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k-Primal Spaces

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Abstract

A function $f : X \to X$ determines a topology $\mathcal{P}(f) = \{U \subseteq X : f^{-1}(U) \subseteq U\}$. A topological space (X, τ) is *primal* (or *functional Alexandroff*) if $\tau = \mathcal{P}(f)$ for some function f, and is k*-primal* if τ is the supremum of a set of k primal topologies on X. Using the associated specialization quasiorder, we give necessary and sufficient conditions for a finite topological space to be k-primal. We show that the k-primal topologies on a finite set X form a lattice and discuss lattice complements.

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1. Introduction

In 1937, Alexandroff [1] studied topologies whose closed sets also form a topology, or equivalently, topologies in which arbitrary intersections of open sets are open. Such topologies are now called Alexandroff topologies. Since every topology on a finite set is obviously an Alexandroff topology, Alexandroff spaces are widely used in computer sciences.

The most fundamental property of Alexandroff spaces is that the category **Alx** of Alexandroff spaces with continuous maps is isomorphic to the category **Qos** of qosets (that is, quasiordered sets) with order-preserving maps. Any Alexandroff topology τ on X gives a quasiorder $\leq_\tau$ on X by taking

$$x \leq_\tau y \text{ if and only if } x \text{ is in the } \tau\text{-closure } \overline{\{y\}}^{\tau} \text{ of } y,$$

and by the same correspondence every quasiorder $\leq$ on X gives an Alexandroff topology $\tau_\leq$ on X. The τ-closed sets are the $\leq$-decreasing sets, and in particular, $\overline{\{x\}} = \downarrow x = \{y \in X : y \leq x\}$. Every point x in an Alexandroff topology has a

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smallest neighborhood $N(x) = \uparrow x = \{y \in X : x \leq y\}$. Given an Alexandroff space X, we will interchangeably identify it by its topology or its associated "specialization" quasiorder. The specialization quasiorder $\leq_\tau$ is a partial order if and only if τ is T_0. For further details and applications, see [1, 14, 15, 24, 25].

Any function $f : X \to X$ determines a topology $\mathcal{P}(f)$ on X by taking the open sets to be those sets $U \subseteq X$ with $f^{-1}(U) \subseteq U$, or equivalently, by taking the closed sets to be those sets C with $f(C) \subseteq C$. Thus, the $\mathcal{P}(f)$ closed subsets of X are those sets invariant under f. A topological space (X, τ) is *primal* if $\tau = \mathcal{P}(f)$ for some function $f : X \to X$. Primal spaces were introduced independently by Ayatollah Zadeh Shirazi and Golestani [2] in 2011 and by Echi [9] in 2012. In [2], primal spaces are called *functional Alexandroff space*. Properties of these spaces have been extensively studied in the ten years since their introduction [3, 5, 10, 12, 18, 19, 20, 21, 22]. In a primal space $(X, \mathcal{P}(f))$, the specialization quasiorder is denoted $\leq_f$, and the closure of a point x is the orbit $\{f^n(x) : n \in \mathbb{N}\}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Here, we study k-primal topologies, which are the supremum of k primal topologies. In Section 2, we recall the Dushnik–Miller dimension, introduce the primal dimension of a finite partially ordered set and show some connections between them. Section 3 shows that a topology with finite specialization qoset $(X, \leq)$ is k-primal for some integer k if and only if for any cyclic point a and any $x, y \in X$ such that $a \leq y$ and $x \leq y$, we have $a \leq x$. Section 4 shows that the topologies on a finite set which are k-primal for some positive integer k form a complemented lattice.

2. Counting and dimension

In this paper, we consider k-primal spaces, defined below.

Definition 2.1. *For any positive integer k and any family $\{f_i : 1 \leq i \leq k\}$ of functions from a set X to itself, let $\leq_{f_i}$ be the specialization quasiorder for $(X, \mathcal{P}(f_i))$. We define the k-primal topology $\mathcal{P}(f_1, \ldots, f_k)$ to be the topology having specialization quasiorder $\leq_{f_1,\ldots,f_k} = \bigcap\{\leq_{f_i} : 1 \leq i \leq k\}$. Thus, $x \leq_{f_1,\ldots,f_k} y$ if and only if $x \leq_{f_i} y$ for each $i = 1, \ldots, k$. A k-primal space is a set X with a k-primal topology.*

Clearly, if $k = 1$ we have the notion of primal spaces, and for any natural number $k \neq 0$, a k-primal space is $(k+1)$-primal.

Since $\leq_{f_1,\ldots,f_k}$ is the intersection of the quasiorders $\leq_{f_i}$ for $i = 1, \ldots, k$, it is also a quasiorder, so k-primal spaces are Alexandroff. Since $N(x) = \uparrow x$ and $\overline{\{x\}} = \downarrow x$ in any Alexandroff space, it follows that $\overline{\{x\}}^{\mathcal{P}(f_1,\ldots,f_k)} = \bigcap_{i=1}^{k} \overline{\{x\}}^{\mathcal{P}(f_i)}$, and $N_{\mathcal{P}(f_1,\ldots,f_k)}(x) = \bigcap_{i=1}^{k} N_{\mathcal{P}(f_i)}(x)$ for any $x \in X$.

In the lattice $TOP(X)$ of topologies on a set X, recall that $\tau_1 \vee \tau_2$ has a basis of sets of form $U_1 \cap U_2$ where $U_i \in \tau_i$ $(i = 1, 2)$. For Alexandroff topologies, $\tau_1 \vee \tau_2$ has a basis of minimal neighborhoods $N_1(x) \cap N_2(x)$, where $N_i(x)$ is the minimal neighborhood of x in τ_i. It follows that $\mathcal{P}(f_1, \ldots, f_k) = \bigvee_{i=1}^{k} \mathcal{P}(f_i)$, where the supremum is taken in the lattice $TOP(X)$.

If κ is any non-zero cardinal number, Definition 2.1 could be generalized by declaring an Alexandroff topology τ on X to be κ-primal if $\tau = \bigvee_{i \in I} \mathcal{P}(f_i)$ where $|I| = \kappa$, and the supremum is taken in the lattice of Alexandroff topologies on X. We will not need this generality in this paper.

We have an immediate result.

