

T_i -ordered reflections

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ABSTRACT. We present a construction which shows that the T_i -ordered reflection ($i \in \{0, 1, 2\}$) of a partially ordered topological space (X, τ, \leq) exists and is an ordered quotient of (X, τ, \leq) . We give an explicit construction of the T_0 -ordered reflection of an ordered topological space (X, τ, \leq) , and characterize ordered topological spaces whose T_0 -ordered reflection is T_1 -ordered.

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1. INTRODUCTION

The T_0 -, T_1 -, and T_2 -reflections of a topological space have long been of interest to categorical topologists. Methods of constructions of these in the category TOP are described in the references given in [3] (see p. 302). Here, we consider the corresponding concepts of T_i -ordered reflections in the category ORDTOP of partially ordered topological spaces with continuous increasing functions as morphisms. In Section 2, we construct the T_i -ordered reflection ($i = 0, 1, 2, S2$) of a partially ordered topological space (X, τ, \leq) . The construction is extrinsic, occurring in the category PREORDTOP of preordered topological spaces with continuous increasing functions as morphisms, which contains ORDTOP as a subcategory. In Section 3, we give an intrinsic construction of the T_0 -ordered reflection of a partially ordered space (X, τ, \leq) and examine some properties of this reflection.

A preordered topological space (X, τ, \preceq) is a set X with a topology τ and a preorder \preceq . Following the notation of Nachbin ([7]), for $A \subseteq X$, the increasing hull of A is $i(A) = \{y \in X : \exists a \in A \text{ with } a \preceq y\}$. A set A is an *increasing set*

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if $A = i(A)$. The closed increasing hull $I(A)$ of $A \subseteq X$ is the smallest closed increasing set containing A . Decreasing sets, decreasing hulls $d(A)$, and closed decreasing hulls $D(A)$ are defined dually. A set is *monotone* if it is increasing or decreasing. There are many compatibility conditions between the topology and order of a preordered topological space which one may stipulate. These include the convex topology condition (τ has a subbase of monotone open sets) or the ordered separation axioms, some of which are defined below. Our preordered spaces need not satisfy any of these compatibility conditions.

Suppose (X, τ, \preceq) is a preordered topological space. It is well-known that the preorder \preceq induces an equivalence relation $\sim = G(\preceq) \cap G(\preceq^{-1})$ on X defined by $x \sim y$ if and only if $x \preceq y$ and $y \preceq x$. The preordered topological space (X, τ, \preceq) is said to be *T_0 -preordered* if any of the following equivalent statements holds.

- (a) $x \not\sim y \Rightarrow [I(x) \neq I(y) \text{ OR } D(x) \neq D(y)]$
- (b) If $I(x) = I(y)$ and $D(x) = D(y)$, then $x \sim y$.
- (c) If $x \not\sim y$, there exist a monotone open neighborhood of one of the points which does not contain the other point.

Observe that if \preceq is a partial order, then the relation \sim is equality.

A preordered topological space (X, τ, \preceq) is *T_1 -preordered* if $i(x)$ and $d(x)$ are closed for every $x \in X$, or equivalently, if $x \not\preceq y$ in X implies there exists an open increasing neighborhood of x which does not contain y and there exists an open decreasing neighborhood of y which does not contain x .

A preordered topological space (X, τ, \preceq) is *T_2 -preordered* if there is an increasing neighborhood of x disjoint from some decreasing neighborhood of y whenever $x \not\preceq y$. Equivalently, (X, τ, \preceq) is *T_2 -preordered* if the preorder \preceq is closed in $(X, \tau) \times (X, \tau)$.

A preordered topological space (X, τ, \preceq) is *strongly T_2 -preordered*, or for notational convenience, *T_{S2} -preordered*, if there is an increasing *open* neighborhood of x disjoint from some decreasing *open* neighborhood of y whenever $x \not\preceq y$.

If \preceq is a partial order, then (X, τ, \preceq) is a partially ordered topological space, or simply an ordered topological space. If the preorder of a T_i -preordered topological space (X, τ, \preceq) is a partial order, we will say (X, τ, \preceq) is *T_i -ordered*. We will typically denote preorders by \preceq and partial orders by \leq . To avoid confusion when indicating inclusions, we may represent a preorder \sqsubseteq by its graph $G(\sqsubseteq)$.

