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COMPLEMENTS OF TOPOLOGIES WITH SHORT SPECIALIZATION QUASIORDERS

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ABSTRACT. By identifying a topology τ on a finite set X with its specialization quasiorder $\leq$, we investigate the complements of τ in the lattice of topologies on X in cases where the heights of the specialization posets are small.

Dedicated to Ralph Kopperman, a pioneer in asymmetric topology.

1. Introduction

If τ is a topology on a finite set X, associating the specialization quasiorder defined by $x \leq y$ if and only if $x \in cl\{y\}$ gives a one-to-one correspondence between the topologies and quasiorders on X. We interchangeably denote a topology τ by its specialization quasiorder $\leq$. Thus, the smallest neighborhood $N(x)$ of x corresponds to $\uparrow x = \{y \in X : x \leq y\}$, and we will use $N(x)$ and $\uparrow x$ interchangeably. A quasiorder $\leq$ on X gives an equivalence relation $\approx$ defined by $x \approx y$ if and only if $x \leq y$ and $y \leq x$, and gives a partial order on the $\approx$-equivalence classes defined $[x] \leq' [y]$ if and only if $x \leq y$. To draw the Hasse diagram for a quasiorder, we draw the Hasse diagram for the related partial order $\leq'$ and represent each point $[x] \in X/\approx$ by the set of points of X which make up $[x]$. In this context, we will call the set of points of $[x]$ a *cloud*. A quasiorder $\leq$ is a partial order if and only if each cloud is a singleton, or equivalently, if τ is T_0. A quasiorder $\leq$ is a total order if and only if τ is an irreducible T_0, T_5 topological space (see [6]) or equivalently, if τ is T_0 and is a nested collection of open sets. The height of a quasiordered set $(X, \leq)$ is the

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maximum number of distinct points in a chain in $(X, \leq)$. In the lattice of topologies on X, τ' is a complement of τ if $\tau \vee \tau' = \tau_D$ and $\tau \wedge \tau' = \tau_I$, where τ_D is the discrete topology on X and τ_I is the indiscrete topology. For finite sets, $\tau \vee \tau' = \tau_D$ if and only if $\uparrow x \cap \uparrow' x = \{x\}$ (that is, $N(x) \cap N'(x) = \{x\}$) for all $x \in X$. If $\leq$ is a connected partial order on X (that is, the Hasse diagram for $\leq$ is a connected graph), then $\geq$ is a complement to $\leq$. In 1958, Hartmanis proved that every topology on a finite set has a complement, and in 1966 Anne Steiner proved that every Alexandroff topology has an Alexandroff complement. See [10, 15, 8]. Further results on complementation in lattices of topologies are given in [1, 4, 12, 16].

2. Totally Ordered Results

Theorem 2.1. *If τ is a topology on X whose quasiorder $\leq$ is a partial order of height 2 or 3, then τ admits a complement τ' whose quasiorder $\leq'$ is a total order.*

Proof. Let $K = \{x \in X : x \parallel y \quad \forall y \neq x\}$, $T = \{x \in X : x \text{ is maximal in } X - K\}$, $M = \{m \in X : b < m < t \text{ for some } b, t \in X\}$, and $B = \{x : x \text{ is minimal in } X - K\}$. If the height of $\leq$ is 2, then $M = \emptyset$. Note that M is an antichain, for otherwise the height of X would be greater than 3. Also, B is clearly seen to be an antichain. Put total orders on the sets $T, M \cup K, B$, and then form the total order $\leq'$ on X by taking $T <' (M \cup K) <' B$. We will show that $\leq'$ is a complement of $\leq$.

For $x \in T \cup K$, $\uparrow x = \{x\}$, so $\uparrow x \cap \uparrow' x = \{x\}$. For $m \in M$, $\uparrow m \subseteq \{m\} \cup T$ and $\uparrow' m \subseteq M \cup K \cup B$, so $\uparrow m \cap \uparrow' m = \{m\}$. For $b \in B$, $\uparrow b \subseteq \{b\} \cup M \cup T$ and $\uparrow' b \subseteq B$, so $\uparrow b \cap \uparrow' b = \{b\}$. Thus, $\tau \vee \tau' = \tau_D$.