Proposition 2.2. *If X is infinite, then the indiscrete topology τ_I on X is not k-primal.*

Proof: Suppose to the contrary X is infinite and $\tau_I = \mathcal{P}(f_1, \ldots, f_k)$. For any fixed $x \in X$, the intersection of the $\mathcal{P}(f_i)$-closures of x is the τ_I-closure of x, which is X. Thus, for every $x \in X$ and every $i \in \{1, \ldots, k\}$, the $\mathcal{P}(f_i)$ closure of x is X. This shows that each topology $\mathcal{P}(f_i)$ is indiscrete. However, this contradicts the fact ([22], Prop. 3) that the indiscrete topology on an infinite set is not primal. □

Recall that a *minimal point* in a qoset $(X, \leq)$ is an $x \in X$ satisfying the property: for each $y \in X$, if $y \leq x$ then $x \leq y$. A *cyclic point* in $(X, \leq)$ is an element a such that there exists $b \neq a$ with $a \leq b$ and $b \leq a$. If a is a cyclic point, the associated *cycle* is the set $\{b \in X : a \leq b \text{ and } b \leq a\}$.

The motivation for k-primal spaces is that many Alexandroff spaces are not primal but are k-primal. The following example illustrates such a situation. Let $X = \{0, 1, 2\}$ equipped with the quasi-order $\leq$ where $1 \leq 0$ and $2 \leq 0$, as seen in Figure 1.

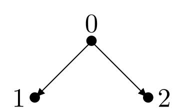


Fig. 1. k-primal but not primal

Using the characterization of primal spaces in [9], [2], or [22], $(X, \leq)$ is not primal, but $(X, \leq) = (X, \mathcal{P}(f_1, f_2))$ where $f_1(0) = 1, f_1(1) = f_1(2) = 2$ and $f_2(0) = 2, f_2(1) = f_2(2) = 1$. Thus, $(X, \leq)$ is 2-primal. This example points out that Theorem 3 of [22] is incorrect: The supremum in $TOP(X)$ of two primal spaces need not be primal. (The error in Theorem 3 of [22] does not impact the other results given there.)

Counting topologies (with certain properties) on finite sets is an old and challenging question [11]. The number of topologies on an n-element set is only known for $n \leq 18$. Table 1 lists the number of distinct k-primal topologies and the number of inequivalent (i.e. nonhomeomorphic) k-primal topologies on small sets. Figures 2–5 show the associated digraphs of the inequivalent k-primal topologies for sets with $2, 3, 4$ and 5 elements. Recall that a directed edge $x \to y$ implies that $x \geq y$. So, the closure of a point x is the set of all vertices lying in a directed path starting from x. Table 1 and Figures 2–5 were produced by an exhaustive computer algorithm. (See http://people.wku.edu/tom.richmond/k-Primal_Spaces.nb for the programming code.)

n	Distinct topologies	Inequivalent topologies	Distinct k-primal topologies	Inequivalent k-primal topologies
1	1	1	1	1
2	4	3	4	3
3	29	9	26	8
4	355	33	279	25
5	6942	139	4937	88
6	209527	718	141831	398
7	9535241	4535	6418715	2327

Table 1: Enumerating k-primal topologies

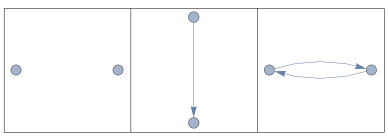


Fig. 2. 2-node nonhomeomorphic k-primal topologies

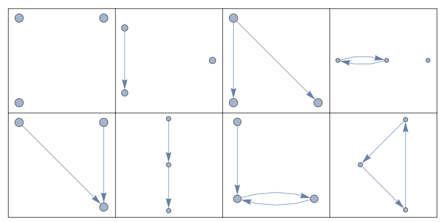


Fig. 3. 3-node nonhomeomorphic k-primal topologies

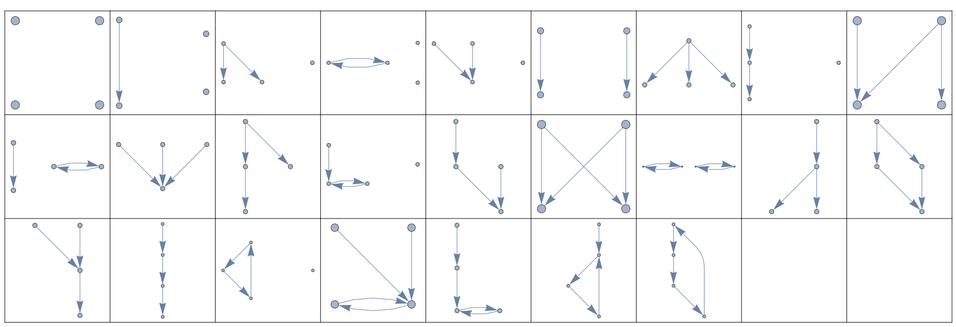


Fig. 4. 4-node nonhomeomorphic k-primal topologies

Recall that an element x in a primal space $(X, \mathcal{P}(f))$ is said to be a periodic point if $f^n(x) = x$ for some $n \in \mathbb{N}, n \neq 0$. The least $n \in \mathbb{N}$ such that $f^n(x) = x$ is called the period of x. Clearly, periodic points with period $\neq 1$ are cyclic points. Because of its importance, we recall the following lemma as well as its proof.

Lemma 2.3. *[9, Lemma 2.1]. Let $f : X \to X$ be a function. Then $x \in X$ is a minimal point of $(X, \leq_f)$ if and only if x is a periodic point of f.*

Proof: Suppose that x is minimal in $(X, \leq_f)$. As $f(x) \leq_f x$, we get $x \leq_f f(x)$. Hence, x is periodic.

