))]^{\nabla} = \frac{\delta - r}{1+\delta\nu(t)}u'(C(t)). \tag{3.6}$$

This equation conveys the same intuition as discrete and continuous models (equations (3.3) and (3.5) respectively). The left-hand side of the equation is the ∇-derivative of $u'(C(t))$. In the numerator of the right-hand side, $\nu(t)$ is the backward graininess function $\nu(t) = t - \rho(t)$. If consumption rises between t and $t+\nu(t)$, then $[u'(C(t))]^{\nabla} < 0$. That will be the case if $\delta - r < 0$. Therefore, we obtain a result which covers both the discrete and continuous cases.

3.4 Comparing Discrete, Continuous and Time Scale models

One of the main advantages of time scale models is that they unify discrete and continuous models in a more general framework. Using definition the nabla-derivative and the jump operators, it is straightforward to show that continuous and discrete models are special cases of the time scale calculus model.

First, consider the discrete time model. When $\mathbb{T} = \mathbb{Z}$, the backward graininess function becomes $\nu(t) = 1$. The ∇-derivative of marginal utility becomes the difference in marginal utilities between $t-1$ and t: $[u'(C(t))]^{\nabla} = u'(C(t)) - u'(C(t-1))$. Thus, equation (3.6) can be written as

$$u'(C(t)) - u'(C(t-1)) = \frac{\delta - r}{1+\delta} u'(C(t)). \tag{3.7}$$

Rearranging equation (3.7) yields the same expressions as equation (3.3). Thus, setting $\mathbb{T} = \mathbb{Z}$ converts the time scale calculus model into the discrete time model.

Next, consider the continuous model. When $\mathbb{T} = \mathbb{R}$, the backward graininess function is $\nu(t) = 0$ and the ∇-derivative of marginal utility is the conventional derivative with respect to t: $[u'(C(t))]^{\nabla} = \frac{d(u'(C(t)))}{dt}$. Substituting these two expressions into equation (3.6) yields equation (3.5). Therefore, the continuous time model is a special case of the time scale model.

Equation (3.6) also demonstrates another potential advantage of time scale calculus models. If we parametrize the utility function, we can use equations (3.3), (3.5) and (3.6) to solve for the growth rate of consumption. Discrete and continuous time models would show that the growth rate of consumption for $u(C) = \ln C$ is constant because it is determined by δ and r. The time scale calculus model implies that the growth rate of consumption can change depending on the time scale because of $\nu(t)$. So, if consumption data are collected at fixed intervals but the time scale is such that consumption occurs with varying frequency, even if δ and r are constant, we would see fluctuations in the observed growth rate of consumption.

Unification of the discrete and continuous models and ability to work with time-varying frequencies are just a couple of many potential advantages time scale calculus brings to economics. In the following section we discuss some other potential contributions of time scale calculus to economics.

4 Examples of Time Scales

In this section we solve the model presented in the previous section on several different time scales and try to draw parallels with the discrete and continuous cases. We use time scale version of Euler equation 3.6 to solve for the optimal consumption paths for different time scales. The aim of these examples is to demonstrate how time scale calculus can be used to study the problems in which timing of events is different from conventional discrete (or, more precisely, evenly spaced over time) and continuous time models.

4.1 Example 1: $\mathbb{T} = \mathbb{W}h$

This time scale is very similar to the conventional discrete model. As a matter of fact, time scale techniques are not required to solve and analyze this model (e.g. see Obstfeld and Rogoff 1998, pp 745-747). However, it is a good illustration of how time scale calculus works.

