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Ordered Compactifications, Galois Connections, and Quasi-uniformities

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An ordered topological space $(X, \tau, \leq)$ is a set X with a topology τ and a partial order $\leq$. We usually assume some forms of compatibility between the topology and order, such as *convexity* of the topology (τ has a base of $\leq$-convex sets) or the T_2-ordered condition (the graph of $\leq$ is closed in the product $(X, \tau) \times (X, \tau)$). We assume both these conditions for all ordered spaces considered here. The study of ordered topological spaces may be considered more general than the study of topological spaces, for any topological space (X, τ) may be thought of as an ordered space $(X, \tau, =)$ trivially ordered by equality. The classical introduction to ordered topological spaces is Leopoldo Nachbin's *Topology and Order* [12], written in Portuguese in 1950 and translated into English in 1965. Besides the classical approach to ordered spaces which I usually use, Brümmer, Künzi, Nailana, Salbany, and others have used more sophisticated approaches, including bitopological techniques [13, 16], quasi-uniform techniquess [14, 9], nonstandard analysis techniques [17], and categorical techniques [2].

An *ordered compactification* of an ordered topological space $(X, \tau, \leq)$ is a compact T_2-ordered space $(X', \tau', \leq')$ such that (X, τ) is dense in (X', τ') and the restriction of $\leq'$ to X agrees with $\leq$. The classical study of compactifications of topological spaces amounts to the study of trivially ordered compactifications of trivially ordered topological spaces. Richard Chandler's *Hausdorff Compactifications* [3] provides an excellent introduction to theory of topological compactification. Only the completely regular topological spaces have (Hausdorff) compactifications; Nachbin [12] showed that an ordered space has an ordered compactification if and only if it is *completely regular ordered*.

The set $K_o(X)$ of all ordered compactifications of a completely regular ordered space X has a natural order relation defined by agreeing that for $X', X'' \in K_o(X), \quad X' \geq X''$ if and only if there is an increasing function $f : X' \to X''$ with $f(x) = x \ \forall x \in X$. With this order relation, $K_o(X)$ is a complete upper semilattice (that is, every subset of $K_o(X)$ has a supremum). In particular, a completely regular ordered space X has a largest ordered compactification $\beta_o X$ called the *Stone-Čech ordered compactification* or the *Nachbin compactification*.

Analogous results hold for the collection $K(X)$ of compactifications of a completely regular topological space X. Determining which results for topological compactifications carry over in some form to the more general setting of ordered compactifications has been an important part of the study of ordered topological spaces. Consider the following well known topological result.

Proposition: For a completely regular topological space X, the following are equivalent:

(1) The complete upper semilattice $K(X)$ is a complete lattice.

(2) X has a smallest compactification.

(3) X has a one-point compactification.

(4) X has a finite-point compactification.

(5) X is locally compact.

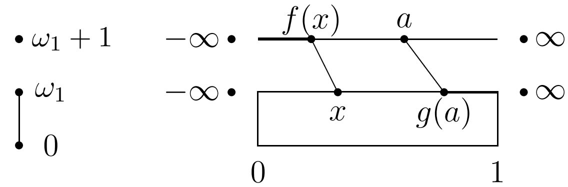
For ordered compactifications, the situation is much more complicated. It is a simple lattice theoretic result that a complete upper semilattice is a complete lattice if and only if it has a smallest element, so the statement for ordered compactifications analogous to the implication (1) $\iff$ (2) above holds. Besides the trivial implication (3) $\Rightarrow$ (4), none of the other implications above hold in the setting of ordered spaces.

For example, consider the real line $(\mathbb{R}, \tau, =)$ with the usual topology and the trivial order. Given the trivial order, the one-point compactification $\mathbb{R}^*$ $= \mathbb{R} \cup \{\infty\}$ is an ordered compactification ($\mathbb{R}^*$, τ^*, $=^*$) of $(\mathbb{R}, \tau, =)$, but there are other closed orders on $(\mathbb{R}^*, \tau^*)$ which extend the original order $=$ on $\mathbb{R}$. The order $\leq'$ on $\mathbb{R}^*$ defined by $x \leq' y$ if and only if $x = y$ or $[x = \infty$ and $y = 0]$ gives an ordered compactification $(\mathbb{R}^*, \tau^*, \leq')$ of $(\mathbb{R}, \tau, =)$ in which $\infty < 0$. Similarly, the closed order $\leq''$ on $\mathbb{R}^*$ defined by $x \leq'' y$ if and only if $x = y$ or $[x = 0$ and $y = \infty]$ gives an ordered compactification $(\mathbb{R}^*, \tau^*, \leq'')$ of $(\mathbb{R}, \tau, =)$ in which $0 < \infty$. There can be no ordered compactification below $(\mathbb{R}^*, \tau^*, \leq')$ and $(\mathbb{R}^*, \tau^*, \leq'')$. Thus, $(\mathbb{R}, \tau, =)$ has a one-point ordered compactification (indeed, infinitely many of them) but no smallest ordered compactification. For more details on the failure of the other equivalences above in the ordered setting, see [15].

