Topologies arising from metrics valued in abelian ℓ-groups

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We dedicate this paper to Mel Henriksen. He passed away at age 82 on October 15, 2009. We want to thank him for many useful conversations on this particular paper, as well as his work with and support for us over the years. We will miss him as a warm friend and a valued coworker.

Abstract. This paper considers metrics valued in abelian ℓ-groups and their induced topologies. In addition to a metric into an ℓ-group, one needs a filter in the positive cone to determine which balls are neighborhoods of their center. As a key special case, we discuss a topology on a lattice ordered abelian group from the metric d_G and the filter of positives consisting of the weak units of G; in the case of $\mathbb{R}^n$, this is the Euclidean topology. We also show that there are many Nachbin convex topologies on an ℓ-group which are not induced by any filter of the ℓ-group.

1. Introduction

We bring the following definitions from [2].

Definition 1.1. A *lattice ordered group* (or *ℓ-group*) is a group G which is also a lattice under a partial order $\leq$, in which for all $a, b, x, y \in G$, $x \leq y \Rightarrow a + x + b \leq a + y + b$.

The *positive cone* of the ℓ-group G is $G^+ = \{x \in G \mid x \geq 0\}$. For every a in an ℓ-group G, the *positive part* of a is $a \vee 0 \in G^+$ and it is denoted by a^+, and the *negative part* of a is $-a \vee 0 \in G^+$ and it is denoted by a^-. We have $a = a^+ - a^-$, which shows that every element can be written as the difference of two positives. Also the *absolute value* of a is denoted by $|a|$ and is defined by $|a| = a^+ + a^-$

Absolute value has the following properties. For every $x, y \in G$,

(i) $|x| = |-x|$,
(ii) $|x| \geq 0$,
(iii) $|x| = 0$ if and only if $x = 0$.
Also,
(iv) $|x + y| \leq |x| + |y|$ if and only if G is abelian (see [7]).

2000 *Mathematics Subject Classification*: Primary: 06F20; Secondary: 54H12, 54F05.

Key words and phrases: Lattice ordered groups, generalized metric, metric, partial metric, quasimetric, valued in an ℓ-group. Positive filter, strong units, weak units. Induced topology, ℓ-group topology, (Nachbin) order convexity.

The first author wishes to acknowledge the support of this research by PSC-CUNY grants #69487-00 38 and #61743-00 39.

Definition 1.2. Given an ℓ-group G, a *G-metric* on a set X is a map $d\colon X \times X \to G$ satisfying the usual axioms for a metric, except that G need not be $\mathbb{R}$: namely for every $x, y, z \in X$,

(sym) $d(x, y) = d(y, x)$,
(pos) $0 \leq d(x, y)$,
(sd0) $d(x, x) = 0$,
(sep) if $d(x, y) = 0$ then $x = y$,
(tri) $d(x, z) \leq d(x, y) + d(y, z)$.

A map $q\colon X \times X \to G$ is called a *G-quasimetric* if it satisfies *pos*, *sd*0, *tri*, and the following asymmetric version of sep, which we call *t0*: if $q(x, y) = q(y, x) = 0$ then $x = y$.

Each G-quasimetric q has a *dual, q^**, defined by $q^*(x, y) = q(y, x)$, which is easily seen to be a G-quasimetric.

The above of course are the usual metric axioms: symmetry, positivity, 0 self-distance, separation and triangularity. Now consider the ℓ-group G and define $d_G\colon G \times G \to G$ by $d_G(x, y) = |x - y|$. Then for every $x, y, z \in G$, we have:

(i) $d_G(x, y) = |x - y| = |y - x| = d_G(y, x)$
(ii) $d_G(x, y) = |x - y| \geq 0$
(iii) $d_G(x, y) = 0$ if and only if $x = y$, and
(iv) G is abelian if and only if $d_G(x, z) = |x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z| = d_G(x, y) + d_G(y, z)$ (again by [7]).

Thus d_G is a G-metric (on the set G) if and only if G is abelian (see [6] and see [5] for a generalization). In this paper we study topologies on abelian ℓ-groups which arise from such internal "distances". In the next section, we concentrate on the G-metric d_G while in later sections we consider quasimetrics as well. For this reason, we note even in the next section when our results hold for G-quasimetrics.

2. Topologies induced by metrics and positive filters in abelian ℓ-groups

Because of the above discussion, we need and we henceforth assume that G is an abelian ℓ-group unless the contrary is stated explicitly. Thus d_G will be a G-metric.

If $G = \mathbb{R}$ is the real line, the usual way of describing a metric topology is to choose a base consisting of "balls" of positive radius about each point. In what follows, we generalize this process beginning with a more precise description of what we mean by positivity. Throughout, $\mathbb{N}$ will denote the set of positive integers $\{1, 2, 3, \dots\}$.

Definition 2.1. A subset P of G^+ is called a *positive filter* on G if for every $r, s \in P$, $t \geq r \wedge s$ implies $t \in P$. A positive filter P is called *Archimedean* if whenever $a \in G^+$ is such that $na \leq p$ for each $n \in \mathbb{N}$ and each $p \in P$, then $a = 0$.

Example 2.2. (i) For every ℓ-group G, G^+ is the largest positive filter on G and $\emptyset$ is the smallest positive filter on G.

(ii) In the ℓ-group $\mathbb{Z}$ of integers, the set $\mathbb{N}$ is an Archimedean positive filter.

(iii) For any abelian ℓ-group F, let $NA(F)$ denote the product $\mathbb{Q} \times F$ ($\mathbb{Q}$ the rationals), equipped with the lexicographic order defined by: $(q, f) \geq (0, 0)$ if $q > 0$, or $q = 0$ and $f \geq 0$. Let $V_F = \{(q, f) \in \mathbb{Q} \times F : q > 0\}$. Then V_F is a positive filter on $NA(F)$ which is not Archimedean, since for each $n \in \mathbb{N}$, $x \in F^+$, and $r > 0$, $(0, nx) \leq (r, 0)$. When an element a of $NA(F)$ belongs to V_F we write $a \gg 0$.

We now consider how topologies arise from G-quasimetrics and positive filters.

Definition 2.3. Suppose G is an abelian ℓ-group, q is a G-quasimetric on a set X, and P is a positive filter on G. For any $r \in P$, $n \in \mathbb{N}$ and $x \in X$, we let $N^q_{r/n}(x) = \{y \in X : nq(x, y) \leq r\}$, and we define $\tau_{q,P} = \{T \subseteq X :$ if $x \in T$ then $N^q_{r/n}(x) \subseteq T$ for some $r \in P$ and $n \in \mathbb{N}\} \cup \{X\}$.

When there is no danger of confusion we denote $N^q_{r/n}(x)$ by $N_{r/n}(x)$.