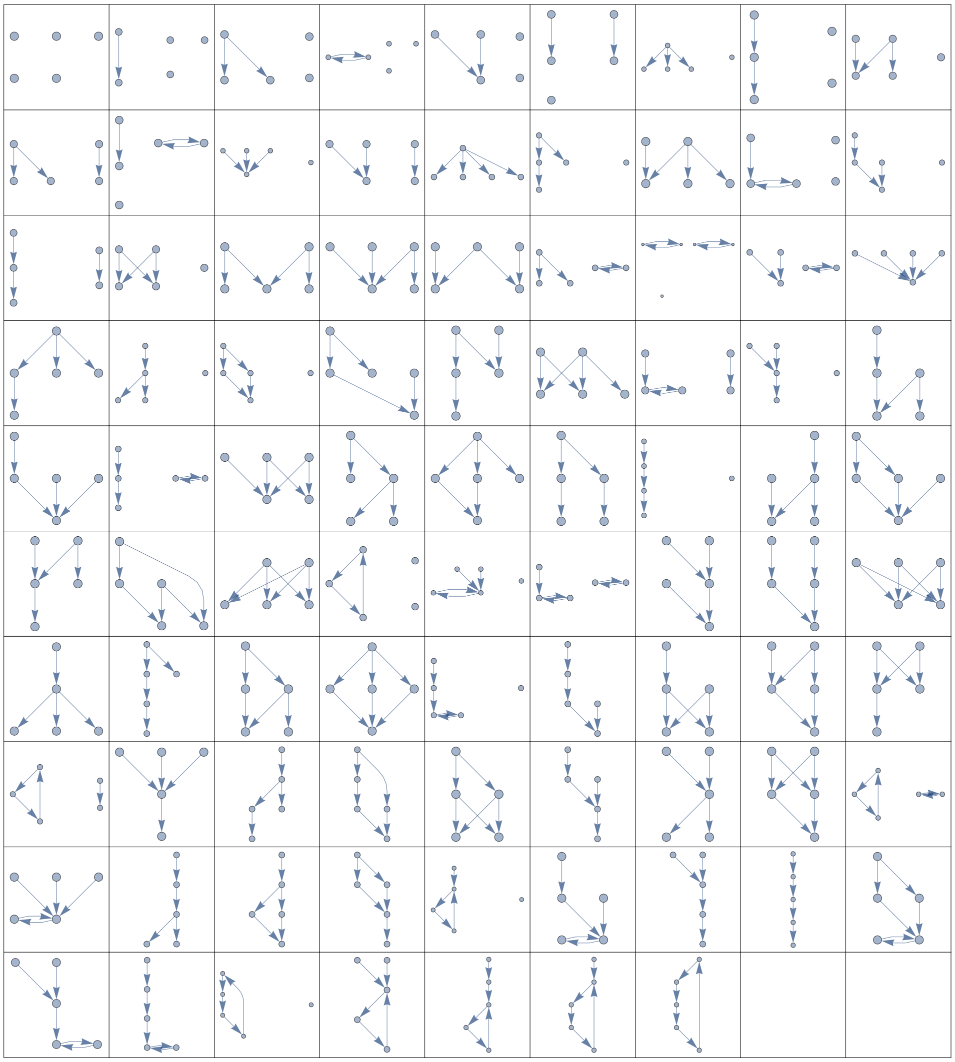


Fig. 5. 5-node nonhomeomorphic k-primal topologies

Conversely, if x is periodic and $y \leq_f x$, then there exists $n \in \mathbb{N}$ such that $y = f^n(x)$. But, as x is periodic, we may write $x = f^m(y)$ for some $m \in \mathbb{N}$, so $x \leq_f y$, showing that x is minimal. □

If $P = (X, \leq)$ is a partially ordered set, recall that $\leq^*$ is called a linear extension of $\leq$ if and only if $\leq^*$ is a total order on X which contains $\leq$. Szpilrajn's theorem [28] guarantees that every partial order is contained in a linear order.

In order dimension theory, the (Dushnik–Miller) dimension of a partially ordered set P is the least natural number t for which there exists a family $\{\leq_1, \ldots, \leq_t\}$ of linear extensions of P whose intersection is the ordering of P.

Such linear extensions form a *realization* of P. We denote the dimension of P by $\dim(P)$.

One of the major results of dimension theory is Hiraguchi's Theorem [13], which gives the best upper bound on the dimension of a partially ordered set in terms of its cardinality. It states that the dimension of a partially ordered set $P = (X, \leq)$ is at most $\left\lfloor \frac{|X|}{2} \right\rfloor$, provided $|X| \geq 4$.

Since every partial order on a finite set is an intersection of a finite number of linear orders and every linear order is a quasiorder, we have the following result.

Proposition 2.4. *Every finite partially ordered set P is a k-primal space for some positive natural number k.*

Proof: Let $P = (X, \leq)$ be a finite partially ordered set and let $(X, \sqsubseteq) = \{x_n \sqsubseteq \cdots \sqsubseteq x_1\}$ be a linear extension of $(X, \leq)$. Then, we can define a map $f_{\sqsubseteq} \in X^X$ by $f_{\sqsubseteq}(x_i) = x_{i+1}$ and $f_{\sqsubseteq}(x_n) = x_n$.

For any realization $\{\sqsubseteq_1, \ldots, \sqsubseteq_k\}$ of P, we may form $f_{\sqsubseteq_i}$ as above for each $i = 1, \ldots, k$, and this gives a family of primal spaces $\{(X, \mathcal{P}(f_{\sqsubseteq_i})) : i = 1, \ldots, k\}$ such that

$$(X, \leq) = \left(X, \bigvee_{i=1}^{k} \mathcal{P}(f_{\sqsubseteq_i}) \right) = (X, P(f_{\sqsubseteq_1}, ..., f_{\sqsubseteq_k})).$$

Therefore, $(X, \leq)$ is a k-primal space. □

Using Proposition 2.4 and the Dushnik–Miller dimension, we introduce the notion of primal dimension of a partially ordered set as follows.

Definition 2.5. *The primal dimension of a finite partially ordered set $P = (X, \leq)$, denoted $\dim_{\mathcal{P}}(P)$, is the least natural number k such that P is k-primal.*

The following result is immediate.

Proposition 2.6. *For any finite partially ordered set P, we have $\dim_{\mathcal{P}}(P) \leq \dim(P)$.*

Since $\dim_{\mathcal{P}}(P) \leq \dim(P)$ and $\dim(P)$ is at most $\left\lfloor \frac{|X|}{2} \right\rfloor$, we have $\dim_{\mathcal{P}}(P) \leq \left\lfloor \frac{|X|}{2} \right\rfloor$ for any $P = (X, \leq)$ with $|X| \geq 2$. The following examples show that the previous inequality may be strict.

Examples 2.7.

1. *The 2-dimensional partially ordered set ($\dim(P) = 2$) illustrated in Figure 6 is 1-primal ($\dim_{\mathcal{P}}(P) = 1$).*
2. *A nontrivial example showing that the inequality in Proposition 2.6 is strict is shown in Figure 7. The 3-dimensional partially ordered set ($\dim(P) = 3$) illustrated is 2-primal but not primal ($\dim_{\mathcal{P}}(P) = 2$).*

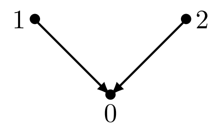
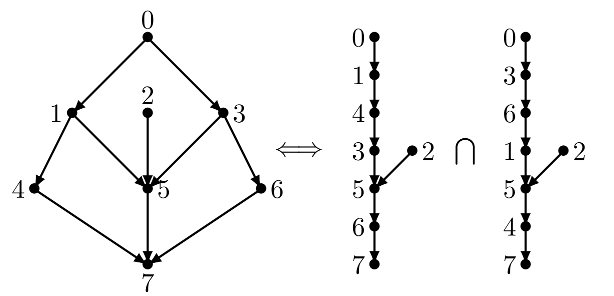