2. EXISTENCE OF T_i -ORDERED REFLECTIONS

A special quotient. The definition of a T_0 -preordered topological space (X, τ, \preceq) involved the equivalence relation \sim on X defined by $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$. For any preordered topological space (X, τ, \preceq) , we obtain a partially ordered topological space by giving X/\sim the quotient topology τ/\sim and the order \leq defined by $[a] \leq [b]$ if and only if $a \preceq b$. The following properties of this quotient construction are easily verified.

Proposition 2.1. *Suppose \preceq is a preorder on a set X , \sim is the equivalence relation $G(\preceq) \cap G(\preceq^{-1})$ on X , and \leq is the partial order on X/\sim defined by $[a] \leq [b]$ if and only if $a \preceq b$.*

- (a) *Any \preceq -increasing or \preceq -decreasing set is \sim -saturated.*
- (b) *The quotient map $q : X \rightarrow X/\sim$ carries increasing (decreasing) sets to increasing (decreasing) sets. Specifically, $q(i_{\preceq}(A)) = i_{\leq}(q(A))$ for any subset $A \subseteq X$, and dually.*
- (c) *If \preceq^* is a preorder on X with $G(\preceq) \subseteq G(\preceq^*)$ and \sim_* is defined from \preceq^* as \sim is defined from \preceq , then the \sim_* -equivalence classes are \sim -saturated.*

Preorders induced by functions. Any continuous increasing function $f : (X, \tau, \leq) \rightarrow (Y, \gamma, \sqsubseteq)$ between two partially ordered topological spaces induces a preorder \preceq_f on X defined by $a \preceq_f b$ if and only if $f(a) \sqsubseteq f(b)$. Furthermore, $G(\leq) \subseteq G(\preceq_f)$.

Suppose (Y, γ, \sqsubseteq) is T_i -ordered for some $i \in \{0, 1, 2, S2\}$. Now $G(\preceq_f) = (f^{-1} \times f^{-1})(\sqsubseteq)$. Noting that f^{-1} and $f^{-1} \times f^{-1}$ carry open (respectively, closed, \sqsubseteq -increasing, \sqsubseteq -decreasing, disjoint) sets to open (respectively, closed, \preceq_f -increasing, \preceq_f -decreasing, disjoint) sets and that the T_i -(pre)ordered properties ($i \in \{0, 1, 2, S2\}$) are defined in terms of open/closed/increasing/decreasing/disjoint sets, it follows that (X, τ, \preceq_f) is T_i -preordered.

Also, if \sim_f is the equivalence relation $G(\preceq_f) \cap G(\preceq_f^{-1})$, observe that the \sim_f -equivalence class of $a \in X$ is $[a] = f^{-1}(f(a))$, a fiber of the map f . Thus, \preceq_f is a partial order if and only if f is injective.

We now apply the special quotient construction described above to the preorder \preceq_f induced by a function f . Suppose $f : (X, \tau, \leq) \rightarrow (Y, \gamma, \sqsubseteq)$ is a continuous increasing function between partially ordered spaces (X, τ, \leq) and (Y, γ, \sqsubseteq) , \preceq_f is the preorder on X defined by $a \preceq_f b$ if and only if $f(a) \sqsubseteq f(b)$, \sim_f is the equivalence relation on X defined by $a \sim_f b$ if and only if $a \preceq_f b$ and $b \preceq_f a$, and \leq_f is the partial order on X/\sim_f defined by $[a] \leq_f [b]$ if and only if $a \preceq_f b$. We now show that if (Y, γ, \sqsubseteq) is T_i -ordered for some $i \in \{0, 1, 2, S2\}$, then the partially ordered space $(X/\sim_f, \tau/\sim_f, \leq_f)$ is also T_i -ordered. Define $h : (X/\sim_f, \tau/\sim_f, \leq_f) \rightarrow (Y, \gamma, \sqsubseteq)$ by $h([a]) = h(f^{-1}f(a)) = f(a)$, that is, $h([a]) = fq^{-1}[a]$ where $q : X \rightarrow X/\sim_f$ is the natural quotient map. Now h is continuous, for if U is open in Y , then $h^{-1}(U) = qf^{-1}(U)$ which is open since f is continuous and the quotient map carries saturated open sets to open sets. The definitions of \leq_f, \preceq_f, h , and \preceq_h respectively imply the following implications:

$$[a] \leq_f [b] \iff a \preceq_f b \iff f(a) \sqsubseteq f(b) \iff h([a]) \sqsubseteq h([b]) \iff [a] \preceq_h [b].$$

Thus, h is increasing and $\preceq_h = \leq_f$. By the remarks of the second paragraph after Proposition 2.1, it follows that $(X/\sim_f, \tau/\sim_f, \leq_f)$ is T_i -ordered.