By the above, $\emptyset$ is certainly open. If our positive filter P is $\emptyset$, then $\tau_{q,P} = \{\emptyset, X\}$, because when $T \subseteq X$ satisfies the condition that $x \in T$ implies that $N_{r/n}(x) \subseteq T$ for some $r \in P$ and $n \in \mathbb{N}$, then $T = \emptyset$.

In the next theorem we show that $\tau_{q,P}$ is always a topology, which we call the *induced topology* resulting from q and P.

Earlier references such as [8] used distance functions into certain ordered semigroups, and filters on these semigroups with additional properties which were called "sets of positives". The stronger algebraic properties of ℓ-groups allow us to work with all positive filters. We accomplish this without requiring divisibility, by using the properties of abelian ℓ-groups and the sets $N_{r/n}(x)$.

Theorem 2.4. *Let G be an ℓ-group, $q\colon X \times X \to G$ be a G-quasimetric on a set X, and P be a positive filter on G. Then the set $\tau_{q,P}$ is a topology on X, and $\{N_{r/n}(x) : r \in P,\ n \in \mathbb{N}\} \cup \{X\}$ is a neighborhood base at each $x \in X$.*

Proof. If $P = \emptyset$, then we are done by the remarks above. If $P \neq \emptyset$, the usual proof for metrics that those sets which contain a "closed" ball of positive radius centered about each point (an $N_{r/n}(x),\ r \in P,\ n \in \mathbb{N}$) form a topology in which those balls are a neighborhood base at x is easily transferred to this situation (but note that for positive integers m, n and $r, s \in P$, $N_{r \wedge s/m+n}(x) \subseteq N_{r/n}(x) \cap N_{s/m}(y)$). $\square$

The reader can verify that for each $x \in X$, the $T_{r/n}(x) = \{y \in X :$ for some $k \in \mathbb{N}$, $N_{r/k}(y) \subseteq N_{r/n}(x)\}$ form an open base about x in $\tau_{q,P}$.

Recall that if τ is a topology on a group $(G, +)$ such that the mapping $(x, y) \to x + y$ is continuous and inversion is continuous, then $(G, +, \tau)$ is called a *topological group*, and τ is called a *group topology*.

Let $(G, +, \vee, \wedge)$ be an ℓ-group and τ a group topology on $(G, +)$. Then τ is called an *ℓ-group topology* if the lattice operations $\vee, \wedge : (G, \tau) \times (G, \tau) \to (G, \tau)$ are continuous, and in this case $(G, +, \vee, \wedge, \tau)$ is called a *topological ℓ-group*.

If we consider any abelian ℓ-group G, then $d_G \colon G \times G \to G$ is a G-metric. In this case $N_{r/n}(x) = \{y \in X : n|x - y| \leq r\}$, and we denote the topology induced by d_G and a positive filter P by $\tau_{d_G,P}$ or $\tau_{G,P}$.

Theorem 2.5. *Let G be an abelian ℓ-group, and P be a positive filter on G. Then $(G, \tau_{G,P})$ is a topological ℓ-group.*

Proof. As a G-metric, d_G is a G-quasimetric, so by Theorem 2.4, $\tau_{G,P}$ is a topology on X, and $\{N_{r/n}(x) : r \in P,\ n \in \mathbb{N}\} \cup \{X\}$ is a neighborhood base at each $x \in X$. Now we show that $+ \colon G \times G \to G$ is continuous. Assume $x + y \in T \in \tau_{G,P}$. Then there are $r \in P,\ n \in \mathbb{N}$ so that $N_{r/n}(x + y) \subseteq T$. Using *tri*, one can easily verify that $N_{r/2n}(x) + N_{r/2n}(y) \subseteq N_{r/n}(x + y) \subseteq T$, so $+$ is continuous since the $N_{r/n}(x)$ form a neighborhood base about x. Also one can see if $x \in T \in \tau_{G,P}$ then $-x \in -T$. But $T \in \tau_{G,P}$ iff $-T \in \tau_{G,P}$ because $N_{r/n}(-x) = -N_{r/n}(x)$.

To see the continuity of $\vee, \wedge$, note that these are both order preserving operations which satisfy the rule $(x * y) + s = (x + s) * (y + s)$, where $*$ denotes either of these lattice operations. Given $r \in P,\ n \in \mathbb{N}$, let $u \in N_{r/n}(x),\ v \in N_{r/n}(y)$. Then $nx * ny - r = (nx - r) * (ny - r) \leq nu * nv \leq (nx + r) * (ny + r) = nx * ny + r$, so $u * v \in N_{r/n}(x * y)$. □

Definition 2.6. A subset U of a partially ordered set $(P, \leq)$ is said to be *order convex* if whenever $u, v \in U$ and $u \leq t \leq v$, then $t \in U$. A *Nachbin convex ℓ-group topology* is an ℓ-group topology with a base of order convex sets. That is, if $x \in T \in \tau$ then $x \in U \subseteq T$ for some order convex $U \in \tau$. (In his monograph [11], Nachbin called such topologies "locally convex" topologies.)

Theorem 2.7. *Let G be an abelian ℓ-group and P be a positive filter on G. Then $(G, \tau_{d_G,P})$ is a Nachbin convex topological ℓ-group.*

Proof. We already know that $\tau_{d_G,P}$ is an ℓ-group topology. For Nachbin convexity let $q_G(x, y) = (x - y) \vee 0 = (x - y)^+$. Then for $r \in P,\ n \in \mathbb{N},\ x \in X$, consider $N^q_{r/n}(x) = \{y : nq_G(x, y) \leq r\}$ and $N^{q^*}_{r/n}(x) = \{y : nq_G(y, x) \leq r\}$. Of course, $q_G(x, y), q_G(y, x) \leq q_G(x, y) + q_G(y, x) = d_G(x, y) = q_G(x, y) \vee q_G(y, x)$. By these inequalities:

if $nq_G(x, y) \leq r$ and $nq_G(y, x) \leq r$ then $nd_G(x, y) \leq r$, so $N^q_{r/n}(x) \cap N^{q^*}_{r/n}(x) \subseteq N_{r/n}(x)$, and

if $nd_G(x, y) \leq r$ then $nq_G(x, y) \leq r$ and $nq_G(y, x) \leq r$ so $N_{r/n}(x) \subseteq N^q_{r/n}(x) \cap N^{q^*}_{r/n}(x)$.

Similarly to the definition of $T_{r/n}(x)$ in the proof of 2.4, we let $U_{r/n}(x) = \{y \in G : \text{there is a } k \in \mathbb{N} \text{ such that } N^q_{r/k}(y) \subseteq N^q_{r/n}(x)\}$, and $L_{r/n}(x) = \{y : \text{there is a } k \in \mathbb{N} \text{ such that } N^{q^*}_{r/k}(y) \subseteq N^{q^*}_{r/n}(x)\}$.