Suppose $U \in \tau \cap \tau'$ and $x \in U$. If $x \in T$, $M \cup K \cup B \subseteq \uparrow' x$ and $X = \uparrow (M \cup K \cup B) \subseteq \uparrow\uparrow' x \subseteq U$, so $U = X$. If $x \in M \cup K \cup B$, then $\uparrow' x$ contains some $b \in B$, $\uparrow b$ contains some $t \in T$, $\uparrow' t$ contains $M \cup K \cup B$, and $\uparrow (M \cup K \cup B) = X$. Thus, the only nonempty set open in τ and τ' is X. □

Suppose τ is a topology on X whose quasiorder $\leq$ is a partial order of height 2 or 3. With the notation of the proof above, if $t = |T|, m = |M \cup K|$, and $b = |B|$, the arbitrary choice of the total orders on $T, M \cup K$, and B show that τ admits at least $t!m!b!$ complements τ' with total quasiorder $\leq'$.

The next theorem shows that the condition in Theorem 2.1 that the quasiorder $\leq$ is a partial order is necessary.

Theorem 2.2. *If $(X, \leq)$ is a quasiorder which is not a partial order, then it has no totally ordered complement.*

Proof. Under the hypotheses, there exists $a \neq b$ with $a \leq b, b \leq a$. If $\leq'$ is a complement in which a and b are related, without loss of generality assume $a \leq' b$. Now $\{a, b\} \subseteq \uparrow \{a\} \cap \uparrow' \{a\}$, so $\uparrow a \cap \uparrow' a \neq \{a\}$. □

Theorem 2.3. *If $\tau \neq \tau_D$ is a topology on a finite set X, then τ has a T_0 complement τ' whose specialization quasiorder is a partial order of height $h \leq 3$.*

Proof. Let $T = \{x \in X : [x] \text{ is maximal in } X/\approx\}$ where $a \approx b$ if and only if $a \leq b$ and $b \leq a$, $[x]$ is the $\approx$-equivalence class of x, and the quotient space $X/\approx$ is the T_0-reflection of $(X, \leq)$. Let $B = \{x \in X : [x] \text{ is minimal in } X/\approx\}$, and $M = X - (T \cup B)$. Make T, M, and B antichains in $\leq'$, with $T <' M <' B$. Thus, $t <' m <' b$ for all $b \in B, m \in M, t \in T$ and $x \leq' x$ for all $x \in X$. Now $\leq'$ has height less than three and is a complement of $\leq$. □

Note that this result gives a converse to Theorem 2.1: If $\leq$ is a total order on X, then $\leq$ has a T_0 complement whose quasiorder is a partial order of height $h \leq 3$. We can improve on this.

Theorem 2.4. *If $\leq$ is a total order on X and $|X| > 1$, then $\leq$ has a T_0 complement $\leq'$ where $\leq'$ is a partial order of height 2.*

Proof. If the order on X is $1 < 2 < \cdots < n$, let $A = \{1, 2, \ldots, n-1\}$. Define $\leq'$ by making A an antichain and putting $n <' A$. Now $\leq'$ is a complement of $\leq$ with height 2. □

3. Submaximal, Door, and Whyburn Spaces

A topological space X is *submaximal* if every dense subset is open, is a *door space* if every set is either open or closed, and is *Whyburn* if for every non-closed $A \subseteq X$ and every $x \in clA - A$, there exists $B \subseteq A$ with $clB - B = \{x\}$. Submaximal spaces were investigated in [2, 3, 5]. Edwin Hewitt [11] used the term "MI space" for submaximal spaces with no isolated points; Theorem 3.3 below concerns such spaces. Whyburn spaces ([13]) are called *AP*-spaces in [14]. Further references on these spaces can be found in [7], where the following results are given.