Proposition 2.2. *Suppose (X, τ, \preceq) is a preordered topological space, \sim is the equivalence relation on X defined by $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$, and \leq is the partial order on X/\sim defined by $[a] \leq [b]$ if and only if $a \preceq b$. Then for $i \in \{0, 1, S2\}$, (X, τ, \preceq) is T_i -preordered if and only if $(X/\sim, \tau/\sim, \leq)$ is T_i -ordered.*

Proof. Suppose $i \in \{0, 1, S2\}$ and $(X/\sim, \tau/\sim, \leq)$ is T_i -ordered. The two paragraphs following Proposition 2.1 remain valid for a function f from a preordered space to a partially ordered space, and if f is taken to be the quotient map q from (X, τ, \preceq) to $(X/\sim, \tau/\sim, \leq)$, then $\preceq_f = \preceq$. Thus, (X, τ, \preceq) is T_i -preordered.

For the converse, first suppose that (X, τ, \preceq) is T_0 -preordered. If $[a] \neq [b]$ in X/\sim , then there is a \preceq -monotone open neighborhood N of one of the points a or b which does not contain the other. Now $q(N)$ is a \leq -monotone open neighborhood of one of the points $[a]$ or $[b]$ in X/\sim which does not contain the other, so X/\sim is T_0 -ordered. Now suppose (X, τ, \preceq) is T_1 -preordered. For $[x] \in X/\sim$, we have $i_{\preceq}([x]) = i_{\preceq}(q(x)) = q(i_{\preceq}(x))$ by Proposition 2.1 (b). Since $i_{\preceq}(x)$ is closed and saturated, $q(i_{\preceq}(x))$ will be closed in X/\sim . With the dual argument, this shows that X/\sim is T_1 -ordered. Finally, suppose (X, τ, \preceq) is T_{S2} -preordered. If $[a] \not\leq [b]$ in X/\sim , then $a \not\preceq b$ in X , so there exist a \preceq -increasing τ -open neighborhood N_a of a and a \preceq -decreasing τ -open neighborhood N_b of b in X which are disjoint. By Proposition 2.1 (a) and (b), it follows that $q(N_a)$ and $q(N_b)$ are the required \leq -monotone τ/\sim -open neighborhoods separating $[a]$ and $[b]$ in X/\sim . \square

The result of Proposition 2.2 does not hold for $i = 2$. While the reasoning of the first paragraph of the proof shows that if $(X/\sim, \tau/\sim, \leq)$ is T_2 -ordered then (X, τ, \preceq) is T_2 -preordered, the example below shows that the converse fails.

Example 2.3. If (X, τ, \preceq) is a T_2 -preordered space, \sim is the equivalence relation on X defined by $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$, and \leq is the partial order on X/\sim defined by $[a] \leq [b]$ if and only if $a \preceq b$, then $(X/\sim, \tau/\sim, \leq)$ need not be T_2 -ordered.

Let γ be the Euclidean topology on \mathbb{R} . Define a topology τ on $X = \mathbb{R}$ as follows: Each point of $\mathbb{Q} \setminus \{0\}$ is isolated. For $x \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{0\}$, a τ -neighborhood of x is $\{x\} \cup (U \cap \mathbb{Q})$ where U is a γ -neighborhood of x .

Define a preorder \preceq on X by $a \preceq b$ if and only if $a = b$ or $\{a, b\} \subseteq \mathbb{R} \setminus \mathbb{Q}$. (In fact, \preceq is already an equivalence relation, so $\sim = \preceq$.) The graph of \preceq is $(\mathbb{R} \setminus \mathbb{Q})^2 \cup \Delta_X$. Because each γ -neighborhood of $x \in X$ contains a τ -neighborhood of x , it follows that (X, τ) is T_2 , and thus Δ_X is closed in $X \times X$. Observe that \mathbb{Q} is a neighborhood of each of its points, so $\mathbb{R} \setminus \mathbb{Q}$ is closed in X . It follows that $G(\preceq) = (\mathbb{R} \setminus \mathbb{Q})^2 \cup \Delta_X$ is closed in $X \times X$, so \preceq is T_2 -preordered.