As the example above shows, if $(X', \tau', \leq')$ is an ordered compactification of $(X, \tau, \leq)$, there may be many other orders $\leq^*$ on (X', τ') for which $(X', \tau', \leq^*)$ becomes an ordered compactification. If two such "topologically equivalent" ordered compactifications $(X', \tau', \leq')$ and $(X', \tau', \leq^*)$ are comparable, the larger ordered compactification is the one with the smaller order. (Refer to the definition of the order on $K_o(X)$.)

The semilattice $K_o(X)$ has been described for some restrictive types of ordered spaces. Blatter [1] and later Kent and Richmond [7] describe $K_o(X)$ if X is a totally ordered space. The cases described in [10] and [11] show that $K_o(X)$ may be complicated for relatively simple ordered spaces X.

In all that follows, we will focus on a single example of the type described in [11]. Let X be the unit interval $[0, 1]$ and $Y = [0, \omega_1) \cup \{\omega_1 + 1\}$ be the ordinals less than the first uncountable ordinal together with an isolated top element $\omega_1 + 1$. Give X and Y each their usual topology and order, and $X \times Y$ the product topology and the product order $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$. In general, $\beta_o(X \times Y) \neq \beta_o X \times \beta_o Y$, but as X and Y are totally ordered, the necessary and sufficient conditions of [6] show that in our case equality does hold, so $\beta_o(X \times Y) = [0, 1] \times [0, \omega_1] \cup \{\omega_1 + 1\}$. All smaller ordered compactifications of $X \times Y$ are topologically equivalent to $\beta_o(X \times Y)$ and are obtained from $\beta_o(X \times Y)$ by imposing a larger closed order relation on $\beta_o(X \times Y)$, without introducing any new order on the base space $X \times Y$. This may be done by making a point x of $X \times \{\omega_1\}$ greater than a point $f(x)$ of $X \times \{\omega_1 + 1\}$ (and by transitivity, x must also be greater than all points less than $f(x)$). Dually, order may be added by making a point a of $X \times \{\omega_1 + 1\}$ less than each point of an increasing segment $[g(a), 1]$ of $[0, 1] \times \{\omega_1\}$. The figure below suggests the possible additional order.



Thus, any ordered compactification of $X \times Y$ determines a pair of functions f and g where, for $x \in X \times \{\omega_1\}$, $f(x)$ is the greatest element of $X \times \{\omega_1 + 1\}$ which is less than x, with $f(x) = -\infty$ if x is not greater than any points of $X \times \{\omega_1 + 1\}$; and for $x \in X \times \{\omega_1 + 1\}$, $g(x)$ is the least element of $X \times \{\omega_1\}$ which is greater than x, with $g(x) = \infty$ if x is not less than any elements of $X \times \{\omega_1\}$. Now f and g may be considered to be functions on $X \cup \{\pm\infty\}$, where $\pm\infty$ are topologically isolated fixed points of f and g, with $-\infty < x < \infty \ \forall x \in X$. One may show that f and g are increasing functions, f is continuous from the right and g continuous from the left, and that f and g satisfy the inequality

$$f(x) < g(f(x)) \leq x \leq f(g(x)) < g(x).$$

Now any element of $K_o(X \times Y)$ determines a pair of functions (f, g) as above, and conversely any such pair of functions determines an ordered compactification of $X \times Y$.

Such pairs of functions are examples of a structure known as a *Galois connection.* Details on Galois connections, including the definition and proposition below, may be found in [4]. (A symmetic but contravariant form of the definition appears in the literature as well; we use the covariant form.)

Definition: Suppose $(P, \leq)$ and $(Q, \leq')$ are partially ordered sets. If $f : P \to Q$ and $g : Q \to P$ are functions such that for all $p \in P$ and all $q \in Q$,

$$p \leq g(q) \iff f(p) \leq' q,$$

then the quadruple (P, f, g, Q) is called a *Galois connection.*

Proposition: (See [4].) Let $(P, \leq)$ and $(Q, \leq')$ be partially ordered sets and $f : P \to Q$ and $g : Q \to P$ be functions. Then the following are equivalent:

(1) (P, f, g, Q) is a Galois connection.

(2) f is increasing, and $g(q) = \max\{z \in P : f(z) \leq' q\}$ for each $q \in Q$.

(3) f and g are increasing, $x \leq' f(g(x))$ for all $x \in Q$ and $g(f(x)) \leq x$ for all $x \in P$.

