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ON RESOLVABLE PRIMAL SPACES

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ABSTRACT A topological space is called resolvable if it is a union of two disjoint dense subsets, and is n-resolvable if it is a union of n mutually disjoint dense subsets. Clearly a resolvable space has no isolated points. If f is a selfmap on X, the sets $A \subseteq X$ with $f(A) \subseteq A$ are the closed sets of an Alexandroff topology called the primal topology $\mathcal{P}(f)$ associated with f. We investigate resolvability for primal spaces $(X, \mathcal{P}(f))$. Our main result is that an Alexandroff space is resolvable if and only if it has no isolated points. Moreover, n-resolvability and other related concepts are investigated for primal spaces.

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Introduction. A topological space is called an *Alexandroff space* if arbitrary nonempty intersections of open sets are open. P. Alexandroff [1] introduced these spaces in 1937, with the name of *Diskrete Räume*. In 1966, A. K. Steiner [20] called these spaces *principal spaces*. Alexandroff spaces play an important role in several areas including digital topology and computer science.

In 1998, Richmond [18] studied the Alexandroff topologies on a set X with particular attention to their lattice structure by considering them as partially ordered partitions. Uzcátegui and Vielma [23] gave interesting results about Alexandroff spaces from a different perspective by viewing them as closed sets of the Cantor cube 2^X.

An important property of Alexandroff spaces is that the closure of the union of a family of subsets in such a space is the union of the closures of these subsets. In particular, if X is an Alexandroff space and A a subset of X, then $\overline{A} = \bigcup\{\overline{\{x\}} : x \in A\}$.

For a set X and a selfmap $f : X \longrightarrow X$, we obtain an Alexandroff topology $\mathcal{P}(f)$ on X by taking the closed sets to be the subsets A which are f-invariant, (i.e, the subsets A with $f(A) \subseteq A$). This concept was introduced independently by Shirazi and Golestani [2] in 2011 and Echi [9] in 2012. Shirazi and Golestani called such spaces *functional Alexandroff spaces*, and Echi called them *primal spaces*. We will follow Echi's terminology.

In 1943, Hewitt [13] defined *resolvable spaces* to be those topological spaces X having two complementary subsets, each of which is dense in X. Equivalently, X is resolvable if and only if X contains two disjoint dense subsets. Hewitt presented several classes of topological spaces which are resolvable.

After Hewitt, in 1964, J. G. Ceder [6] considered the following question: how many pairwise disjoint dense subsets may a topological space X contain? Ceder generalized W. Sierpinski's 1949 result [19] that if X is a metric space with the property that each nonempty open set U has $|U| \geq m \geq \aleph_0$, then X contains m pairwise disjoint subsets each intersecting every open set in at least m points. Ceder also introduced the notions of *maximally resolvable spaces* and *n-resolvable spaces*. The related concepts of *exactly n-resolvable spaces* and *strongly exactly n-resolvable spaces*, discussed in Section 4 below, were introduced later (see [4, 5, 11]).

From [9] and [12], it is known that a topological space is primal if and only if its $\mathbf{T}_0$-reflection is a primal space. In Section 1, for a given primal space $(X, \mathcal{P}(f))$, we give an explicit construction (up to homeomorphism) of the $\mathbf{T}_0$-reflection $\mathbf{T}_0(X, \mathcal{P}(f))$ as a primal space $(Y, \mathcal{P}(g))$, where Y is a subset of X and $g : Y \longrightarrow Y$ is a selfmap defined in terms of f.

In Section 2, we investigate resolvability of Alexandroff spaces. Based on a result of A. H. Stone [21], we characterize the n-resolvable Alexandroff spaces. The proof of the needed result from [21] is lengthy and complicated. In the class of primal spaces, we characterize n-resolvable spaces and provide elementary, direct proofs. With $n = 2$, the definition of n-resolvable reduces to that of resolvable. In particular, we show that an Alexandroff space is resolvable if and only if it has no isolated points.

In Section 3, primal spaces whose $\mathbf{T}_0$-reflection are resolvable, that is, the *$\boldsymbol{T}_0$-resolvable primal spaces*, are characterized.

In the final section, exactly n-resolvable spaces and strongly exactly n-resolvable spaces are characterized in the class of primal spaces.

The set of integers is denoted by $\mathbb{Z}$. We use $\mathbb{N}$ to denote the set $\{0, 1, 2, 3, \ldots\}$, and we take $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ and $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$.

1. The $\mathbf{T}_0$-reflection of a primal space. Let $\mathcal{C}$ be a category. Following Kennison [15], a *flow* in $\mathcal{C}$ is a couple (X, f) where X is an object of $\mathcal{C}$ and f

is a morphism from X to itself. Now, let (X, f) be a flow in the category **Set** of sets. O. Echi has defined the associated primal topology $\mathcal{P}(f)$ on X, whose closed sets are exactly those sets A which are f-invariant (that is, the sets with $f(A) \subseteq A$). A *primal space* is a topological space (X, τ) such that there is some mapping $f : X \to X$ with $\tau = \mathcal{P}(f)$.