To see that $(X/\sim, \tau/\sim, \leq)$ is not T_2 -ordered, suppose to the contrary that it is. Now $\pi \not\preceq 0$ in X , so $[\pi] \not\leq [0]$ in X/\sim , and thus there exist disjoint sets M, N , where M is a \leq -increasing τ/\sim -neighborhood of $[\pi]$ and N is a \leq -decreasing τ/\sim -neighborhood of $[0]$. If $q : X \rightarrow X/\sim$ is the quotient map,

then $q^{-1}(N)$ contains a τ -neighborhood $\{0\} \cup (B \cap \mathbb{Q})$ of 0, where B is a γ -open neighborhood of 0. For any $b \in B \setminus \mathbb{Q}$, we have $b \in \mathbb{R} \setminus \mathbb{Q} = [\pi] \subseteq q^{-1}(M)$. Now $q^{-1}(M)$ is a \preceq -increasing τ -neighborhood of b . But a \preceq -increasing τ -neighborhood of $b \in \mathbb{R} \setminus \mathbb{Q}$ has form $(\mathbb{R} \setminus \mathbb{Q}) \cup U$ where U is a γ -neighborhood of b . Now $U \cap B \neq \emptyset \Rightarrow U \cap (B \cap \mathbb{Q}) \neq \emptyset \Rightarrow q^{-1}(M) \cap q^{-1}(N) \neq \emptyset$, contrary to $M \cap N = \emptyset$. Thus, $(X/\sim, \tau/\sim, \leq)$ is not T_2 -ordered.

The existence construction. Suppose (X, τ, \leq) is a given partially ordered topological space. For $i \in \{0, 1, 2, S2\}$, let

$$\mathcal{P}_i = \{ \preceq^* \quad : \quad \preceq^* \text{ is a preorder on } X, G(\leq) \subseteq G(\preceq^*), \\ \text{and } (X/\sim_*, \tau/\sim_*, \leq^*) \text{ is } T_i\text{-ordered} \}.$$

Note that $\mathcal{P}_i \neq \emptyset$ since $X \times X$ belongs to it. If $i \in \{0, 1, S2\}$, by Proposition 2.2, we have

$$\mathcal{P}_i = \{ \preceq^* : G(\leq) \subseteq G(\preceq^*) \text{ and } (X, \tau, \preceq^*) \text{ is } T_i\text{-preordered} \}.$$

Let $G(\preceq^i) = \bigcap \mathcal{P}_i$. As an intersection of preorders containing $G(\leq)$, \preceq^i is also a preorder containing $G(\leq)$.

Proposition 2.4. *For $i \in \{0, 1, 2, S2\}$, $(X/\sim_i, \tau/\sim_i, \leq^i)$ is T_i -ordered.*

Proof. $i = 0$: Suppose $[x] \neq [y]$ in $(X/\sim_0, \tau/\sim_0, \leq^0)$. Then $x \not\sim_0 y$ in X , so for some $\preceq^* \in \mathcal{P}_0$, there exists a \preceq^* -monotone open neighborhood N of a which does not contain b , where $\{a, b\} = \{x, y\}$. Applying Proposition 2.1 (b), we obtain a \preceq^* -monotone open neighborhood N' of $[a]_{\sim_*}$ which does not contain $[b]_{\sim_*}$. Since $G(\leq^0) \subseteq G(\preceq^*)$, Proposition 2.1 (c) implies the existence of a natural increasing quotient map $q : X/\sim_0 \rightarrow X/\sim_*$, and $q^{-1}(N')$ is a \leq^0 -monotone open neighborhood of $[a]_{\sim_0}$ which does not contain $[b]_{\sim_0}$. Thus, $(X/\sim_0, \tau/\sim_0, \leq^0)$ is T_0 -ordered.

$i = 1$: Because the increasing hull of x in $\preceq^1 = \bigcap \mathcal{P}_1$ is the intersection of the increasing hulls of x in each \preceq^* in \mathcal{P}_1 , and each of these latter increasing hulls is closed, it follows that $i_{\preceq^1}(x)$ is closed for any $x \in X$. With the dual argument, we have (X, τ, \preceq^1) is T_1 -preordered. By Proposition 2.2, $(X/\sim_1, \tau/\sim_1, \leq^1)$ is T_1 -ordered.