Fig. 6. A 2-dimensional partially ordered set which is 1-primal



$$(X, \leq) \quad = \quad (X, \leq_{f_1}) \ \bigcap \ (X, \leq_{f_2})$$

Fig. 7. A 3-dimensional partially ordered set which is 2-primal

Remark 2.8. *Hiraguchi's bound is the best possible (i.e. there exists a partially ordered set* $P = (X, \leq)$ *such that* $\dim(P) = \left\lfloor \frac{|X|}{2} \right\rfloor$*). For instance, consider the ordered set* $St_n = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ *ordered as follows:*

$\{a_1, \ldots, a_n\}$ *and* $\{b_1, \ldots, b_n\}$ *are both antichains.*

No b_i *is a lower bound of an* a_j*.*

$a_i \leq b_j$ *if and only if* $i \neq j$*.*

It is known that $Dim(St_n) = n$ *(for more information, see [8, Theorem 4.1]). The* 5*-dimensional ordered set* St_5 *is illustrated in Figure 8.*

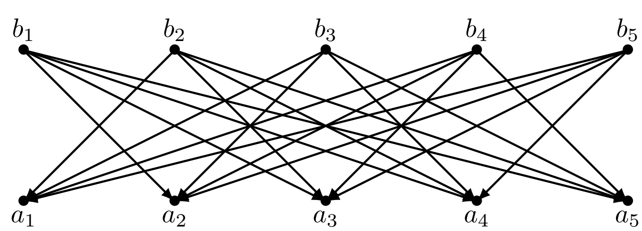


Fig. 8. The 5-dimensional ordered set St_5; $dim_{\mathcal{P}}(St_5) = dim(St_5)$

Using the example of Remark 2.8, we deduce the following proposition.

Proposition 2.9. *Hiraguchi's bound is also the best possible bound for the primal dimension (i.e. there exists a partially ordered set* $P = (X, \leq)$ *such that* $\dim_{\mathcal{P}}(P) = \left\lfloor \frac{|X|}{2} \right\rfloor$*).*

Proof: A similar argument can be used to show that the primal dimension $\dim_{\mathcal{P}}(St_n) = n$ (for $n \geq 3$). In fact, fix a family $I = \{f_1, \ldots, f_k\}$ of functions from St_n to itself such that $\bigcap_{i=1}^{k} \leq_{f_i} = \leq$.

For any fixed $i \in \{1, \ldots, n\}$, there exists $f_l \in I$ such that $a_i \nleq_{f_l} b_i$. For $j \neq i$, then we have $a_i < b_j$ and $a_j < b_i$. For $k \neq i, j$, we have $a_i < b_k$ and $a_j < b_k$. Thus, $\{a_i, a_j\} \subseteq (\downarrow b_k) \subseteq \overline{\{b_k\}}^{\mathcal{P}(f_l)}$. Hence, a_i and a_j are comparable in $(X, \leq_{f_l})$. If $a_i <_{f_l} a_j$, then $a_i <_{f_l} b_i$ which is absurd. Then, $a_j <_{f_l} a_i$ and consequently $a_j <_{f_l} b_j$. So we have obtained that every $\leq_{f_l}$ $(f_l \in I)$ contains at most one pair (a_i, b_i) such that $a_i \nleq_{f_l} b_i$. Thus there exists an injective mapping $\varphi : \{1, \ldots, n\} \to I$ which assigns to every i a map $f_l \in I$ such that $a_i \nleq_{f_l} b_i$. Therefore $n \leq |I|$ follows.

Hence, the Hiraguchi's bound is also the best possible bound for the primal dimension. □

3. A characterization of *k*-primal spaces

Recall that a qoset $(X, \leq)$ is connected if and only if for any $a, b \in X$, there exists a finite sequence $(a = x_1, x_2, ..., x_{n-1}, x_n = b)$ such that x_i and x_{i+1} are comparable for every $1 \leq i \leq n-1$. Hence, it is clear that every qoset $(X, \leq)$ can be written as a disjoint union of connected qosets. Now, suppose that $X = \bigcup_{i=1}^{k} C_k$ is the decomposition of X into k disjoint connected qosets. Then, the induced quasi-order by $\leq$ on every C_i will be denoted by $\leq_{C_i}$. One can see easily that for every $x \in X$, the downset of x in $(X, \leq)$ is exactly the downset of x in $(C_j, \leq_{C_j})$, where C_j is the unique connected qoset of X containing x. In this case, we write $(\downarrow x) = (\downarrow_{C_j} x)$. Finally, by a *strict qoset* we mean any qoset $(X, \leq)$ which is not a partially ordered set.

Now, we are in a position to give the main result of this paper. We show that a finite qoset $(X, \leq)$ is k-primal for some k if and only if whenever a cyclic point a is a lower bound of y, it is below every other lower bound of y.

Theorem 3.1. *Let* $(X, \leq)$ *be a finite qoset. Then,* $(X, \leq)$ *is a* k*-primal space for some* $k \in \mathbb{N}, k \neq 0$ *if and only if for every cyclic point* a *and every* $x, y \in X$ *we have*

$$\left\{ \begin{array}{l} a \leq y \\ x \leq y \end{array} \right. \implies a \leq x.$$

Proof: The proof will be divided into many steps.

Step 1: First, we remark that the condition in Theorem 3.1 implies that every cyclic point is minimal. Indeed, let a be a cyclic point. Consider $x \in X$ with $x \leq a$. Since $a \leq a$ and $x \leq a$, then by hypothesis $a \leq x$. Therefore, a is a minimal point.

Step 2: Let $(X, \leq)$ be a finite strict qoset which is connected and satisfies the condition in Theorem 3.1. Let us show that $(X, \leq)$ has a unique cycle and every minimal point is a cyclic point.

Indeed, by the hypothesis, $\leq$ is a strict quasiorder, so there exists a cyclic point $a \in X$ (a is also a minimal point by Step 1). Let b be a minimal point in X. We must show that $b \in (\downarrow a)$.

Since $(X, \leq)$ is connected, there exists a finite sequence $(a = v_1, \ldots, v_p = b)$ with $\{v_1, v_2, \ldots, v_p\} \subseteq X$, such that v_i and v_{i+1} are always comparable. If there exists a point v_j such that $v_j \ngeq a$, then take v_{i_0} ($i_0 > 1$) to be a first such point. Hence, $v_{i_0-1} \geq a$ and the two points v_{i_0-1} and v_{i_0} are comparable. So, we have two cases:

(1) $v_{i_0-1} \leq v_{i_0} \Rightarrow a \leq v_{i_0}$.