Clearly $\mathcal{P}(f)$ is an Alexandroff topology on X. Given a subset A of X, the closure of A in $(X, \mathcal{P}(f))$ will be denoted $\overline{A}^X$. It is easily seen that $\overline{A}^X$ is exactly $\bigcup\{f^n(A) : n \in \mathbb{N}\}$ and in particular for any point $x \in X$, $\overline{\{x\}} = \{f^n(x) : n \in \mathbb{N}\}$. That is, the closure of a point x is the orbit of x by f, denoted $\mathcal{O}_f(x)$. Recall that by convention, $f^0(x) = x$.

The smallest open set containing x is

$$\mathcal{V}_f(x) = \{y \in X : f^n(y) = x \text{ for some } n \in \mathbb{N}\},$$

and the family $\mathcal{B} = \{\mathcal{V}_f(x) : x \in X\}$ is a basis for the open sets of $\mathcal{P}(f)$.

The construction of the $\mathbf{T}_0$-reflection $\mathbf{T}_0(X)$ of a topological space X, defined below, is well-known. In this section, we give an explicit construction of $\mathbf{T}_0(X, \mathcal{P}(f))$ for a primal space $(X, \mathcal{P}(f))$ and show that $\mathbf{T}_0(X, \mathcal{P}(f))$ is homeomorphic to a primal space $(Y, \mathcal{P}(g))$, where Y and g are derived from X and f. The properties of this construction will be used later. This construction defines $\mathbf{T}_0(X, \mathcal{P}(f))$ as a subset Y of X and g as a restriction of f with minor identifications. All periodic points of X become fixed points of g.

Recall that a reflective subcategory $\mathbf{A}$ of $\mathbf{B}$ is a full subcategory such that the embedding functor $\mathbf{A} \longrightarrow \mathbf{B}$ has a left adjoint functor. This means that for each object X of $\mathbf{B}$, there exist an object $\mathbf{F}(X)$ of $\mathbf{A}$ (called the $\mathbf{A}$-reflection of X) and a morphism $\mu_X : X \longrightarrow \mathbf{F}(X)$ in $\mathbf{B}$ such that for each object Y in $\mathbf{A}$ and each morphism $f : X \longrightarrow Y$ in $\mathbf{B}$, there exists a unique morphism $\widetilde{f} : \mathbf{F}(X) \longrightarrow Y$ in $\mathbf{A}$ such that $\widetilde{f} \circ \mu_X = f$ (for more information see [17, page 89]).
Further, recall that for all $i = 0, 1, 2, 3, 3\frac{1}{2}$, the subcategory $\mathbf{Top}_i$ of T_i-spaces is reflective in the category **Top** of all topological spaces.

Now, we recall the particular case of the $\mathbf{T}_0$-reflection of a topological space X. Define an equivalence relation on X by

$$x \sim y \text{ if and only if } \overline{\{x\}} = \overline{\{y\}}.$$

The resulting quotient space $X/_\sim$ is the $\mathbf{T}_0$-reflection $\mathbf{T}_0(X)$ of X. In particular, $\mathbf{T}_0(X)$ is a $\mathbf{T}_0$-space with the following properties:

- The canonical onto map $\mu_X : X \longrightarrow \mathbf{T}_0(X)$ is a *quasihomeomorphism*; that is, μ_X^{-1} provides a bijection between the open sets of $\mathbf{T}_0(X)$ and those of X (see [12, 22]).
- For any $\mathbf{T}_0$-space Y and any continuous map f from X to Y, there exists a unique map $\widetilde{f}$ which makes the following diagram commute.

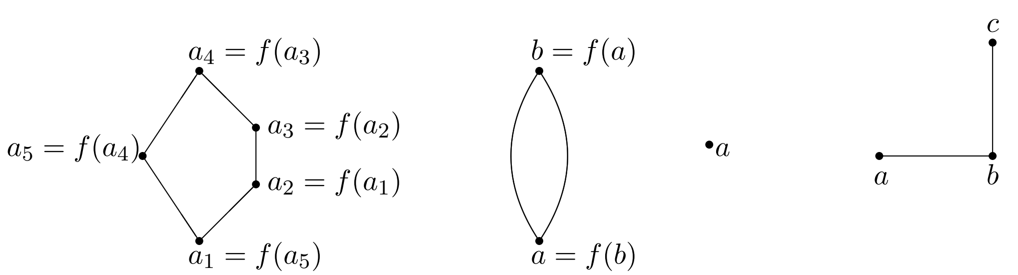
$$\begin{tikzcd}
X \arrow[rr, "\boldsymbol{\mu}_X"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\nabla"] & \mathbf{T}_0(X) \arrow[dl, "\widetilde{f}"] \\
& Y &
\end{tikzcd}$$

Because $\mu_X : X \longrightarrow \mathbf{T}_0(X)$ is a quasihomeomorphism, it follows that X is an Alexandroff space if and only if $\mathbf{T}_0(X)$ is an Alexandroff space.

Before giving the main result of this section, let us introduce some definitions, notations, and remarks.