$i = 2$: Suppose $[a] \not\leq [b]$ in $(X/\sim_2, \tau/\sim_2, \leq^2)$. Then there exists $\preceq^* \in \mathcal{P}_2$ such that $[a]_{\sim_*} \not\leq [b]_{\sim_*}$ in the T_2 -ordered space $(X/\sim_*, \tau/\sim_*, \leq^*)$. Let N_a and N_b be disjoint τ/\sim_* neighborhoods of $[a]_{\sim_*}$ and $[b]_{\sim_*}$ respectively, with N_a being \leq^* -increasing and N_b being \leq^* -decreasing. Since $G(\leq^2) \subseteq G(\preceq^*)$, the natural quotient map q from X/\sim_2 to X/\sim_* yields $q^{-1}(N_a)$ and $q^{-1}(N_b)$ as oppositely directed monotone neighborhoods separating $[a]$ and $[b]$ in X/\sim_2 . Thus, $(X/\sim_2, \tau/\sim_2, \leq^2)$ is T_2 -ordered.

Taking N_a and N_b above to be open sets proves the case $i = S2$. □

We are now ready for the main result of this section.

Theorem 2.5. *Suppose (X, τ, \leq) is a partially ordered topological space and $i \in \{0, 1, 2, S2\}$. Then the T_i -ordered reflection of (X, τ, \leq) is $(X/\sim_i, \tau/\sim_i, \leq^i)$.*

Proof. Suppose $i \in \{0, 1, 2, S2\}$ is given, (Y, γ, \sqsubseteq) is T_i -ordered, and $f : (X, \tau, \leq) \rightarrow (Y, \gamma, \sqsubseteq)$ is continuous and increasing. From the paragraph preceding Proposition 2.2, it follows that $(X/\sim_f, \tau/\sim_f, \leq_f)$ is a T_i -ordered space with $G(\leq) \subseteq G(\leq_f)$. From the definition of \leq^i , we have $G(\leq^i) \subseteq G(\leq_f)$. From Proposition 2.1 (c), there is a natural continuous increasing quotient map $q : (X/\sim_i, \tau/\sim_i, \leq^i) \rightarrow (X/\sim_f, \tau/\sim_f, \leq_f)$ which carries $[a]_{\sim_i}$ to $[a]_{\sim_f}$. We have shown above that there is a continuous increasing function $h : (X/\sim_f, \tau/\sim_f, \leq_f) \rightarrow (Y, \gamma, \sqsubseteq)$. Now $hq : X/\sim_i \rightarrow Y$ is continuous and increasing. Thus, each continuous increasing function $f : X \rightarrow Y$ can be lifted through X/\sim_i , and from the construction, this lifting is unique. Thus, X/\sim_i is the T_i -ordered reflection of X . \square

It is easy to verify that this construction gives the property Q reflection of (X, τ, \leq) as a quotient for any property Q for which

- (a) $(X/\sim_Q, \tau/\sim_Q, \leq^Q)$ has property Q where $a \sim_Q b$ if and only if $a \leq_Q b$ and $b \leq_Q a$; $G(\leq_Q) = \bigcap \{G(\leq^*) : G(\leq) \subseteq G(\leq^*) \text{ and } (X/\sim_*, \tau/\sim_*, \leq^*) \text{ is an ordered space with property Q}\}$; and $[a] \leq^Q [b]$ in X/\sim_Q if and only if $a \leq_Q b$ in X .
- (b) If (Y, γ, \sqsubseteq) has property Q and $f : (X, \tau, \leq) \rightarrow (Y, \gamma, \sqsubseteq)$ is continuous and increasing, then $(X/\sim_f, \tau/\sim_f, \leq_f)$ has property Q.

Furthermore, the methods of this section can be used to find the T_i reflection ($i = 0, 1, 2$) of a topological space (X, τ) by considering (X, τ) as a discretely ordered topological space $(X, \tau, =)$ and taking all preorders to be equivalence relations, so that the resulting quotients are discretely ordered.

3. THE T_0 -ORDERED REFLECTION

The construction of the T_i -ordered reflections in the previous section was an extrinsic construction—working from outside the space (X, τ, \leq) —which produced the T_i -ordered reflection as a quotient based on the intersection of all suitable preorders on X for which the indicated quotient construction would yield a T_i -ordered space. In this section, we present an intrinsic construction of the T_0 -ordered reflection and discuss some other properties of the T_0 -ordered reflection. Intrinsic constructions of the other T_i -ordered reflections ($i > 0$) studied in the previous section appear to be much more complicated.