Theorem 3.1 (Lazaar, Sabri, Tahri)**.** *Suppose τ is a topology on a finite set with specialization quasiorder $\leq$. Let h be the height of $\leq$.*

(1) *τ is submaximal if and only if $\leq$ is a partial order and $h \leq 2$.*
(2) *τ is a door space if and only if $h \leq 2$ and all chains of length 2 share a common point, which is necessarily a maximal or minimal point.*
(3) *τ is Whyburn if and only if $|\downarrow x| \leq 2$ for all $x \in X$.*

We note that the condition to be a door space implies that the quasiorder $\leq$ is a partial order. Furthermore, (X, τ) is a door space $\Rightarrow (X, \tau)$ is submaximal, and T_0 + Whyburn $\Rightarrow$ submaximal.

If $(X, \leq)$ is submaximal, we take $X = T \cup B \cup K$ where $T = \{t \in X : b < t \text{ for some } b \in X\}$, $B = \{b \in X : b < t \text{ for some } t \in X\}$, and $K = X - (T \cup B)$.

Theorem 3.2. *Suppose $(X, \leq)$ is submaximal and not discrete. If $\leq'$ is a complement of $\leq$, then*

(a) *every minimal $\leq'$-cloud contains some $t \in T$ and no points $b < t$, and*
(b) *every maximal $\leq'$-cloud contains an element of B.*

Proof. First note that any set $A \subseteq X$ with $T \subseteq A$ is $\leq$-increasing. If C is a minimal $\leq'$-cloud which is disjoint from T, then we have the contradiction that $C \notin \{\emptyset, X\}$ and $X - C$ is $\leq$-increasing and $\leq'$-increasing. Thus, every minimal $\leq'$-cloud C contains some $t \in T$. If C also contains $b < t$, then we have the contradiction $t \in \uparrow b \cap \uparrow' b = \{b\}$. This proves (a). Suppose there is a maximal $\leq'$-cloud $D \neq \emptyset$ containing no point of B. Then $D \in \tau \cap \tau'$. If $D = X$, for any $b < t$ we have the contradiction $t \in \uparrow b \cap \uparrow' b$. Thus, $X \neq D \in \tau \cap \tau'$, contrary to $\leq'$ being a complement of $\leq$. This proves (b). $\square$

A submaximal space $X = T \cup B \cup K$ has many complements. For one, make T and B antichains in $\leq'$ and put $t <' k_1 \leq' k_2 <' b$ for every $t \in T$, $k_1, k_2 \in K$, and $b \in B$. For another, take $t_1 \leq' t_2 <' k_1 \leq' k_2 <' b_1 \leq' b_2$ for all $t_i \in T$, $k_i \in K$, and $b_i \in B$. This quasiorder has three clouds, $T <' K <' B$. For a third complement, consider clouds T and $K \cup B$ with $T <' K \cup B$. Note that, except in the simplest cases, the complements described here involve clouds with more than one point (so, they are not T_0-complements), and have height $h > 2$ (so they are not submaximal complements).

Theorem 3.3. *Suppose $(X, \leq)$ is submaximal and not discrete. Then $\leq$ has a submaximal complement if and only if it has no isolated points.*

Proof. Suppose $\leq$ and $\leq'$ are complements and each is submaximal. All the clouds in $\leq$ are singletons, so Theorem 3.2 implies that every $\leq$-minimal point is an element of T and every $\leq$-maximal element is an element of B. Any isolated point would thus be in $T \cap B$, a contradiction. Thus, $\leq$ (and similarly $\leq'$) can have no isolated points.