(2) $v_{i_0} \leq v_{i_0-1} \Rightarrow a \leq v_{i_0}$ (by our condition, since $a \leq v_{i_0-1}$).

In both cases, $a \leq v_{i_0}$. Finally, $a \leq b$ and by the minimality of b, $b \leq a$. So b is in the same cycle with a.

Step 3: Let $(X, \leq)$ be a finite connected strict qoset which satisfies the condition of Theorem 3.1. Then, $(X, \leq)$ is k-primal. Indeed, let $(\downarrow v_1) = \{v_1, \ldots, v_p\}$ be the unique cycle in $(X, \leq)$. Define the subset $X^* = (X - (\downarrow v_1)) \cup \{v_1\}$ endowed with the induced quasi-order $\leq_{X^*}$. Then, $(X^*, \leq_{X^*})$ is a partial ordered set (note that v_1 is the least element in X^*). So, by Proposition 2.4, $(X^*, \leq_{X^*})$ is k-primal. Hence, there exist $f_1^*, \ldots, f_k^* \in {X^*}^{X^*}$ such that

$$\leq_{X^*} \; = \; \leq_{f_1^*, \ldots, f_k^*} .$$

For any $x \in X^*$, if we denote by $(\downarrow_{X^*} x)$ the downset of x in X^*, we can write:

$$(\downarrow x) = (\downarrow_{X^*} x) \cup (\downarrow v_1) = \bigcap_{i=1}^{k} \overline{\{x\}}^{\mathcal{P}(f_i^*)} \cup (\downarrow v_1).$$

For each function f_i^* we define the function $f_i \in X^X$ by:

$$\begin{cases} f_i(x) = f_i^*(x) \;, \forall x \in X^* - \{v_1\} \\ f_i(v_j) = v_{j+1} \text{ if } 1 \leq j \leq p-1 \text{ and } f_i(v_p) = v_1 \end{cases} .$$

So, it is clear that $\forall x \in X^*$, $\overline{\{x\}}^{\mathcal{P}(f_i^*)} \subseteq \overline{\{x\}}^{\mathcal{P}(f_i)}$. Since $\forall x \in X^*$, $v_1 \in \bigcap_{i=1}^{k} \overline{\{x\}}^{\mathcal{P}(f_i^*)}$, then for each f_i and for each $x \in X^*$, $v_1 \in \overline{\{x\}}^{\mathcal{P}(f_i)}$ and consequently

$$\forall x \in X^*, \; \overline{\{x\}}^{\mathcal{P}(f_i)} = \overline{\{x\}}^{\mathcal{P}(f_i^*)} \cup (\downarrow v_1),$$

and if $x \in (\downarrow v_1)$, then by the construction $\overline{\{x\}}^{\mathcal{P}(f_i)} = \{v_1, \ldots, v_p\} = (\downarrow v_1) = (\downarrow x)$.

In summary, we get

$$\forall x \in X^*, \bigcap_{i=1}^{k} \overline{\{x\}}^{\mathcal{P}(f_i)} = \bigcap_{i=1}^{k} \overline{\{x\}}^{\mathcal{P}(f_i^*)} \cup (\downarrow v_1) = (\downarrow x).$$

$$\forall x \in (\downarrow v_1), \bigcap_{i=1}^{k} \overline{\{x\}}^{\mathcal{P}(f_i)} = \bigcap_{i=1}^{k} (\downarrow x) = (\downarrow x).$$

Hence, $\leq = \leq_{f_1,\ldots,f_k}$ and $(X, \leq)$ is k-primal.

Step 4: The proof of the main theorem.
Let $(X, \leq)$ be a k-primal space.
Suppose $a, x, y \in X$, a is a cyclic point, and $a \leq y$, $x \leq y$. We have

$$\begin{aligned} &(\downarrow a) \cup \{x\} \subset (\downarrow y) \\ \Longrightarrow\ &(\downarrow a) \cup \{x\} \subset \overline{\{y\}}^{\mathcal{P}(f_i)}, \ \forall i = 1, \ldots, k. \end{aligned}$$

But $\left(\overline{\{y\}}^{\mathcal{P}(f_i)}, \leq_{f_i}\right)$ is totally quasi-ordered (because $(X, \leq_{f_i})$ is a primal space). So, a and x are comparable in $(X, \leq_{f_i})$ and thus by Lemma 2.3, $\forall i = 1, \ldots, k$, $a \leq_{f_i} x$. Finally, $a \leq x$.

Conversely, let $\{C_1, \ldots, C_p\}$ be the family of connected components of $(X, \leq)$ each endowed with the quasi-order induced by that of X.

Since each connected quasi-ordered set $(C_i, \leq_{C_i})$ fulfills our condition, then by Proposition 2.4 and Step 3 each $(C_i, \leq_{C_i})$ is k_i-primal. That is, $\forall i = 1, \ldots, p$, there exist k_i functions $f_{i,1}, \ldots, f_{i,k_i}$ from C_i to C_i such that

$$\leq_{C_i} = \leq_{f_{i,1},\cdots,f_{i,k_i}}.$$

Let $k = \max\{k_i : i = 1, \ldots, p\}$. For each $i = 1, \ldots, p$ and $k_i < j \leq k$ define the functions $f_{i,j} = f_{i,1}$.

We define the functions $g_t \in X^X$ $(t = 1, \ldots, k)$ by

$$g_t(x) = f_{i,t}(x) \text{ if } x \in C_i.$$

Thus, g_t restricted to component C_i is the map $f_{i,t}$.

Now, let $x \in X$. Since x belongs to a unique C_n, we have

$$(\downarrow x) = (\downarrow_{C_n} x) = \bigcap_{j=1}^{k_n} \overline{\{x\}}^{\mathcal{P}(f_{n,j})} = \bigcap_{j=1}^{k} \overline{\{x\}}^{\mathcal{P}(f_{n,j})} = \bigcap_{j=1}^{k} \overline{\{x\}}^{\mathcal{P}(g_j)}.$$

In conclusion, $(X, \leq)$ is a k-primal space for $k = \max\{k_i : i = 1, \ldots, p\}$. □

The general form of a k-primal space is suggested in Figure 13 and the first paragraph of the proof of Theorem 4.3