With $P = Q = X \cup \{\pm\infty\}$, we see that each ordered compactification of $X \times Y$ corresponds to a Galois connection (P, f, g, Q), and, by (2) above, the second function g is in fact determined by the first function f. For our space $X \times Y$, it follows that $K_o(X \times Y)$ is isomorphic to the collection of functions $\mathcal{F} = \{f : X \cup \{\pm\infty\} \to X \cup \{\pm\infty\} | f$ is increasing, continuous from the right, strictly below the diagonal on X, with $\pm\infty$ as fixed points$\}$. The order on $\mathcal{F}$ is the pointwise order on functions: $r \leq s$ if and only if $r(x) \leq s(x)\ \forall x$.

Further details on the connection between Galois connections and elements of $K_o(X \times Y)$ where X and Y are arbitrary totally ordered spaces are given in [11]. In the more general setting described in that paper, the elements of $K_o(X \times Y)$ need not all be topologically equivalent. Geometric descriptions of possible identifications of points of $\beta_o X \times \beta_o Y \setminus (X \times Y)$ are given. What follows is a preliminary report on another interesting structure possessed by $K_o(X \times Y) \approx \mathcal{F}$, where X and Y are the specific spaces given above.

There is a close connection between ordered topological spaces and quasi-uniform spaces, the details of which can be found in Peter Fletcher and Bill Lindgren's monograph [5], as well as in the works of Hans-Peter Künzi. A quasi-uniformity $\mathcal{U}$ on X is *compatible* with an ordered space $(X, \tau, \leq)$ if the intersection of the entourages of $\mathcal{U}$ gives the partial order $\leq$ and the symmetric topology generated by the associated uniformity $\mathcal{U} \cup \mathcal{U}^{-1}$ is τ. There is a one-to-one correspondence (via completion) between the compatible totally bounded quasi-uniformities on X and the ordered compactifications of $(X, \tau, \leq)$.

In his doctoral work at Technische Universität Dresden, Ralph Kummetz [8] defines a directed family $\mathcal{F}$ of functions on a poset X to be an *F-poset* if each $f \in \mathcal{F}$ is increasing and below the diagonal, and for every $f \in \mathcal{F}$, there exists $g \in \mathcal{F}$ with $f \leq g \circ g$. An F-poset is *approximating* if $\sup \mathcal{F} = \Delta_X$. Given an F-poset $\mathcal{F}$, the collection $\{U_f : f \in \mathcal{F}\}$, where $U_f = \{(x, y) : y \geq f(x)\}$ is a basis for a quasi-uniformity $\mathcal{U}_{\mathcal{F}}$ on X. If $\mathcal{F}$ is an approximating F-poset, the quasi-uniformity determines the order on X. In our example above, $K_o(X \times Y) \approx \mathcal{F} = \{f : X \cup \{\pm\infty\} \to X \cup \{\pm\infty\} | f$ is increasing, continuous from the right, strictly below the diagonal on X, with $\pm\infty$ as fixed points$\}$. It can be shown that $\mathcal{F}$ is an F-poset on $X \cup \{\pm\infty\}$, and thus gives a quasi-uniformity $\mathcal{U}_{\mathcal{F}}$ on $X \cup \{\pm\infty\}$. One might hope that the order

$\bigcap \mathcal{U}_{\mathcal{F}}$ on $X \cup \{\pm\infty\}$ and the uniform topology from $\mathcal{U}_{\mathcal{F}} \cup \mathcal{U}_{\mathcal{F}}^{-1}$ on $X \cup \{\pm\infty\}$ would give, when restricted to X, the original order on X. This is almost true, except that there is a problem at the endpoints 0 and 1 of X. To recover the original topology and order on X from the topology and order on $X \cup \{\pm\infty\}$, the introduced points $\pm\infty$ can not merely be discarded by taking X as a subspace of $X \cup \{\pm\infty\}$. However, by forming a quotient space identifying the sets $\{-\infty, 0\}$ and $\{1, \infty\}$, we do recover the original topology and order on X.

Thus, the complete upper semilattice $K_o(X \times Y)$ of totally bounded compatible quasi-uniformities on $X \times Y$ may be viewed as a collection of compatible totally bounded quasi-uniformities on $X \times Y$ or a collection of Galois connections $(X \cup \{\pm\infty\}, f, g, X \cup \{\pm\infty\})$. Since one function of a Galois connection is determined by the other, $K_o(X \times Y)$ may be viewed as the collection $\mathcal{F}$ as above, and this collection is an F-poset on $X \cup \{\pm\infty\}$ which determines a single quasi-uniformity on $X \cup \{\pm\infty\}$. This quasi-uniformity gives, after appropriately eliminating the introduced points $\pm\infty$, the original topology and order on the compact factor X of $X \times Y$.

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