Suppose $(X, \leq)$ has no isolated points, so $K = \emptyset$ and X is partitioned into $T \cup B$. Let $C_1, \ldots, C_n$ be the connected components of the graph of $\leq$ and for each C_j, pick $b_j \in B \cap C_j$. Let $\leq'$ be the partial order $\geq$, except that the points $b_1, b_2, \ldots, b_n$ are permuted to $b_2, \ldots, b_n, b_1$, as suggested in Figure 1. Formally,

$$x \leq' y \iff \begin{cases} y = b_j & \text{and } x = b_j \quad (j = 1, \ldots, n) \\ y = b_j & \text{and } x > b_{j-1} \ (j = 2, \ldots, n) \\ y = b_1 & \text{and } x > b_n \\ y \notin \{b_1, \ldots, b_n\} & \text{and } x \geq y. \end{cases}$$

Now every point $x \in C_j - \{b_j\}$ is connected by a $\leq \cup \geq$ path in $C_j - \{b_j\}$ to a point $t \geq b_j$, and such a path is also a $\leq \cup \leq'$ path. Now suppose $t_1 > b_1$, so $t_1 \in C_1 - \{b_1\}$. Now $t_1 <' b_2 < t_2$ for some $t_2 \in C_2 - \{b_2\}$. We have $t_2 <' b_3 < t_3$ for some $t_3 \in C_3 - \{b_3\}$. Continuing in this manner, we find a $\leq \cup \leq'$ path from t_1 to $b_2, \ldots, b_3, \ldots, b_1 < t_1$. With a shift of indices, we can similarly find a $\leq \cup \leq'$ path from any $t_j > b_j$ through all the points $b_{j+1}, \ldots, b_n, b_1, \ldots, b_j$ and back to t_j. Since every point $x \in C_j - \{b_j\}$ is connected to a point $t_j > b_j$ it is connected to every point of $\bigcup_{j=1}^{n} C_j - \{b_j\}$ by a path which contains all the points $b_1, b_2, \ldots, b_n$. Thus, X is $\leq \cup \leq'$ connected, and it follows that $\tau \cap \tau' = \{\emptyset, X\}$.

From the construction, every point of X is maximal in $\leq$ or in $\leq'$, so either $\uparrow x = \{x\}$ or $\uparrow' x = \{x\}$. Thus, $\leq'$ is a complement of $\leq$. Clearly $\leq'$ is a partial order with height 2, so $\leq'$ is submaximal. □

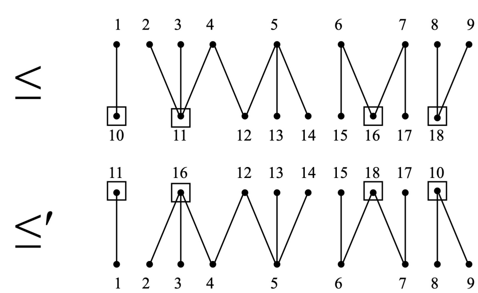

FIGURE 1. To obtain a complement of $\leq$, take $\leq'$ to be $\geq$ with the boxed elements from $\leq$ cyclically permuted in $\leq'$.

Since door spaces are submaximal, the necessary conditions from Theorem 3.2 for complements of submaximal spaces apply to door spaces. Indeed, those conditions are also sufficient for door spaces.

Theorem 3.4. *Suppose $(X, \leq)$ is a door space, the point t common to all chains of length 2 is maximal, and $B = \{b \in X : b < t\}$. Then $\leq'$ is a complement of $\leq$ if and only if*

(a) *there is a minimum $\leq'$-cloud and it contains t and no points of B, and*
(b) *every maximal $\leq'$-cloud contains an element of B.*

Proof. Suppose $\leq'$ is a complement of $\leq$. By Theorem 3.2, every minimal $\leq'$-cloud contains t, since $T = \{t\}$ for a door space. Thus, every minimal cloud is the cloud containing t, so there is a minimum $\leq'$-cloud. Thus, (a) and (b) follow from Theorem 3.2.