In a T_0 -ordered space, $D(x) = D(y)$ and $I(x) = I(y)$ would imply $x = y$. If our space is not T_0 -ordered, then there may be distinct elements x and y with $D(x) = D(y)$ and $I(x) = I(y)$. Our strategy will be to say two such points are equivalent and mod out by this equivalence relation.

Suppose (X, τ, \leq) is an ordered topological space. For $x, y \in X$, define $x \approx y$ if and only if $D(x) = D(y)$ and $I(x) = I(y)$. Order the set X/\approx of \approx -equivalence classes by the *finite step order*:

$$[z_0] \leq^0 [z_n] \iff \exists [z_1], [z_2], \dots, [z_{n-1}] \text{ and } \exists z'_i, z_i^* \in [z_i] (i = 0, 1, \dots, n) \\ \text{with } z'_i \leq z_{i+1}^* \forall i = 0, 1, \dots, n-1.$$

First note that this is indeed antisymmetric and therefore is a partial order: Suppose $[z_0] \leq^0 [z_n]$ and $[z_n] \leq^0 [z_0]$. Then there exist $[z_i]$ ($i = 1, \dots, n, \dots, m$) with $[z_0] = [z_m]$ and there exist $z'_i, z_i^* \in [z_i]$ such that $z'_i \leq z_{i+1}^*$ for all $i = 0, \dots, m-1$. To show $[z_0] = [z_n]$, suppose not. Then either $D(z_0) \neq D(z_n)$ or $I(z_0) \neq I(z_n)$. Now

$$z'_i \leq z_{i+1}^* \Rightarrow z_{i+1}^* \in I(z'_i) \Rightarrow I(z_{i+1}^*) \subseteq I(z'_i) \Rightarrow I(z_{i+1}) \subseteq I(z_i).$$

Applying this for $i = 0, \dots, m-1$ gives $I(z_0) \supseteq I(z_1) \supseteq \dots \supseteq I(z_m) = I(z_0)$. Thus, $I(z_i) = I(z_0) \forall i \in \{1, \dots, m\}$. Dually, $D(z_i) = D(z_0) \forall i \in \{1, \dots, m\}$. It follows that $z_i \approx z_0 \forall i \in \{1, \dots, m\}$, so $[z_0] = [z_n]$, and \leq is antisymmetric.

In fact, the argument above shows that

$$[x] \leq^0 [y] \Rightarrow I(y) \subseteq I(x) \text{ and } D(x) \subseteq D(y).$$

At this point, one can verify that the equivalence relation \approx agrees with \sim_0 introduced in the previous section and that the finite step order described above agrees with the order \leq^0 defined in the previous section by $[a] \leq^0 [b]$ if and only if $a \leq^0 b$ where $G(\leq^0) = \bigcap \mathcal{P}_0$, and thus the T_0 -ordered reflection of (X, τ, \leq) is $(X/\approx, \tau/\approx, \leq^0)$. However, we will continue our intrinsic approach and prove this directly.

It is easy to show that any closed or open monotone set in X is \approx -saturated and that the quotient map $f : X \rightarrow X/\approx$ carries closed increasing sets to closed increasing sets and open increasing sets to open increasing sets. The dual statement (obtained by replacing “increasing” by “decreasing”) also holds. It follows that f is an *ordered quotient map* as defined in Definition 6 of [6]. It is easily verified that if $\mathcal{D} = \{f^{-1}(y) : y \in X/\approx\}$ is the decomposition of X associated with the quotient map $f : X \rightarrow X/\approx$, then for each $[x] \in \mathcal{D}$ and each increasing (decreasing) open set U containing $[x]$, there exists a saturated increasing (decreasing) open set containing $[x]$ which is contained in U . We have A is closed and increasing in X if and only if $f(A)$ is closed and increasing in X/\approx , and B is closed and increasing in X/\approx if and only if $f^{-1}(B)$ is closed and increasing in X . Furthermore, because $I(x) = \bigcap \mathcal{C}$ where \mathcal{C} is the collection of closed increasing sets containing x and $f(\bigcap \mathcal{C}) = \bigcap f(\mathcal{C})$ for any collection \mathcal{C} of \approx -saturated sets, it follows that $f(I(x)) = I_{X/\approx}([x])$. Dually, $f(D(x)) = D_{X/\approx}([x])$.