Note that $U_{r/n}(x)$ is an upper set, since if $v \in U_{r/n}(x)$ and $v \leq w$, then for some $k \in \mathbb{N}$, $N^q_{r/k}(v) \subseteq N^q_{r/n}(x)$. But $N^q_{r/k}(w) \subseteq N^q_{r/n}(x)$ since $kq_G(y,w) = k(y-w)^+ \leq k(y-v)^+ \leq r$. Thus $w \in U_{r/n}(x)$. Also $U_{r/n}(x)$ is open, for if $y \in U_{r/n}(x)$ then for some $k \in \mathbb{N}$, $N_{r/k}(y) \subseteq N^q_{r/k}(y) \subseteq N^q_{r/n}(x)$. It is clear as well that $U_{r/n}(x) \subseteq N^q_{r/n}(x)$. Similarly, $L_{r/n}(x)$ is a lower set and is an open subset of $N^{q^*}_{r/n}(x)$.

If $x \in T \in \tau_{d_G,P}$, then for some $m \in \mathbb{N}$ and $r \in P$, we have $x \in N_{r/m}(x) \subseteq T$. Thus by the last paragraph, $x \in U_{r/m}(x) \cap L_{r/m}(x) \subseteq N^q_{r/m}(x) \cap N^{q^*}_{r/m}(x) \subseteq N_{r/m}(x) \subseteq T$ and $U_{r/m}(x) \cap L_{r/m}(x) \in \tau_{d_G,P}$. That is, the $U_{r/m}(x) \cap L_{r/m}(x)$ form a base of open neighborhoods of x. They are convex as well, for if $y, z \in U_{r/m}(x) \cap L_{r/m}(x)$ and $y \leq w \leq z$, then $w \in U_{r/m}(x)$ since $w \geq y \in U_{r/m}(x)$. Also $w \in L_{r/m}(x)$ since $w \leq z \in L_{r/m}(x)$.

By the last paragraph $w \in U_{r/m}(x) \cap L_{r/m}(x)$. So the $U_{r/m}(x) \cap L_{r/m}(x)$ form a base of convex open neighborhoods of x, completing our proof. □

We now give an example of a positive filter that exists on each abelian ℓ-group G. An element $w \in G^+$ is called a *weak unit* if for every $g \in G^+$, $w \wedge g = 0$ implies $g = 0$. We denote the set of weak units by $W(G)$. The following is well known and easy to show:

Theorem 2.8. *The set $W(G)$ is a positive filter on the abelian ℓ-group G.*

Notice that $W(G)$ may be empty. For example, this happens if G is a direct sum of an infinite number of nontrivial ℓ-groups. But d_G and the positive filter $W(G)$ always give us an ℓ-group topology, and often a useful one. If G is a totally ordered group, then $W(G) = (0, \infty)$, the set of strictly positive elements of G. Thus $\tau_{d_{\mathbb{R}},W(\mathbb{R})}$ is the Euclidean topology on $\mathbb{R}$. In fact, on finite products $G_1 \times ... \times G_n$ of totally ordered groups, $W(G) = (0,\infty) \times ... \times (0,\infty)$. So also, $\tau_{d_{\mathbb{R}^n},W(\mathbb{R}^n)}$ is the Euclidean topology on $\mathbb{R}^n$. Indeed, a major reason for considering these natural distance structures is that many ℓ-group topologies so arise, and we use this in later sections to analyze them.

Next we show that every Tychonoff space can be derived from a generalized metric in the way described above, and that the converse holds. More precisely:

Theorem 2.9. (a) *For each abelian ℓ-group G, Archimedean positive filter P on G, and G-metric d on a set X, $\tau_{d,P}$ is a Tychonoff topology.*

(b) *Conversely, given a Tychonoff topological space (X, τ) there is an abelian ℓ-group G, an Archimedean positive filter P on G, and a G-metric d on X such that $\tau = \tau_{d,P}$.*

Proof. (a) By Theorem 2.4, $\tau_{d,P}$ is a topology. So, as is well-known, it suffices to show that $(X, \tau_{d,P})$ is completely regular and that singletons are closed. To prove it is completely regular, we must show that for each $x \in T \in \tau_{d,P}$, there is a continuous function $h\colon X \to [0,1]$ such that $h(x) = 1$ and $h(a) = 0$ if $a \notin T$.

Since T is open and $x \in T$, there are $r \in P$ and $n \in \mathbb{N}$ so that $N_{r/n}(x) \subseteq T$. Define $f_{x,r/n}\colon X \to G$ by letting $f_{x,r/n}(y) = r - nd(x,y)$. Whenever y and z are in X, we have $d_G(f_{x,r/n}(y), f_{x,r/n}(z)) = |(r - nd(x,y)) - (r - nd(x,z))| = n|d(x,y) - d(x,z)| \le nd(y,z)$. We use this to show that $f_{x,r/n}$ is continuous: For any $y \in X$ and basic neighborhood $N_{s/m}(f(y))$, where $s \in P$ and $m \in \mathbb{N}$, if $z \in N_{s/mn}(y)$ then $mnd(y,z) \le s$, so $md_G(f_{x,r/n}(y), f_{x,r/n}(z)) \le s$, that is, $f(z) \in N_{s/m}(f(y))$.

Note that $f_{x,r/n}(x) = r$. Now since $f_{x,r/n}(y) = r - nd(x,y)$, we have the following equivalences:

$$2f_{x,r/n}(y) \not\geq r \Leftrightarrow 2r - 2nd(x,y) \not\geq r \Leftrightarrow r \not\geq 2nd(x,y) \Leftrightarrow y \notin N_{r/2n}(x).$$

Going now to G, note that $N_{r/2}(r)$ is a basic neighborhood of r not containing 0. Since $\tau_{d_G,P}$ is a group topology, it is completely regular. Thus there is a continuous $g\colon G \to [0,1]$ such that $g(r) = 1$ and $g(t) = 0$ whenever $t \notin N_{r/2}(r)$. As a result, if $y \notin T$ then $y \notin N_{r/2n}(x)$, so by the displayed equivalences, $2f_{x,r/n}(y) \not\geq r$. Thus $2(r - f_{x,r/n}(y)) = 2r - 2f_{x,r/n}(y) \not\leq 2r - r = r$. Therefore $2|r - f_{x,r/n}(y)| \not\leq r$ and hence $f_{x,r/n}(y) \notin N_{r/2}(r)$. So $g(f_{x,r/n}(y)) = 0$. Also, $g \circ f_{x,r/n}\colon X \to [0,1]$ is continuous and $g \circ f_{x,r/n}(x) = g(r) = 1$. This shows that $\tau_{d,P}$ is completely regular by taking $h = g \circ f_{x,r/n}$.

Now we show that singletons are closed. If $y \in \mathrm{cl}(\{x\})$ then each basic neighborhood $N_{r/n}(y)$ of y meets $\{x\}$. The latter occurs if and only if $nd(x,y) \le r$ for each $n \in \mathbb{N}$ and $r \in P$. But P is an Archimedean positive filter, so this last statement implies $d(x,y) = 0$, hence $x = y$. Thus (a) holds.