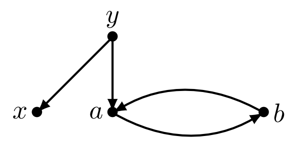


Fig. 9. A qoset which is not k-primal

Examples 3.2. 1. *Let $X = \{x, y, a, b\}$ equipped with the quasiorder defined as in Figure 9.*
Suppose that $(X, \leq)$ is k-primal for some integer k. Hence, $\leq$ is generated by the k functions $f_i : i = 1, ..., k$. Since $(\downarrow a) = (\downarrow b) = \{a,b\}$, then for any function f_i we have $\overline{\{a\}}^{\mathcal{P}(f_i)} = \overline{\{b\}}^{\mathcal{P}(f_i)}$ and thus a, b are cyclic points in $(X, \leq_{f_i})$, which implies by Lemma 2.3 that a and b are minimal points in $(X, \leq_{f_i})$.
Now, since $(\downarrow y) = X$, then $\overline{\{y\}}^{\mathcal{P}(f_i)} = X$ for each $i = 1, ..., k$. Hence $(X, \leq_{f_i})$ is a totally quasi-ordered set containing the minimal points a and b, so $\{a, b\} \subseteq \overline{\{x\}}^{\mathcal{P}(f_i)}$ for each $i = 1, ..., k$. Therefore $\{a, b\} \subseteq \overline{\{x\}}^{\tau_{\leq}}$. This is a contradiction because $\overline{\{x\}}^{\tau_{\leq}} = \{x\}$.
In this example, we have a is a cyclic point, $a \leq y$ and $x \leq y$ but x and a are not comparable.

2. *An infinite qoset $(X, \leq)$ satisfying the condition of the main result need not be k-primal for any integer k.*
For this, take the partially ordered set $(\mathbb{N}, \vdash)$ where the partial order $\vdash$ is defined by : $\forall n \neq 0, n \vdash 0$ and the numbers $1, 2, \dots$ are pairwise incomparable, as shown in Figure 10.

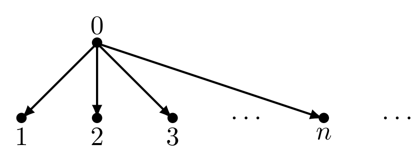


Fig. 10. An infinite partially ordered set which is not k-primal

Then, $(\downarrow 0) = \mathbb{N}$ and for each $a \neq 0$, $(\downarrow a) = \{a\}$. Suppose $(\mathbb{N}, \vdash)$ is k-primal, with $\vdash$ generated by the k functions $f_i, (i = 1, \dots, k)$. Then, $(\downarrow 0) = \bigcap_{i=1}^{k} \overline{\{0\}}^{\mathcal{P}(f_i)} = \mathbb{N}$. So, for any function f_i we have $\overline{\{0\}}^{\mathcal{P}(f_i)} = \mathbb{N}$ and consequently,

$$\forall i = 1, \dots, k, \ \forall n \in \mathbb{N}^* = \mathbb{N} - \{0\}, \ \exists! \ \alpha_{i,n} \in \mathbb{N}^* : n = f_i^{\alpha_{i,n}}(0).$$

Now, for each f_i we have

$$\begin{aligned}\overline{\{1\}}^{\mathcal{P}(f_i)} &= \{1, f_i(1), \ldots, f_i^n(1), \ldots\} \\ &= \left\{f_i^{\alpha_{i,1}}(0), f_i^{\alpha_{i,1}+1}(0), \ldots, f_i^{\alpha_{i,1}+n}(0), \ldots\right\} \\ &\qquad\qquad\qquad\qquad \text{(for some unique } \alpha_{i,1} \in \mathbb{N}^*) \\ &= \mathbb{N} - \left\{0, f_i(0), \ldots, f_i^{\alpha_{i,1}-1}(0)\right\}.\end{aligned}$$

Hence, $\overline{\{1\}}^{\tau_\leq} = \bigcap_{i=1}^{k} \overline{\{1\}}^{\mathcal{P}(f_i)}$ *is an infinite set, which is impossible. While the partially ordered set* $(\mathbb{N}, \vdash)$ *has no primal dimension, we note that it is* 2*-dimensional: for the two linear extensions* $\leq_1$ *and* $\leq_2$ *of* $\vdash$ *given by*

$$\begin{aligned}&n+1 \leq_1 n \quad \forall n \in \mathbb{N} \\ &n \leq_2 n+1 \quad \forall n \in \mathbb{N}^* \quad \textit{and } n \leq_2 0 \quad \forall n \in \mathbb{N},\end{aligned}$$

the family $\{\leq_1, \leq_2\}$ *is a realization of* $(\mathbb{N}, \vdash)$*.*

4. Complementation in the lattice $\boldsymbol{KPTOP(X)}$

The study of complementation in the lattice $TOP(X)$ of all topologies on a set X was advanced by A. Steiner [27] and variations on this theme have been studied extensively since. (See [6, 7, 17, 22, 23, 25, 26].) Recall that τ and τ^* are complements if $\tau \vee \tau^*$ is the discrete topology and $\tau \wedge \tau^*$ is the indiscrete topology. For a positive integer k, let k-$PTOP(X)$ be the collection of k-primal topologies on X. Let $KPTOP(X) = \bigcup_{k=1}^{\infty} k\text{-}PTOP(X) = \{\tau \in TOP(X) : \tau$ is k-primal for some $k \in \mathbb{N}\}$, ordered by $\subseteq$. Recall that $\tau \in KPTOP(X)$ if and only if there exists a natural number k and primal topologies $\mathcal{P}(f_i)$ $(i = 1, \ldots, k)$ with $\tau = \bigvee_{i=1}^{k} \mathcal{P}(f_i)$, where the supremum is taken in $TOP(X)$. It follows that $KPTOP(X)$ is closed under finite suprema. Since the indiscrete topology on $X = \{x_0, \ldots, x_{n-1}\}$ is $\mathcal{P}(f)$ where $f(x_i) = x_{(i+1 \mod n)}$, it is the smallest element of $KPTOP(X)$. Thus, if X if finite, $KPTOP(X)$ is a (complete) lattice. Any lattice which contains the primal topologies must contain finite suprema of primal topologies. We record this as a theorem.

Theorem 4.1. *If X is finite, $KPTOP(X)$ is a lattice, and is the smallest lattice in TOP(X) which contains all the primal spaces.*

We note that if $X = \{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\}$ is countably infinite, $KPTOP(X)$ is not a lattice. With $f(a_i) = a_{i+1}$ and $g(a_i) = a_{i-1}$, we find that the only lower bound of $\mathcal{P}(f)$ and $\mathcal{P}(g)$ in $TOP(X)$ is the indiscrete topology, so the only candidate for $\mathcal{P}(f) \wedge \mathcal{P}(g)$ in $KPTOP(X)$ is the indiscrete topology. However, by Proposition 2.2, the indiscrete topology X is not k-primal, so $KPTOP(X)$ is not a lattice.