Theorem 3.1. *Suppose (X, τ, \leq) is a partially ordered topological space, and $a \approx b$ if and only if $D(a) = D(b)$ and $I(a) = I(b)$. Then X/\approx with the quotient topology and the finite-step order is the T_0 -ordered reflection of X .*

Proof. First we will show that X/\approx is T_0 -ordered. Suppose $I_{X/\approx}([x]) = I_{X/\approx}([y])$ and $D_{X/\approx}([x]) = D_{X/\approx}([y])$. If $f : X \rightarrow X/\approx$ is the natural ordered quotient map, then we have $f(I(x)) = f(I(y))$ and $f(D(x)) = f(D(y))$. Applying f^{-1} to the equalities above and recalling that $I(x)$ and $D(x)$ are saturated, we have $I(x) = I(y)$ and $D(x) = D(y)$, which implies $[x] = [y]$. Thus, X/\approx is T_0 -ordered.

Now suppose Y is any T_0 -ordered space and $g : X \rightarrow Y$ is continuous and increasing. We will show that $\{g^{-1}(y) : y \in Y\}$ is saturated with respect

to $\mathcal{D} = \{f^{-1}([x]) : [x] \in X/\approx\}$. Suppose to the contrary that there exists $y \in Y$ such that $g^{-1}(y)$ is not \mathcal{D} -saturated. Then there exist $b \in g^{-1}(y)$ and $a \in X \setminus g^{-1}(y)$ such that $[a] = [b]$ (that is, $f(a) = f(b)$).

Now $g^{-1}(I_Y(g(b)))$ is a closed increasing set in X which contains $g^{-1}(g(b))$ and therefore contains b . But

$$\begin{aligned} [a] = [b] &\Rightarrow I(a) = I(b) \\ &\Rightarrow a \text{ is an element of every closed increasing set containing } b \\ &\Rightarrow a \in g^{-1}(I_Y(g(b))) \\ &\Rightarrow g(a) \in I_Y(g(b)) \\ &\Rightarrow I_Y(g(a)) \subseteq I_Y(g(b)). \end{aligned}$$

Repeating the argument of the last paragraph with a and b interchanged shows the reverse inclusion, so $I_Y(g(a)) = I_Y(g(b))$. The dual argument shows that $D_Y(g(a)) = D_Y(g(b))$. Since Y is T_0 -ordered, this implies $g(a) = g(b)$, contrary to $a \in X \setminus g^{-1}(y)$ and $b \in g^{-1}(y)$.

Now since $\{g^{-1}(y) : y \in Y\}$ is \mathcal{D} -saturated, there is a natural quotient map h from $X/\mathcal{D} = X/\approx$ to Y , and $hf = g$. From the definition of the finite step order on X/\approx , it is clear that h is increasing, and h is clearly unique from the construction. Thus, X/\approx is the T_0 -ordered reflection of X . \square

The theorem below characterizes those spaces whose T_0 -ordered reflections are T_1 -ordered. Similar results for the non-ordered setting can be found in [1], where a $T_{(i,j)}$ -space is defined to be one whose T_i -reflection satisfies the T_j separation axiom ($0 \leq i < j \leq 2$). Comparing Theorem 3.5(iv) of [1] with Theorem 2(b) of [2], we note that $T_{(0,1)}$ -spaces have been studied by Davis and others subsequently under the name of R_0 -spaces.

Theorem 3.2. *The following are equivalent.*

- (a) *The T_0 -ordered reflection X/\approx of X is T_1 -ordered.*
- (b) *$[x] \not\leq^0 [y]$ in X/\approx implies there exists an open increasing neighborhood of x not containing y and there exists an open decreasing neighborhood of y not containing x .*
- (c) *$i([x]) = \bigcap\{N : N \text{ is an open increasing neighborhood of } x\}$ for any $x \in X$, and*
- $d([x]) = \bigcap\{N : N \text{ is an open decreasing neighborhood of } x\}$ for any $x \in X$.*

Proof. (a) \Rightarrow (c): Because closed or open increasing sets are saturated, we have $i([x]) \subseteq \bigcap\{N : N \text{ is an open increasing neighborhood of } x\}$. Suppose $M = \bigcap\{N : N \text{ is an open increasing neighborhood of } x\} \not\subseteq i([x])$. Then there exists $y \in M \setminus i([x])$, and since M is saturated, $[y] \not\subseteq i([x])$. In particular, $[x] \not\leq^0 [y]$ in the T_1 -ordered space X/\approx , so there exists an increasing open neighborhood J of $[x]$ in X/\approx disjoint from $[y]$. Now if $f : X \rightarrow X/\approx$ is the quotient map, $f^{-1}(J)$ is an open increasing neighborhood of x disjoint from y . This contradicts $y \in M$. This proves that $i([x]) = \bigcap\{N : N \text{ is an open increasing neighborhood of } x\}$ for any $x \in X$. The other statement is proved dually.