(b) Conversely, assume (X,τ) is Tychonoff. If $I = \{(x,T) : x \in T \in \tau\}$, then whenever $i = (x,T) \in I$, there is an $f_i\colon X \to [0,1]$ such that $f_i(x) = 1$ and $f_i(y) = 0$ when $y \notin T$. We must find a lattice ordered group G, an Archimedean positive filter P on G, and a G-metric on X such that $\tau = \tau_{d,P}$. As in Example 2.2(iii), let $G_i = NA(\mathbb{R}) = \mathbb{Q} \times \mathbb{R}$ for each $i \in I$, with the lexicographic order ($(q,r) \le (s,t)$ if $q < s$ or $q = s$ and $r \le t$), and for $x = (q,r)$, $x \gg 0$ denotes $q > 0$. Then G_i is an abelian ℓ-group.

Now let $G = \prod_{i \in I} G_i$ and define $d\colon X \times X \to G$ by letting $d(x,y)_i = (0, |f_i(x) - f_i(y)|)$ for each $i \in I$. One can easily see that $d(x,x) = 0$ and $d(x,y) = d(y,x)$. Also d satisfies the triangle inequality because $|f_i(x) - f_i(y)| \le |f_i(x) - f_i(z)| + |f_i(z) - f_i(y)|$ for each $i \in I$.

To see that d is a G-metric on X, we need only note that $d(x,y) = 0$ implies $x = y$. But if $x \ne y$, then $j = (x, X \setminus \{y\}) \in I$. Thus $f_j(x) \ne f_j(y)$. As a result, $d(x,y)_j \ne 0$, so $d(x,y) \ne 0$.

For each $i \in I$, let $P_i = (0,\infty)$, and let $P = \sum_{i \in I} P_i = \{r \in G : \text{each } r_i > 0$ and $r_i \gg 0$ for all but a finite number of $i\}$. Certainly P is a positive filter. To see that P is Archimedean, note that if $na \le r$ for each $r \in P$, then $a_i \le (0,s)$ for each strictly positive s. Thus each $a_i \le (0,0)$, so $a \le 0$.

It remains to show that $\tau = \tau_{d,P}$. To see that $\tau \supseteq \tau_{d,P}$, it suffices to show that for each $x \in X$, the arbitrary basic $\tau_{d,P}$-neighborhood $N_{r/n}(x)$, for $r \in P,\ n \in \mathbb{N}$, is a τ-neighborhood of x. To see this, let F be the finite set of $j \in I$ such that $r_j \not\gg 0$. Let for $j \in F$, $r_j = (0, t_j)$ for some $t_j \in (0, \infty)$. Next note that the τ-open set $V_x = \bigcap_{j \in F} f_j^{-1}\big((f_j(x) - t_j/n, f_j(x) + t_j/n)\big)$ is a subset of $N_{r/n}(x)$ containing x. For if $y \in V_x$, then $nd_j(x, y) = n|f_j(y) - f_j(x)| < n(0, t_j/n) = (0, t_j) = r_j$ for each $j \in F$. Also $nd_j(x, y) = (0, n|f_j(x) - f_j(y)|) \leq r_j$ for each $j \in I \setminus F$, since $r_j \gg 0$. Thus $nd_j(x, y) \leq r_j$ for each $j \in I$. So $nd(x, y) \leq r$ and therefore $y \in N_{r/n}(x)$. So each $\tau_{d,P}$-open set is a τ-neighborhood of each of its points, and thus is τ-open.

To see the reverse inclusion, note first that by definition of I, if $x \in T \in \tau$, there is a $k \in I$ such that $f_k(x) = 1$ and $f_k(y) = 0$ if $y \notin T$. Choose r so that $r_k = (0, .5)$ and $r_i \gg 0$ if $i \neq k$; then $r \in P$. Now $z \in N_{r/1}(x)$ implies $f_k(x) - f_k(z) \leq |f_k(x) - f_k(z)| \leq .5$, which implies $f_k(z) \geq .5$ so $z \in T$. Thus if $x \in T \in \tau$, then $N_{r/1}(x) \subseteq T$ for some $r \in P$. So $T \in \tau_{d,P}$. $\square$

Remark 2.10. There are easy examples of metrics and positive filters that are not Archimedean, but whose induced topology is Tychonoff. For example, using notation from Example 2.2(iii), define $d\colon \mathbb{Q} \times \mathbb{Q} \to NA(\mathbb{R})$ by $d(x, y) = (|x - y|, 0)$ and let $P = V_F$. We already know that V_F is not Archimedean. But if $y \in N_{r/1}(x)$ for each $r \in V_F$, then $(|x - y|, 0) \leq (q, 0)$ for each $q > 0$. So $x = y$, which makes our topology T_1, and hence Tychonoff.

But $\tau_{G,P}$ is Tychonoff if and only if P is Archimedean. For if $\tau_{G,P}$ is Tychonoff and $a \in G^+$ is such that $na \leq p$ for each $n \in \mathbb{N}$ and $p \in P$ then we have $nd_G(a, 0) = n|a - 0| = na \leq p$. This means $a \in N_{r/n}(0)$ for each basic neighborhood of 0. Hence 0 is in the closure of a. But $\tau_{d_G,P}$ is Tychonoff, so $a = 0$. The converse, that if P is Archimedean, then $\tau_{G,P}$ is Tychonoff, was shown in Theorem 2.9(a).

3. Examples

Here we give some examples of topologies on ℓ-groups that are induced by G-metrics and positive filters. When $a < b$ in an ℓ-group, we use standard notations for open, closed, and half-open intervals, when it is clear from context. E.g., $(a, b) = \{x \in G : a < x < b\}$. Note also that to establish properties of bases of neighborhoods of the set of points of a topological ℓ-group, it suffices to show that they hold at 0.

Examples 3.1. (a) Given any abelian ℓ-group, consider the weakest topology, η, in which (a, b) is an open set whenever $a < b$. We will show that this topology is $\tau_{G,P}$ for some positive filter P in G.

Let $G^> = \{x \in G : x > 0\}$. There are two possibilities. Either $a \wedge b \in G^>$ for each $a, b \in G^>$, or $a \wedge b \not> 0$ for some $a, b > 0$.

In the latter case, since $a, b \geq 0$ and $a \wedge b \geq 0$, we must have $a \wedge b = 0$. Then the open sets $(-a, a)$ and $(-b, b)$ have only 0 in common. It follows that 0 is an isolated point and hence that η is the discrete topology. But the discrete topology is τ_{G,G^+}, so $\eta = \tau_{G,G^+}$.