Now suppose (a) and (b) hold. Suppose $U \in \tau \cap \tau'$ and $x \in U$. By (b), $\uparrow' x$ contains a point $b \in B$, so $\uparrow\uparrow' x$ contains t, and $\uparrow'\uparrow\uparrow' x$ contains $X = \uparrow t$. Thus, $\tau \cap \tau' = \{\emptyset, X\}$. If $x \in X - B$, $\uparrow x = \{x\}$. If $x \in B$, by (a), $t \notin \uparrow' x$, and since $\uparrow x = \{x, t\}$, $\uparrow x \cap \uparrow' x = \{x\}$. Thus, $\leq'$ is a complement of $\leq$. □

Corollary 3.5. *Suppose $(X, \leq)$ is a door space with n elements, the point t common to all chains of length 2 is maximal, $B = \{b \in X : b < t\}$, and $|B| = k$. Then $\leq$ admits exactly $k(n-2)!$ totally ordered complements and at least $k \cdot T(n-2)$ complements, where $T(j)$ is the number of topologies on j points.*

Proof. The totally ordered complements of $\leq$ have form $t <' x_1 <' x_2 <' \cdots <' x_{n-2} <' b$ where b is one of the k elements of B and $x_1 <' \cdots <' x_n$ is one of the $(n-2)!$ total orders on $X - \{b, t\}$. If $b \in B$ and Q is any quasiorder on $X - \{t, b\}$, then $t <' Q <' b$ gives a complement of $\leq$. There are k choices for b and $T(n-2)$ choices for Q. □

If $(X, \leq)$ is a Whyburn space, let $C = \{x \in X : x < y < x$ for some $y \in X\}$, $T = \{t \in X - C : x < t$ for some $x \in X\}$, $B = \{b \in X - C : b < x$ for some $x \in X\}$, and $K = X - (T \cup B \cup C)$, as suggested in Figure 2.

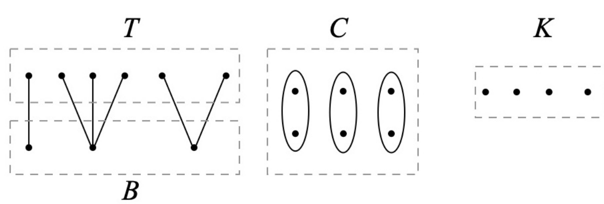

FIGURE 2. A generic Whyburn space.

Some Whyburn spaces have Whyburn complements.

Example 3.6. Suppose a finite Whyburn space $(X, \leq)$ has $K = \emptyset$ and $|T| = |B|$. (Thus, X has no isolated points and $|\uparrow x| \leq 2$ and $|\downarrow x| \leq 2$ for every $x \in X$.) As suggested in Figure 3, inverting the order $\leq$ and cyclically moving each element of $B \cup C$ one place to the right gives a complement $\leq'$. Replacing each cloud $\{2k, 2k+1\}$ in $\leq'$ by the segment $2k+1 \leq'' 2k$ gives a T_0 complement $\leq''$.

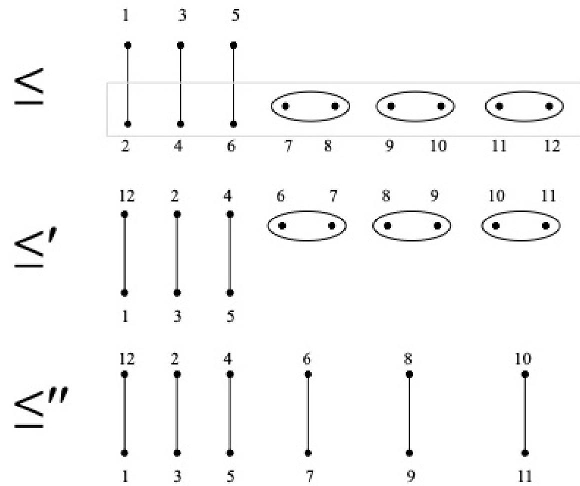

FIGURE 3. To obtain a complement of $\leq$, take $\leq'$ to be $\geq$ with the boxed elements from $\leq$ cyclically permuted in $\leq'$. $\leq''$ is a T_0 complement of $\leq$.