Theorem 4.2. *The following are equivalent.*

(a) $KPTOP(X)$ is a sublattice of $TOP(X)$.
(b) $KPTOP(X)$ is a distributive lattice.
(c) $|X| \leq 2$.

Proof: If $|X| \leq 2$, then $KPTOP(X) = TOP(X)$, so (c) implies (a) and (b). Suppose $X = \{1, 2, \ldots, n\}$ for $n \geq 3$. The example used in Theorem 3(c) of [22] shows (a) implies (c). Specifically, consider the topologies τ_1 and τ_2 on X whose specialization quasiorders are $\leq_1$ on X defined by $i \leq_1 i$ for all $i \in X$ and $3 \leq_1 2 \leq_1 1$, and $\leq_2$ on X defined by $i \leq_2 i$ for all $i \in X$ and $3 \leq_2 1 \leq_2 2$. Figure 11 shows τ_1, τ_2, and $\tau_1 \wedge \tau_2$, where the inf is taken in $TOP(X)$. By Theorem 3.1, this inf is not in $KPTOP(X)$, so $KPTOP(X)$ is not a sublattice of $TOP(X)$. In $TOP(X)$, $\tau_1 \wedge \tau_2$ has basis of minimal neighborhoods $\{\{1,2\}, \{1,2,3\}\} \cup \{\{i\} : i \in X - \{1,2,3\}\}$. In $KPTOP(X)$, $\tau_1 \wedge \tau_2$ has basis $\{\{1,2,3\}\} \cup \{\{i\} : i \in X - \{1,2,3\}\}$.

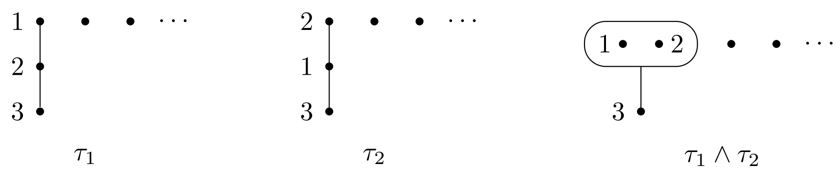


Fig. 11. $\tau_1 \wedge \tau_2$ is not in $KPTOP(X)$

To see (b) implies (c), suppose $X = \{1, 2, 3\} \cup X'$. Consider the topologies τ_i having specialization quasiorders $\leq_i$ as shown in Figure 12. Now $\tau_1 \wedge \tau_2$ is

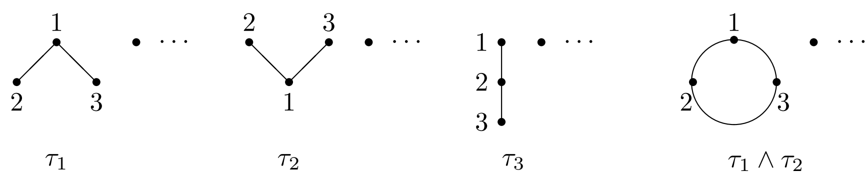


Fig. 12. $\tau_1 \wedge [\tau_2 \vee \tau_3] \neq [\tau_1 \wedge \tau_2] \vee [\tau_1 \wedge \tau_3]$

also shown in Figure 12. It is easy to see that $\tau_1 \wedge \tau_3 = \tau_3$, and then $[\tau_1 \wedge \tau_2] \vee [\tau_1 \wedge \tau_3] = \tau_3$. However, since $\tau_2 \vee \tau_3$ is the discrete topology, $\tau_1 \wedge [\tau_2 \vee \tau_3] = \tau_1 \neq [\tau_1 \wedge \tau_2] \vee [\tau_1 \wedge \tau_3]$. Thus, $KPTOP(X)$ is not distributive. (It is also easy to check that $\tau_3 \vee [\tau_1 \wedge \tau_2] = \tau_3 \neq \tau_1 = [\tau_3 \vee \tau_1] \wedge [\tau_3 \wedge \tau_2]$.) □

By Theorem 3.1, if a finite connected Alexandroff topological space (X, τ) is k-primal, then the associated specialization quasi-order $\lesssim$ on X is either a partially ordered set (if it has no cyclic points), or it is obtained from a partially ordered set with minimum element by splitting the minimum element into several elements in a cycle. That is, if $(X, \lesssim)$ is given the equivalence relation $a \approx b$ if and only if $a \lesssim b$ and $b \lesssim a$, then there is one equivalence class $[x]$ which is not a singleton, and it is minimum in the resulting partial order on the $\approx$-equivalence classes defined by $[a] \leq [b]$ if and only if $a \lesssim b$. This quotient space of $\approx$-equivalence classes, with the specialization topology consisting of the $\leq$-increasing sets, is the T_0-reflection of (X, τ). See [4, 16] for further information on this theme. It is well-known that an Alexandroff space (X, τ) is connected (as a topological space) if and only if the associated quasiorder $(X, \lesssim)$ is connected (as a graph). If $X = (X, \leq)$ is a quasiordered set, by X^{op} we mean the quasiordered set $(X, \geq)$.

Theorem 4.3. *If X is finite, $KPTOP(X)$ is a complemented lattice. That is, if τ is a k-primal topology on a finite set X then τ has a complement which is j-primal for some $j \in \mathbb{N}$.*

Proof: Suppose τ is k-primal with connected components $P_1 \cup C_1, \ldots, P_n \cup C_n, P_{n+1}, \ldots, P_m, L_1, \ldots, L_t, I_1, \ldots, I_j$, where each C_i is a cycle (with more than one point) appearing below the partially ordered set P_i $(1 \leq i \leq n)$, each P_i $(n+1 \leq i \leq m)$ is a partially ordered set with more than one element, each L_i is a cycle with more than one element, and $|I_i| = 1$ $(1 \leq i \leq j)$, as suggested in Figure 13.

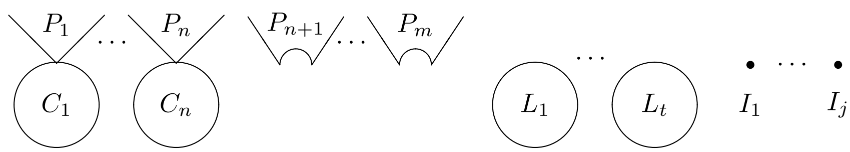


Fig. 13. The quasiorder for an arbitrary k-primal space (X, τ)

For $1 \leq i \leq n$, let $M_i = C_i$, and for $n+1 \leq i \leq m$ let M_i be the set of minimal points in P_i. (Thus, in all cases, M_i is the set of minimal points in its component.) Let $M = \bigcup_{i=1}^{m} M_i$, $L = \bigcup_{i=1}^{t} L_i$, and $I = \bigcup_{i=1}^{j} I_i$.