In the former case, again two possibilities arise. Either $G^>$ has a minimal element (as in the case $G = \mathbb{Z}$) or not. In the first of these cases, suppose a is such a minimal element. Then $\{0\} = (-a, a)$ is an open set. So again $\eta = \tau_{G,G^+}$.

So we are left with the case in which $G^>$ has no minimal element, and $a \wedge b \not> 0$ for each $a, b \in G^>$. Because $G^>$ has no minimal element, for every $m, n \in G^>$ there is an $s > 0$ such that $s < m \wedge n$. Thus for every $m, n \in G^>, m \wedge n \in G^>$. That is to say, $G^>$ is a positive filter in this situation. Now if $0 \in (a, b)$ we have $-a > 0$ and $b > 0$. Thus there is $r \in G^>$ such that $r < (-a) \wedge b$; as a result, $0 \in (-r, r) \subseteq [-r, r] = N_{r/1}(0) \subseteq (a, b)$. Since $G^>$ is a positive filter without a minimal element, we have that $\{N_{r/1}(a) = [a - r, a + r] : r > 0\}$ is a $\tau_{G,G^>}$-neighborhood base about a. Also $\{(a - r, a + r) : r > 0\}$ forms an η-neighborhood base about a. Thus in this case as well, each basic neighborhood in either topology contains one in the other, so $\eta = \tau_{G,G^>}$. Therefore, we always have $\eta = \tau_{G,P}$ for some positive filter on G.

(b) Suppose $G = C(X)$ for a Tychonoff space X. We will describe some ℓ-group topologies on $C(X)$ that have been studied in the past and recast them as the topology $\tau_{C(X),P}$ for appropriately chosen positive filters P.

(i) The m-topology is the one whose open sets are those T such that for each $f \in T$ there are strictly positive functions $g, h \in C(X)$ so that if for each x, $f(x) - g(x) < k(x) < f(x) + h(x)$ then $k \in T$.

The m-topology is important because certain classes of topological spaces X can be characterized by topological properties of $C_m(X)$. For example Hewitt in [4] showed that X is pseudocompact if and only if $C(X)$ with the m-topology is first countable.

The m-topology is given by $d_{C(X)}$ with the set SP of strictly positive functions in $C(X)$, which is a positive filter. To check the above assertions, first note that for each $f \in C(X)$, and each $g, h \in P$, $r = \frac{g \wedge h}{2} \in SP$ and $r + r \leq g, h$, from which it is easy to check that SP is a positive filter, and $\{k : (\forall x)(f(x) - r(x) < k(x) < f(x) + r(x))\} = \{k : (\forall x)(|f(x) - k(x)| < r(x))\} \subseteq \{k : (\forall x)(|f(x) - k(x)| \leq r(x))\} = N_{r/1}(f) \subseteq \{k : (\forall x)(f(x) - g(x) < k(x) < f(x) + h(x))\}$. This shows that each m-topology neighborhood of f contains a $\tau_{C(X),SP}$ neighborhood, and each $\tau_{C(X),SP}$ neighborhood of f contains a m-topology neighborhood. Thus the m-topology is $\tau_{C(X),SP}$.

(ii) In a variation of the m-topology discussed in talks by Iberkleid and McGovern but not published, T is open if for each $f \in T$ there are nonnegative functions g, h with nowhere dense zero sets, so that if for each x, $f(x) - g(x) \leq k(x) \leq f(x) + h(x)$ then $k \in T$.

This collection turns out to be $\tau_{C(X),W(C(X))}$ where, as in Theorem 2.8, $W(C(X))$ is the positive filter of weak units of $C(X)$. To see this, let $ZND(X) = \{r \in C(X) : \forall x,\ r(x) \geq 0$ and $z(r)$ is nowhere dense$\}$, where $z(r) = \{x \in X : r(x) = 0\}$ is the zero set of r, and a set is nowhere dense if the interior of its closure is empty, or equivalently the interior of its complement is dense. We now check that $W(C(X)) = ZND(X)$:

If $r \in ZND(X)$ then $r(x) > 0$ on the complement of some nowhere dense set. If $(r \wedge f)(x) = 0$ for each $x \in X$, then $f(x) = 0$ on the complement of some nowhere dense set, and by its continuity we have that $f(x) = 0$ at every point, so $f = 0$; this shows $r \in W(C(X))$. But if $r \notin ZND(X)$, $r \geq 0$, then $r(x) = 0$ on some $K \subseteq X$ with nonempty interior. Choose a in the interior of K. Then by the complete regularity of X, we can find a continuous $f\colon X \to [0,1]$ so that $f(a) = 1$ and $f(x) = 0$ whenever $x \notin K$. Then $f \in C(X)^+$ and for each $x \in K$, $(r \wedge f)(x) \leq r(x) = 0$ and for each $x \in X \setminus K$, $(r \wedge f)(x) \leq f(x) = 0$, so $r \wedge f \leq 0$, but $f \nleq 0$; thus $r \notin W(G) = W(C(X))$.

The proof that the Iberkleid-McGovern topology is then $\tau_{G,W(G)}$ is then similar to the proof in (i) that the m-topology is $\tau_{G,SP}$.

(iii) The uniform topology on $G = C(X)$ is defined by saying that a set T is open if for each $f \in T$ there is some $\epsilon > 0$ so that if for each x, $f(x) - \epsilon < k(x) < f(x) + \epsilon$ then $k \in T$. This topology is given by $d_{C(X)}$ with the positive filter $UC = \{r \in C(X) : (\exists \epsilon > 0)(\forall x) r(x) \geq \epsilon\}$, of functions that are above a positive constant. The verification is as in (i).

If X is compact, then $C(X)$ becomes a Banach algebra, but unless X is pseudocompact, scalar multiplication fails to be continuous and hence $C(X)$ need not be a topological algebra over $\mathbb{R}$ in any of the three topologies in (b).

Example 3.2 (**Products).** If, for each $i \in I$ we have a set X_i, an abelian ℓ-group G_i, a G_i-quasimetric q_i and a positive filter P_i on G_i, we find an abelian ℓ-group H, containing $\prod_{i\in I} G_i$, an H-quasimetric, $q_H\colon \prod_{i\in I} X_i \times \prod_{i\in I} X_i \to \prod_{i\in I} G_i \subseteq H$, and a positive filter P on H such that $\tau_{q_H,P}$ is the product of the τ_{q_i,P_i}.

Consider the product of abelian ℓ-groups, $H = \prod_i NA(G_i)$, where the $NA(G_i)$ are constructed as in Example 2.2. Use subscripts to denote the operations of $NA(G_i)$, e.g. $+_i, \vee_i$ and $\wedge_i$. Identify G_i with $\{0\} \times G_i$, via the inclusion map $f \to (0, f)$. Define q_H by $q_H(x,y)_i = q_i(x_i, y_i)$. We leave the proof that q_H is an H-quasimetric to the reader. Further, for each i, let $\uparrow P_i$ denote the set of $t \in NA(G_i)$ which are greater than or equal to s for some $s \in P_i$; then $\uparrow P_i$ is a positive filter in $NA(G_i)$ that contains P_i. We can think of $\prod_I G_i$ as a subgroup of H.