For this time scale, the events take place at $t, t+h, t+2h, \ldots, t+nh$ points in time, so it is a slightly modified version of a discrete time model. Using the point classification for time scales, all points are isolated and the jump function $\nu(t) = h$. We can rewrite the Euler equation 3.6 as

$$\frac{u'(C(t)) - u'\left(C\left(t-h\right)\right)}{h} = \frac{\delta - r}{1+\delta h} u'(C(t)). \tag{4.1}$$

If we employ a particular parametrization for the utility function, we can use this equation to find out how income evolves over time. So, if we assume that utility function is the standard constant relative risk aversion utility $u\left(C\left(t\right)\right) = \frac{C^{1-\theta}}{1-\theta}$, then we can find how consumption evolves over time:

$$C\left(t+h\right) = \left(\frac{1+hr}{1+h\delta}\right)^{\frac{1}{\theta}} C\left(t\right) \tag{4.2}$$

If we know the present value of the income stream, we can consumption levels for each period using the fact that the present value of consumption must equal present value of

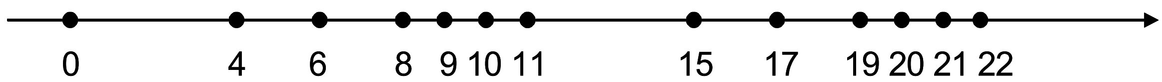

Figure 2: Graph of the Time Scale in Example 2

income. Thus, if we normalize the present value of income to some constant (e.g. 100)

$$C(t) = C(0)\left(\frac{1+hr}{1+h\delta}\right)^{\frac{t/h}{\theta}}$$

where

$$C(0) = 100\left(1 - \left(\frac{1+hr}{1+h\delta}\right)^{\frac{1}{\theta}}\left(\frac{1}{1+hr}\right)\right)$$

4.2 Example 2: $\mathbb{T} = \mathbb{W}h(t)$

In this example we show how time scales can be used to model events whose frequency changes over time. We solve the same model for a particular time scale $\mathbb{T}$ shown in figure 2. The interval between the points in time when consumption takes place varies: four period break is followed by two two-period breaks, followed by four one-period breaks. Again, solution of this model does not require the used of time scale calculus, but the latter makes the solution more compact, for the optimal consumption path is described by the same equation (3.6)

$$[u'(C(t))]^{\nabla} = \frac{\delta - r}{1 + \delta\nu(t)} u'(C(t)),$$

where $\nu(t) = \{1, 2, 4\}$. We use the Euler equation to derive the path of consumption for CRRA utility function as we did in the previous example. First, the present value of the consumption path is

$$PV = C(0) \sum_{i=1}^{\infty} \left(a^i b^{2i} c^{4i} \left(1 + c^{-1} + c^{-2} + c^{-3} + c^{-4}\left(1 + b^{-1} + b^{-2}\right)\right)\right),$$

where $a = \left(\frac{(1+4r)^{1-\theta}}{1+4\delta}\right)^{\frac{1}{\theta}}$, $b = \left(\frac{(1+2r)^{1-\theta}}{1+2\delta}\right)^{\frac{1}{\theta}}$, $c = \left(\frac{(1+r)^{1-\theta}}{1+\delta}\right)^{\frac{1}{\theta}}$. This expression can be approximated numerically for a set of parameters θ, δ and r. Once the present value of consumption path is computed, we can solve recursively for consumption levels at each point of the time scale. For comparison, figure 3 shows four examples of consumption paths: for this time scale, and for the time scale in the previous example where $h = 1$, $h = 2$ and $h = 4$. The graphs were calculated assuming $\delta = 0.15$, $r = 0.1$ and $\theta = 2$.

This graph shows an interesting result: for the time scale in which consumption frequency varies over time, the level of consumption is similar to the case when $h = 2$. The graph shows that the consumer doesn't "stock up" in anticipation of the four-period spell during which nothing happens. On the other hand, during the periods when consumption occurs every period, the consumption level is roughly twice as high