<u>Case 1:</u> $M = \emptyset$. Then there are no components of form $P_i \cup C_i$ or P_j, so $X = L \cup I$. For each $i = 1, \ldots, t$, pick $l_i \in L_i$ and let $U = I \cup \{l_i\}_{i=1}^{t}$. With $\tau^* = \{U\} \cup \{\{x\} : x \in X - U\}$, it is easy to see that τ^* is a complement of τ. Furthermore, since τ^* has one cycle and several isolated points, it is 1-primal.

<u>Case 2:</u> $M \neq \emptyset$. Let (X, τ^*) be the Alexandroff space whose specialization quasiorder $\leq^*$ is the partially ordered set depicted in Figure 14. We describe the order $\leq^*$ on L. Each L_i has at least two elements $l_{i,1}$ and $l_{i,2}$. We form chains $l_{1,1} <^* l_{2,1} <^* \cdots <^* l_{t,1}$ and $l_{t,2} <^* \cdots <^* l_{2,2} <^* l_{1,2}$, with elements of each chain unrelated to elements of the other, and with every element of L not in one of these chains related only to itself. Recall that the smallest τ^*-neighborhood $N^*(x)$ of x is $\uparrow_* x$, and U is τ^*-open if and only if $U = \uparrow_* U = \{y \in X : \exists u \in U, u \leq^* y\}$. In (X, τ), $N(x) = \uparrow x$, or for emphasis, $\uparrow_X x$.

First we show that $N(x) \cap N^*(x) = \{x\}$ for all $x \in X$ by considering cases based on where x lies.

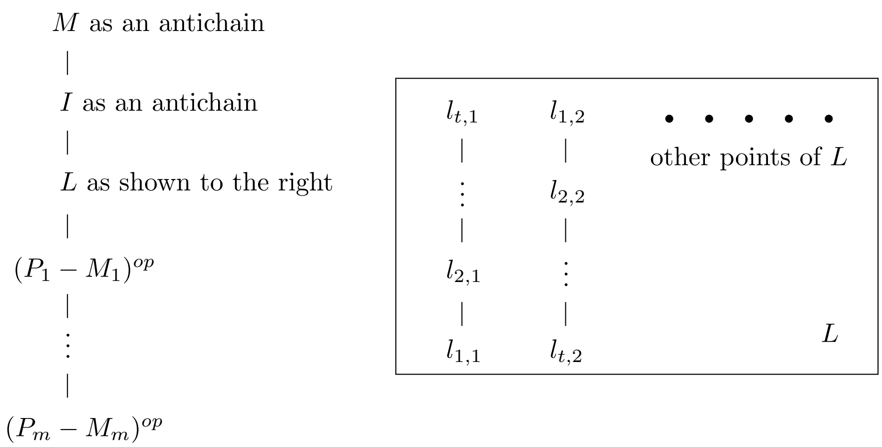


Fig. 14. The quasiorder for the complement (X, τ^*)

If $x \in I$, $N(x) = \{x\}$. If $x \in M$, $N^*(x) = \{x\}$. In each of these cases, $N(x) \cap N^*(x) = \{x\}$.

If $x \in L_i$, then $N(x) \cap N^*(x) = L_i \cap N^*(x) = \{x\}$.

If $x \in P_i - M_i$,

$$N^*(x) = \uparrow_* x = M \cup I \cup L \cup \bigcup_{j=1}^{i-1} P_j \cup \downarrow_{P_i} x = M \cup I \cup L \cup \bigcup_{j=1}^{i-1} P_j \cup \downarrow_X x.$$

Since $N(x) = \uparrow_X x = \uparrow_{P_i} x \subseteq P_i$, we have $N(x) \cap N^*(x) = \uparrow_X x \cap \downarrow_X x = \{x\}$. Thus, $\tau \vee \tau^*$ is the discrete topology.

Next, we show that $\tau \cap \tau^* = \{\emptyset, X\}$. Suppose $U \in \tau \cap \tau^*$ and $x \in U$. Again, we consider cases based on where x lies. Now U is $\leq$-increasing and $\leq^*$ increasing, so we may show $U = X$ by showing that starting from x and iteratively taking $\uparrow$ and $\uparrow_*$, we get every point of X.

If $x \in I$, $M \subseteq \uparrow_* x \subseteq U$, so $\uparrow M \subseteq U$. But since $M \neq \emptyset$, $\uparrow M$ contains $P_i - M_i$ for all $i = 1, \ldots, m$. Now $\uparrow_* (P_m - M_m) = X$. Thus, $U = X$.

If $x \in M_i$ for some i, then $N(x) = \uparrow x \subseteq U$ and $\uparrow x$ contains a point $x' \in P_i - M_i$. Now $\uparrow_* x'$ contains $M \cup I \cup L$, so U contains $M \cup I \cup L$. Now $U = \uparrow U$ must contain $\uparrow(M \cup I \cup L) = X$. Thus, $U = X$.

If $x \in (P_i - M_i)$ for some i, then $M \cup I \cup L \subseteq \uparrow_* x \subseteq U$, and $\uparrow(M \cup I \cup L) \subseteq \uparrow U = U$, so $U = X$.

If $x \in L_i$ for some i, then $L_i = N(x) = \uparrow x \subseteq U$. Considering the order $\leq^*$ on L (as shown in the box in Figure 14), $N^*(x) = \uparrow_* x$ contains one point $a_k \in \{l_{k,1}, l_{k,2}\}$ from each L_k $(1 \leq k \leq j)$, so U must contain $N(a_k) = L_k$

for each k. This shows $L \subseteq U$. Thus, $\uparrow_* L = (M \cup I \cup L) \subseteq U$, and thus $X = \uparrow(M \cup I \cup L) \subseteq U$. Thus, $U = X$.

In all cases, $\tau \cap \tau^* = \{\emptyset, X\}$, so τ^* is a complement of τ.

Finally, since the quasiorder for τ^* is in fact a partial order, Proposition 2.4 or Theorem 3.1 show that $\tau^* \in KPTOP(X)$. □

Since the quasiorder for τ^* described in Figure 14 is a partial order, this construction produces a T_0 complement for X. In particular, together with Proposition 2.4 this shows that every T_0 topology on a finite set has a T_0 complement.

Also, it is easy to see that if X is connected and is a partially ordered set P_1 or a partially ordered set above a cycle $P_1 \cup C_1$, then this algorithm produces a complement which is also connected.

Note that the result of Theorem 4.3 fails if X is infinite: the discrete topology $\mathcal{P}(id)$ is 1-primal, but its only complement is the indiscrete topology, which is not k-primal for any k.

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