As in the proof of Theorem 2.9 (b), let $P = \sum_{i\in I} P_i$ denote the subset of $\prod_i NA(G_i)$ whose elements are those $r = (r_i) \in \prod_i G_i$ such that each $r_i \in \uparrow P_i$ and $r_i \gg 0$ for all but a finite number of $i \in I$. Then P is a positive filter on $\prod_i NA(G_i)$, and it is straightforward to show that the topology $\tau_{q_H,P}$ is the product of the τ_{q_i,P_i}, completing discussion of this example.

As a special case, consider the product $(\mathbb{R}, \varepsilon)^I$, where I is infinite and ε is the usual Euclidean topology on $\mathbb{R}$. As previously noted, $(0,\infty)$ is a positive filter on $\mathbb{R}$, and ε is $\tau_{d_{\mathbb{R}},(0,\infty))}$. By the earlier part of this example, the product topology ε^I equals $\tau_{d_{NA(\mathbb{R})^I,P}}$, where $P = \{r \in NA(\mathbb{R})^I : \text{each } r_i > 0$ and $r_i \gg 0$ for all but a finite number of $i\}$.

In fact ε^I is not of the form $\tau_{d_{\mathbb{R}^I,P}}$ for any positive filter P on $\mathbb{R}^I$. To see this, first note that there is no $s \in (\mathbb{R}^I)^+$ and $n \in \mathbb{N}$ such that $N_{s/n}(0)$ is a neighborhood of 0 in ε^I. This holds since any ε^I neighborhood U of 0 would contain $\{x \in \mathbb{R}^I : x_i \in T_i$ whenever $i \in F\}$ for some finite $F \subseteq I$ and ε-neighborhoods T_i of 0. Since F is finite and I is not, we can choose $j \in I \setminus F$. Then $x \in \mathbb{R}^I$, defined by:

$$x_i = \begin{cases} ns_i & i = j \\ 0 & i \neq j \end{cases}$$

is an element of $U \setminus N_{s/n}(0)$.

As a result, if $P \neq \emptyset$ then for $s \in P$, $N_{s/1}(0)$ is a $\tau_{d_{\mathbb{R}^I},P}$-neighborhood of 0 which is not a ε^I-neighborhood of 0, so $\tau_{d_{\mathbb{R}^I},P} \neq \varepsilon^I$; otherwise $P = \emptyset$, in which case $\tau_{d_{\mathbb{R}^I},P}$ is the indiscrete topology, and ε^I is not.

The above gives an example of a topology on an ℓ-group G which comes from $d_G \colon G \times G \to H$ and a positive filter P on H for some $H \supseteq G$, but does not arise from $d_G \colon G \times G \to G$ and a positive filter P on G.

4. Partial metrics and topologies induced by them

Much more topological structure on an abelian ℓ-group, such as the fact that $\tau_{G,P}$ is the join of an upper and a lower topology with respect to which $+, \vee$ and $\wedge$ are continuous, is brought out using metric-like properties of the supremum function. The appropriate metric-like setting is defined below.

Definition 4.1. A *partial metric* on a set X is a function $p \colon X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$,

(sym) $p(x,y) = p(y,x)$

(ssd) $p(x,x) \leq p(x,y)$ (replacing $p(x,y) \geq 0$)

(tri) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ (replacing the usual triangle inequality)

(sep) $p(x,y) = p(x,x) = p(y,y)$ if and only if $x = y$.

These are the partial metric axioms: symmetry, small self-distance (no element is closer to a given element than it is to itself), triangularity, and separation. Each reduces to one of the metric axioms when $p(x,x) = 0$ is assumed, and implies the usual axiom when that identity no longer holds; for example, separation becomes $d(x,y) = 0$ if and only if $x = y$ in the metric case and implies that the associated topology satisfies the Hausdorff separation property. We are so accustomed to the idea that $d(x,x) = 0$, we fail to note that if the self-distance of a point on the globe, such as New York, were 0, then when we arrived at the airport, we would be at any address in the City.

Example 4.2. The map $\vee \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\vee(x,y) = \max\{x,y\}$ is a partial metric on $\mathbb{R}$ (indeed, we show this for arbitrary abelian ℓ-groups in

Theorem 4.4). It is straightforward to show that $\vee$ satisfies the axioms *sym*, *ssd* and *sep*. For the *tri* axiom, one can show that $p(x,z) \leq p(x,y)+p(y,z)-p(y,y)$ using the three cases: when y is the largest element, when y is the smallest element and when y is between x and z.

Partial metrics were introduced by the computer scientist Steve Matthews in [10] in the early 1990's as a mechanism for studying how to gain increasingly precise, but always partial, knowledge about ideal objects. For example, to study gaining the knowledge that a number is in smaller and smaller intervals, it is useful to study the poset of compact nonempty real intervals, under the partial order $\supset$. While there is no need to be familiar with the terminology, there are three topologies involved: the Scott (in which a set T is open if whenever $\bigvee D \in T$ and D is directed, then D meets T; then all Scott open sets are upper sets), the lower (whose closed sets are generated by sets of the form $[g,\infty) = \{x : x \geq g\}$ for g any element), and their join, the Lawson[1]. It turned out to be convenient in [9] to allow partial metrics to assume values in certain complete lattices with an additional abelian semigroup operation, $+$. For similar reasons in this paper we find it convenient to allow our partial metrics to assume values in lattice ordered groups.

The usual topology on the reals is the join of two one-sided topologies, the *upper topology*, whose open sets are those of the form (a,∞) for $a \in (-\infty,\infty)$, and the dually defined *lower topology*. Note that a set T is open in the upper topology if and only if $-T$ (its set of additive inverses) is open in the lower topology. In general, for any topology τ on a group (in this case additively written) such that the group operation is continuous, its *dual* is the topology whose open sets are those of the form $-T$, for T open in τ, and this dual is also such a topology.

Recall that if $(P,\leq)$ is a partially ordered set and $A \subseteq P$, then $\{x \in P : a \leq x \text{ for some } a \in A\}$ is denoted by $\uparrow A$ and is called the *upper set of* A. Also $\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$ is called the *lower set of* A. A set B is an *upper set* if for some A, $B = \uparrow A$, (equivalently, $B = \uparrow B$), and *lower sets* are defined similarly.

For the lattice ordered groups, the dual of a topology all of whose open sets are upper sets, is one all of whose open sets are lower sets, and the lattice operations are continuous in the dual if and only if they are in the original.