Example 3.7. A Whyburn space with isolated points may also have a Whyburn complement, as seen in Figure 4. Indeed, this shows that for any $n \in \mathbb{N}$, there exists a Whyburn space of cardinality n which has a Whyburn complement.

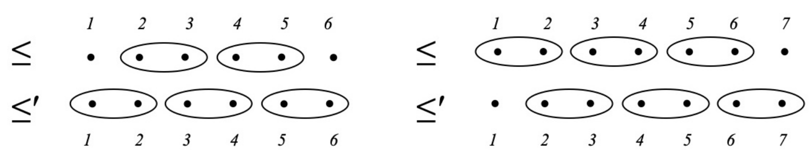

FIGURE 4. Whyburn spaces with isolated points and Whyburn complements.

Some Whyburn spaces do not have Whyburn complements.

Theorem 3.8. (a) *If $(X, \leq)$ is a Whyburn space with $C = \emptyset$ and $|T| \geq |B| > 1$, then $\leq$ has a Whyburn complement if and only if $|B| = |T|$ and $K = \emptyset$.*

(b) *If* $(X, \leq)$ *is a Whyburn space with* $T = B = \emptyset$ *and* $4 \leq |C| \leq |K|$, *then* $\leq$ *has no Whyburn complement.*

Proof. (a) If $\leq'$ is a Whyburn complement of the Whyburn space $(X, \leq)$, the $\leq'$-component of $x \in T \cup K$ is either a 2-element cloud $\{x, b\}$ where $b \in B$ or an antichain of elements of B above x. Thus, each $x \in T \cup K$ contains at least one element of B in its $\leq'$-component, and distinct elements of $T \cup K$ cannot use the same element of B. Thus, $|T| \geq |B| \geq |T| + |K| \geq |T|$ and it follows that $|B| = |T|$ and $|K| = 0$. Conversely, if $(X, \leq)$ is a Whyburn space with $|B| = |T| > 1$ and $K = C = \emptyset$, Example 3.6 shows that $\leq$ has a Whyburn complement.

(b) If $\leq'$ is a Whyburn complement, the $\leq'$-component of $k \in K$ is either a 2-element cloud $\{k, c\}$ where $c \in C$ or an antichain of elements of C above k. Thus, each $k \in K$ contains at least one element of C in its $\leq'$-component, so $|C| \geq |K|$. The hypothesis that $|C| \leq |K|$ implies $|C| = |K|$. Then the $\leq'$-component of each $k \in K$ is either a chain $k < c$ or a cloud $\{k, c\}$ for some $c \in C$. Pick a $\leq$-cloud $\{c, c^*\}$. The $\leq'$-components of c and c^* are either (a) chains $k <' c, k^* <' c^*$, (b) clouds $\{k, c\}, \{k^*, c^*\}$, or (c) one chain $k <' c$ and one cloud $\{k^*, c^*\}$. In case (a) $\{c, c^*\} \in \tau \cap \tau'$; in case (b) $\{c, c^*, k, k^*\} \in \tau \cap \tau'$; in case (c) $\{c, c^*, k^*\} \in \tau \cap \tau'$. Since X has at least 8 elements, none of these sets in $\tau \cap \tau'$ is X, so $\leq'$ is not a complement of $\leq$. □

A complete characterization of Whyburn spaces which have Whyburn complements remains open.

Theorem 3.9. *A Whyburn space without isolated points has a complement with height* $h \leq 2$.

Proof. If $(X, \leq)$ is a Whyburn space with $K = \emptyset$, relabel each cloud $\{c_i, c_i^*\} \subseteq C$ as $\{b_{c_i}, t_{c_i}\}$ and consider b_{c_i} and t_{c_i} to be elements of B and T respectively. Then, the proof of Theorem 3.3 provides a complement $\leq'$ of height $h \leq 2$. □

We note that if $|B| < |T|$, the complement provided by the proof of Theorem 3.9 is not a Whyburn complement.

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