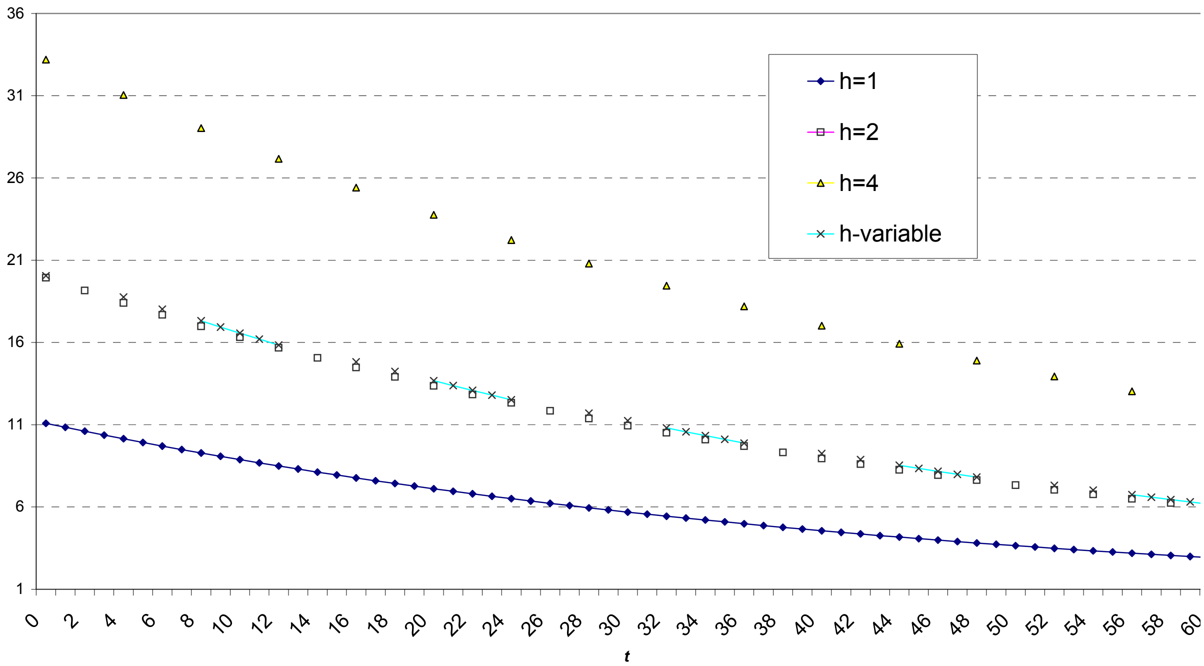

Figure 3: Comparison of consumption paths on the $h = 1$, $h = 2$, $h = 4$ and time variable time scales

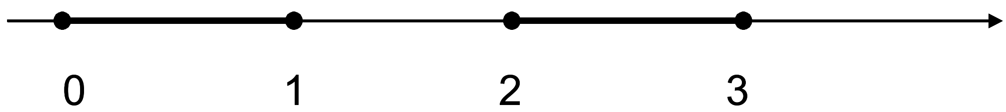

Figure 4: Graph of the $\mathbb{P}_{1,1}$ time scale

as it would be when $h = 1$. So it appears, that the determining factor of consumption level is the average number of times consumption occurs per period. Because the average number of consumption points is the same as when $h = 2$, the consumptions paths for the $h = 2$ time scale and the variable time scale are similar. This is a fairly counterintuitive implication because in reality we would expect consumers to reduce the consumption when it occurs at higher frequency and increase with lower frequency. Most likely, this result is due to additively-separable utility function.

4.3 Example 3: $\mathbb{T} = \mathbb{P}_{1,1}$

The time scale in this example can be described as evenly spaced intervals of real line: $\mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$ where $k \in \mathbb{W}$. The graph of this time scale is shown in figure 4: Thus consumption occurs continuously for one period followed by a period without consumption. Euler equation 3.6 during the consumption periods is the same as in continuous time case (Eq. 3.5). Change in consumption between the end of the last consumption period and beginning of the new consumption period can be described by Euler equation for discrete case (Eq. 3.3). Once again, if assume CRRA utility, we can