We now note that the upper and lower topologies on $\mathbb{R}$ can be obtained by using our partial metric $p = \vee$ in the following way:

We defined G-quasimetrics in Definition 1.2 to be maps $q\colon X \times X \to G^+$ satisfying *sd*0, *t0*, and *tri*. There we also introduced the dual, q^*, of a G-quasimetric q. In Definition 2.3 we defined the closed balls that arise from a G-quasimetric and the topology arising from it and a positive filter.

[1] Textbooks on the subjects of denotational semantics and domain theory discuss these topologies and their uses; the text [1] may be downloaded and printed at the site we give in the references.

Given a partial metric, p, two quasimetrics q_p and its dual, q_p^* are defined: $q_p(x,y) = p(x,y) - p(y,y)$, and $q_p^*(x,y) = q_p(y,x) = p(x,y) - p(x,x)$. For $p = \vee$, $q_p(x,y) = \vee(x,y) - y = (x-y) \vee 0 = (x-y)^+$ and $q_p^*(x,y) = \vee(x,y) - x = (y-x) \vee 0 = (y-x)^+$. Then in the case $G = \mathbb{R}$, the upper topology is the topology arising from $q_\vee$ and $P = (0,\infty)$, and the lower topology the one arising from $q_\vee^*$ and $(0,\infty)$.

Definition 4.3. Given a set X and an abelian ℓ-group G, a *G-partial metric* on X into G is a map $p\colon X \times X \to G$ which satisfies the axioms *sym, ssd, tri,* and *sep*, given in Definition 4.1.

If $p\colon X \times X \to G$ is a G-partial metric on a set X into an abelian ℓ-group G, the function $q_p\colon X \times X \to G$ defined by $q_p(x,y) = p(x,y) - p(y,y)$ is called the *associated quasimetric* of p.

Theorem 4.4. *Suppose G is an abelian ℓ-group.*

(a) *If $q^+(a,b) = (a-b)^+$ for all a, b in G, then $q^+\colon G \times G \to G^+$ is a G-quasimetric.*
(b) *If p is a G-partial metric on a set X, then q_p is a G-quasimetric.*
(c) *The map $\vee\colon G \times G \to G$ is a G-partial metric.*

Moreover, $q_\vee = q^+$.

Proof. (a) That *sd0* holds is obvious. If $(a-b)^+ = (b-a)^+ = 0$, then $a = b$, so *t0* also holds, and *tri* holds because $a-c = a-b+b-c \leq (a-b)\vee 0 + (b-c)\vee 0$ and $0 \leq (a-b)\vee 0 + (b-c)\vee 0$. Thus $q^+(a,c) = (a-c)\vee 0 \leq (a-b)\vee 0 + (b-c)\vee 0 = q^+(a,b) + q^+(b,c)$.

(b) By *ssd*, each $q_p(x,y) \in G^+$, and *sd0* holds by the definition of q_p. If $q_p(x,y) = q_p(y,x) = 0$, then $p(x,y) = p(x,x) = p(y,y)$. So $x = y$ by *sep*, and hence *t0* is established. To verify *tri* for q_p, consider that:

$$
\begin{aligned}
&q_p(x,y) + q_p(y,z) - q_p(x,z) \\
&= [p(x,y) - p(y,y)] + [p(y,z) - p(z,z)] - [p(x,z) - p(z,z)] \\
&= [p(x,y) + p(y,z) - p(y,y)] - p(x,z) \geq 0 \quad \text{by } \textit{tri} \text{ for } p.
\end{aligned}
$$

Thus, q_p is a G-quasimetric.

(c) Certainly, $\vee(a,b) = \vee(b,a) \geq \vee(a,a)$ and if $\vee(a,b) = \vee(a,a) = \vee(b,b)$ we immediately have $a = a \vee a = b \vee b = b$. Finally, for the triangle inequality, it will suffice to show that $\vee(a,c) + \vee(b,b) \leq \vee(a,b) + \vee(b,c)$, but the left side is then $(a \vee c) + b = (a+b) \vee (c+b)$, and $a+b, c+b \leq (a \vee b) + (c \vee b)$, so $(a+b) \vee (c+b) \leq (a \vee b) + (c \vee b)$.

To complete this proof, notice that $q_\vee(x,y) = (x \vee y) - y = (x-y) \vee (y-y) = (x-y)^+ = q^+(x,y)$. □

Definition 4.5. Given a positive filter P on an abelian ℓ-group G, a set X and a G-partial metric on X into G, the topologies $\tau_{q_p,P}$ and $\tau_{q_p^*,P}$ are defined by Definition 2.3. They will be called the *topologies on X induced by p and P,* and will be denoted by $\tau_{p,P}$ and $\tau_{p^*,P}$, respectively.

In Theorem 2.4 we showed that each $\tau_{q,P}$ is a topology and and $\{N_{r/n}(x) : r \in P,\ n \in \mathbb{N}\} \cup \{X\}$ is a neighborhood base at each $x \in X$. Now recall that the *specialization order* of a topological space, (X, τ) is defined by $x \leq_\tau y \Leftrightarrow x$ is in the closure of $\{y\}$. If $\tau = \tau_{q,P}$ for some G-quasimetric $q\colon X \times X \to G$ and Archimedean positive filter P, then x is in the closure of $\{y\} \Leftrightarrow$ for each neighborhood N of x, $\{y\} \cap N \neq \emptyset$, that is, if and only if, for each $r \in P,\ n \in \mathbb{N},\ y \in N^q_{r/n}(x)$, or equivalently, if and only if, for each $r \in P,\ n \in \mathbb{N},\ nq(x, y) \leq r$. Since P is Archimedean, this occurs if and only if, $q(x, y) = 0$.

Theorem 4.6. *Suppose G is an abelian ℓ-group and P a positive filter in G. Then a set T is open in $\tau_{\vee,P}$ if and only if whenever $x \in T$, there is an $r \in P$ and an $n \in \mathbb{N}$, such that $\{y : ny \geq nx - r\} \subseteq T$, and T is open in $\tau_{\vee^*,P}$ if and only if for each $x \in T$, there are $r \in P,\ n \in \mathbb{N}$, such that $\{y : ny \leq nx+r\} \subseteq T$. The operations $+, \vee, \wedge$ are continuous from $(G, \tau_{\vee,P}) \times (G, \tau_{\vee,P})$ to $(G, \tau_{\vee,P})$, and inversion is a homeomorphism between $(G, \tau_{\vee,P})$, and $(G, \tau_{\vee^*,P})$. If P is Archimedean the specialization order on $\tau_{\vee,P}$ is the order on G.*

Proof. To verify the first assertion, it suffices to show that $nN_r(x/n) =\uparrow (nx - r)$ for each $x \in G$, $n \in \mathbb{N}$, and $r \in P$. This follows since, $n(x \vee y) - ny = n((x \vee y) - y) = n(x - y)^+$, and $N_{r/n}(x) = \{y \in G : nq_\vee(x, y) \leq r\} = \{y \in G : n(x \vee y) - ny \leq r\} = \{y \in G : n(x - y)^+ \leq r\} = \{y \in G : nx - ny \leq r\} = \{y \in G : nx - r \leq ny\}$. The proof that $nN_r(x/n) =\downarrow (nx + r)$ is similar.