(c) \Rightarrow (a): Suppose (c). If X/\approx is not T_1 -ordered, then there exist $[x] \not\leq^0 [y]$ such that either (i) every increasing open neighborhood of $[x]$ in X/\approx contains $[y]$, or (ii) every decreasing open neighborhood of $[y]$ in X/\approx contains $[x]$. If (i) holds, then $[y] \in \bigcap \{N : N \text{ is an open increasing neighborhood of } x\} = i([x])$, contrary to $[x] \not\leq^0 [y]$. If (ii) holds, then $[x] \in \bigcap \{N : N \text{ is an open decreasing neighborhood of } y\} = d([y])$, contrary to $[x] \not\leq^0 [y]$.

(a) \Rightarrow (b): Suppose (a). Now $[x] \leq^0 [y]$ in X/\approx implies there exists an open increasing (respectively, decreasing) neighborhood of $[x]$ (respectively, $[y]$) not containing $[y]$ (respectively, $[x]$). Taking f^{-1} of these neighborhoods gives the desired neighborhoods in X .

(b) \Rightarrow (a): If $[x] \not\leq^0 [y]$ in X/\approx , then by (b) there exists an open increasing neighborhood N of x not containing y and there exists an open decreasing neighborhood M of y not containing x . Now M and N are saturated, and since f is an ordered quotient map, $f(M)$ and $f(N)$ are monotone open neighborhoods of $[y]$ and $[x]$, respectively, which show that X/\approx is T_1 -ordered. \square

A set A which satisfies $A = I(A) \cap D(A)$ is called a *c-set*. In [4], maximal filters of *c*-sets are used to construct the Wallman ordered compactification of an ordered space with convex topology. The Wallman ordered compactification w_0X is a universal compact T_1 extension. In [5], conditions involving *c*-sets are given to insure w_0X is T_1 -ordered. Thus, one might expect *c*-sets to play a role in the T_1 -ordered or even T_0 -ordered reflection. Let $C(A) = I(A) \cap D(A)$, that is, let $C(A)$ be the smallest *c*-set containing A .

Proposition 3.3. *Suppose (X, τ, \leq) is an ordered topological space and let \approx be the equivalence relation on X defined by $x \approx y$ if and only if $D(x) = D(y)$ and $I(x) = I(y)$. Then $x \approx y$ if and only if $C(x) = C(y)$.*

Proof. Suppose $C(x) = C(y)$. Then $x \in C(y) \subseteq I(y)$, so $I(x) \subseteq I(y)$. Interchanging x and y shows that $I(y) \subseteq I(x)$, so $I(x) = I(y)$. Dually, $D(x) = D(y)$, so $x \approx y$. The converse is immediate. \square

Thus, the equivalence classes of the T_0 -ordered reflection are determined by the closure operator $C(\cdot)$. If X has a convex topology, this closure operator is especially nice.

Theorem 3.4. *If the ordered topological space (X, τ, \leq) has a convex topology, then the topological space (X', τ') underlying its T_0 -ordered reflection (X', τ', \leq') is simply the T_0 reflection of (X, τ) .*

Proof. Suppose X has a convex topology. We will show that $C(x) = I(x) \cap D(x) = cl\{x\}$. Clearly $y \in cl\{x\} \Rightarrow y \in I(x) \cap D(x)$. For the converse, suppose $y \notin cl\{x\}$. Then there exist an increasing open neighborhood N_y of y and a decreasing open neighborhood M_y of y with $x \notin N_y \cap M_y$. Thus, either $x \notin N_y$ or $x \notin M_y$, and taking complements shows that $y \notin D(x)$ or $y \notin I(x)$, that is, $y \notin I(x) \cap D(x)$, as needed.

By Proposition 3.3, $x \approx y$ if and only if $cl\{x\} = cl\{y\}$. It is well-known that the T_0 reflection of (X, τ) is given by the quotient topology on the quotient set X/\simeq where $x \simeq y$ if and only if $cl\{x\} = cl\{y\}$. \square

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