Table 1: Consumption as a function of time for $\mathbb{P}_{1,1}$

t	$C(t)$	Discounting
$[0,1]$	$C(0)e^{(r-\delta)\frac{t}{\theta}}$	e^{-rt}
$[2,3]$	$C(0)\left(\frac{1+r}{1+\delta}\right)e^{(r-\delta)\frac{t-1}{\theta}}$	$\frac{1}{1+r}e^{-r(t-1)}$
$[4,5]$	$C(0)\left(\frac{1+r}{1+\delta}\right)^2 e^{(r-\delta)\frac{t-2}{\theta}}$	$\left(\frac{1}{1+r}\right)^2 e^{-r(t-2)}$
$[2k,2k+1]$	$C(0)\left(\frac{1+r}{1+\delta}\right)^k e^{(r-\delta)\frac{t-k}{\theta}}$	$\left(\frac{1}{1+r}\right)^k e^{-r(t-k)}$

find the optimal consumption path. Evolution of consumption over the time period is show in table 1. The table also shows how consumption is discounted given interest rate r.

Therefore, we can calculate the present value of consumption as

$$
\begin{aligned}
PV &= C(0)\sum_{k=0}^{\infty}\left(\int_{2k}^{2k+1}\left(\left(\frac{1}{1+\delta}\right)^k \exp\left(-\left((1-\theta)\,r-\delta\right)\frac{k}{\theta}\right)\exp\left(\left((1-\theta)\,r-\delta\right)\frac{t}{\theta}\right)\right)dt\right) \\
&= C(0)\theta\,(1+\delta)\,\frac{1-e^{\frac{-((1-\theta)r-\delta)}{\theta}}}{-((1-\theta)r-\delta)\left(1+\delta-e^{\frac{-((1-\theta)r-\delta)}{\theta}}\right)}
\end{aligned}
$$

From the above expression we can calculate initial level of consumption $C(0)$ and use that value to calculate recursively the rest of the consumption path.

The graph shows that consumption on $\mathbb{P}_{1,1}$ time scale starts on a much higher level and decays more rapidly than continuous time consumption. Therefore, this time scale is equivalent to having a higher internal discount rate δ, so the consumer will consume greater amounts early on.

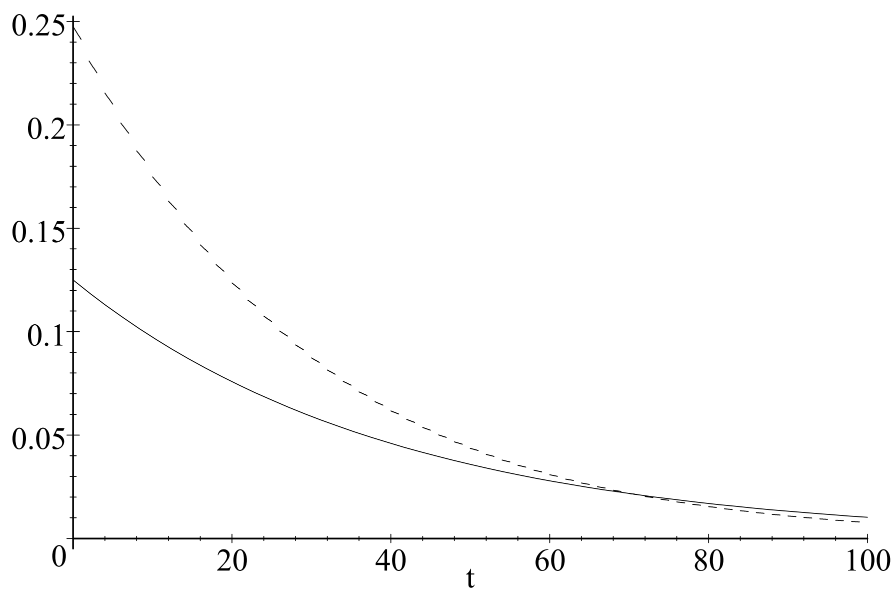

Figure 5: Comparison of the continuous time and the $\mathbb{P}_{1,1}$ time scale consumption paths.

5 Conclusion

The main themes of time scale calculus are unification and extension: it aims to unify existing differential and difference equation techniques into one set of techniques and to expend dynamic models by studying more complex time scales. This paper provides several examples of time scale models with the hope of attracting more economists to time scale models by showing that this relatively simple technique offers rich modeling capabilities.

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