The continuity of $+$, $\vee$, and $\wedge$ are shown as in the proof of Theorem 2.5. So is that of $-$ from $(G, \tau_{\vee,P})$ to $(G, \tau_{\vee^*,P})$ and from $(G, \tau_{\vee^*,P})$ to $(G, \tau_{\vee,P})$.

Finally, we must show that if P is Archimedean, $y \in \mathrm{cl}(x)$ if and only if $y \leq x$. But by the discussion in the last paragraph before this theorem, this holds if and only if $(y - x)^+ = 0$, that is, if and only if $y \leq x$. □

Corollary 4.7. *For any abelian ℓ-group G and positive filter P, $\tau_{q^*_\vee,P} = \{-T : T \in \tau_{q_\vee,P}\}$ and $\tau_{|\ |,P} = \tau_{q_\vee,P} \vee \tau_{(q_\vee)^*,P}$.*

Proof. The first assertion is a restatement of the fact that by Theorem 4.6, inversion is a homeomorphism between $(G, \tau_{\vee,P})$ and $(G, \tau_{\vee^*,P})$. We leave the second to the reader. □

5. The positive filter of strong units

Definition 5.1. In a po-group G, $e \in G$ is called a *strong unit* if for every $a \in G$ there is a positive integer n such that $a < ne$. We denote the set of strong units of G by $S(G)$.

It is obvious that every strong unit should be positive.

Theorem 5.2. *Let G be an abelian ℓ-group. Then $S(G)$ is a positive filter.*

Proof. If $r, s \in S(G)$ and $r \wedge s \leq t$ then $t \in S(G)$, since if $a \in G$ there are positive integers j, k so that $a < jr$ and $a < ks$; thus $a \leq jkr \wedge jks = jk(r \wedge s) < (jk + 1)t$. Thus $t \in S(G)$, and so $S(G)$ is a positive filter. □

We recall that an ℓ-group is Archimedean if whenever $a, b \in G^+$ and $na \leq b$ for every $n \in \mathbb{Z}$, then $a = 0$.

Theorem 5.3. *Let G be an abelian ℓ-group such that $S(G) \neq \emptyset$. Then $S(G)$ is an Archimedean positive filter if and only if G is an Archimedean ℓ-group.*

Proof. It is clear that if G is an Archimedean ℓ-group, then each nonempty positive filter is Archimedean. Conversely, assume that $S(G)$ is Archimedean. To see that G is an Archimedean ℓ-group, consider $a, b \in G$ such that for every $n \in \mathbb{Z},\ na \leq b$, and let $e \in S(G)$. Then there is an $m \in \mathbb{N}$ such that $b \leq me$. Thus we have, for every $n \in \mathbb{Z}$, $na \leq b \leq me$. Now $me \in S(G)$ and for every $n \in \mathbb{N}$ we have $na \leq me \in S(G)$. Thus since $S(G)$ is Archimedean, $a = 0$. □

Theorem 5.4. *For each nonempty positive filter F, $\tau_{G,S(G)} \subseteq \tau_{G,F}$.*

Proof. We show that if $A \in \tau_{G,S(G)}$ and $A \neq G, \emptyset$, then $A \in \tau_{G,F}$. Let $x \in A$; since $A \in \tau_{G,S(G)} \setminus \{G\}$, there are $s \in S(G),\ m \in \mathbb{N}$, such that $N_{s/m}(x) \subseteq A$. Since $F \neq \emptyset$, let $r \in F$. Then there is a positive integer k such that $r < ks$, thus $ks \in F$ and $N_{r/km}(x) \subseteq N_{ks/km}(x) = N_{s/m}(x) \subseteq A$. By the arbitrary nature of $x \in A$, the proof is complete. □

Corollary 5.5. *For $G = C([0,1])$, the pointwise and L^p topologies, $p \geq 1$ are Nachbin convex ℓ-group topologies that do not arise from a G-metric d_G and a positive filter on G.*

Proof. Consider $G = C(X)$ with the usual order, product and addition, where X is a compact T_2 space. Then $S(C(X)) = \{f \in C(X) : \exists \epsilon > 0$ such that for every $x \in X,\ f(x) \geq \epsilon\}$. Thus the topology induced by this positive filter which is the uniform topology is the smallest among topologies induced by positive filters.

The pointwise topology (in which a net $f_n \to f$ when for each $x \in X$, $f_n(x) \to f(x)$) is Nachbin convex by Theorem 2.7 and Example 3.2. By the proof of Theorem 2.5 it is an ℓ-group topology strictly between the indiscrete and the uniform topologies on $C(X)$, so by Theorem 5.4 it is not $\tau_{G,P}$ for any positive filter P on G.

Now further, let $X = [0,1]$, so $G = C([0,1])$, equipped with the usual pointwise operations and order. Then for each $p \in [1, \infty)$, the L^p-topology restricted to $C(X)$ is also weaker than the uniform topology, so it similarly cannot be $\tau_{G,P}$ for any positive filter P on G.

To avoid overuse of the letter p below, let $a = p \in [1, \infty)$; then for $f, g \in X$, define $q_a(f,g) = (\int_0^1 [(f-g)^+(x)]^a dx)^{1/a}$. Clearly $q_a(f,g) = \|(f-g)^+\|_a$. Notice that q_a is a quasimetric: for each $f, g, h \in X,\ q_a(f,f) = \|0\|_a = 0$, and because integration is monotone and $(f-h)^+ \leq (f-g)^+ + (g-h)^+$, $q_a(f,h) = \|(f-h)^+\|_a \leq \|(f-g)^+ + (g-h)^+\|_a \leq \|(f-g)^+\|_a + \|(g-h)^+\|_a = q_a(f,g) + q_a(g,h)$, and if $q_a(g,h) = q_a(h,g) = 0$ then $(g-h)^+ = 0 = (h-g)^+$, so $g = h$.

Next notice that $\leq_{q_a}$ is the group order on $(X, \leq)$: $q_a(f,g) = 0$ iff $\int_0^1 (f-g)^+(x)dx = 0$, iff $(f-g)^+ = 0$, thus iff $f \leq g$. As in the part of the proof of Theorem 2.7 involving $U_{r/n}(x)$ and $L_{r/n}(x)$ this means that $\tau_{q_a^s}$ (where $q_a^s = q_a \vee q_a^*$) is Nachbin convex with respect to the order $\leq_{q_a}$, which we have just shown to be the group order, $\leq$. □

The authors wish to thank Richard Ball, for bringing the L^p examples some time ago to Mel Henriksen's attention and theirs. The proof that they cannot not be of the form $\tau_{G,P}$ is simplified by the later discovery of Theorem 5.4.

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