Definition 1.1. Let (X, f) be a flow in **Set** and $x \in X$. Then x is said to be a *periodic point* of f if $f^n(x) = x$ for some $n \in \mathbb{N}^*$. In this case, $p := \min\{n \in \mathbb{N}^*; f^n(x) = x\}$ is called the *period* of x, and x is said to be a *p-periodic point* of f. The set of all periodic points of X will be denoted by $Per(f)$. In particular, a 1-periodic point of f is called a *fixed* point of f and the set of all fixed points of f will be denoted by $Fix(f)$.

Graphical conventions. In our graphical depictions of primal spaces, generally we will represent $f(a)$ as the vertex above a. There are special cases, such as those shown in Figure 1, which do not follow this general rule. We will represent cycles in the counterclockwise direction. Figure 1(a) represents a cycle with $f(a_i) = a_{i+1}$, for $1 \le i \le n-1$ and $f(a_n) = a_1$. Figure 1(b) represents a cycle of length two, with $f(a) = b$ and $f(b) = a$. Figure 1(c) represents a fixed point a. Sometimes it will be convenient to represent $f(a)$ as the vertex to the right of a. Figure 1(d) represents points a, b, c with $f(a) = b$, $f(b) = c$ and $f(c) = c$.



(a) (b) (c) (d)

FIGURE 1.

Let (X, f) be a flow in **Set** and equip X with the topology $\mathcal{P}(f)$. Under the equivalence relation $\sim$ defined by $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$, the equivalence class of any non-periodic point is a singleton. Let Λ be a complete set of equivalence class representatives from $Per(f)$, so $\{\overline{\{a\}} : a \in \Lambda\}$ is a partition of $Per(f)$.

Let $Y = (X \backslash Per(f)) \sqcup \Lambda$ be the disjoint union of Λ and the non-periodic points of f. We define the map $g : Y \longrightarrow Y$ as follows:

(g_1) If $y \in \Lambda$, then $g(y) := y$.

(g_2) If $y \in X \backslash Per(f)$ and $\overline{\{f(y)\}}^X = \overline{\{a\}}^X$ for some $a \in \Lambda$, then $g(y) := a$, that is, if $y \in X \backslash Per(f)$ and $f(y) \in Per(f)$ then $g(y) = a \in \Lambda$ where $a \sim f(y)$.

(g_3) If $y \in X \backslash Per(f)$ and $\overline{\{f(y)\}}^X \neq \overline{\{a\}}^X$ for each $a \in \Lambda$, then $g(y) := f(y)$, that is, if $y \in X \backslash Per(f)$ and $f(y) \in X \backslash Per(f)$ then $g(y) = f(y)$.

This construction collapses cycles to points, as suggested by Figure 2. Our goal is Theorem 1.10 in which we will show that $(Y, \mathcal{P}(g))$ is the $\mathbf{T}_0$-reflection of $(X, \mathcal{P}(f))$.

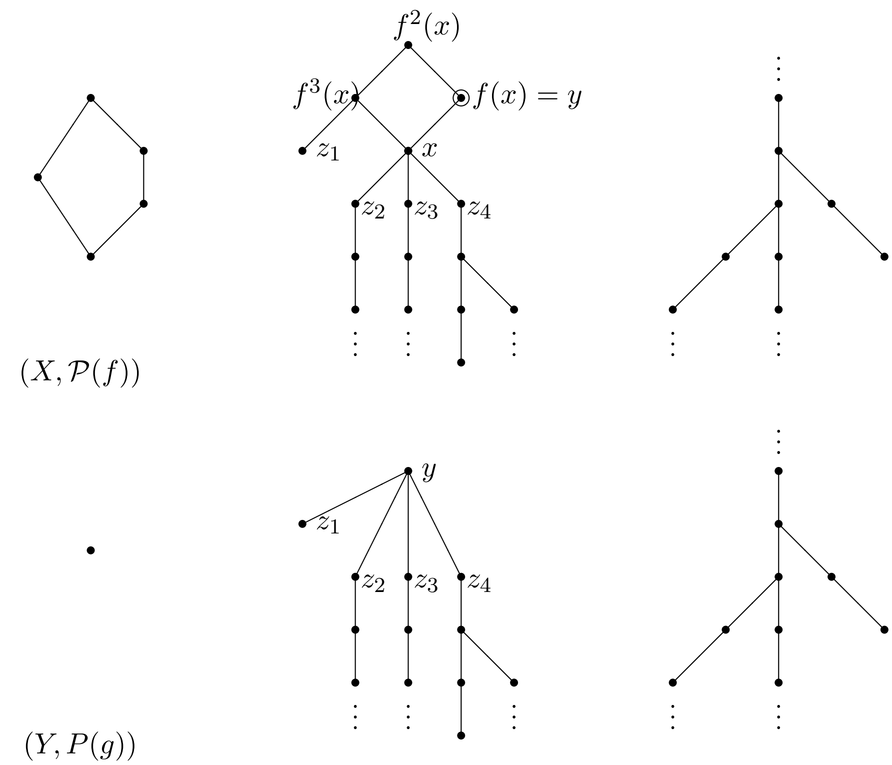


Figure 2. Cycles in $(X, \mathcal{P}(f))$ collapse to points in $(Y, \mathcal{P}(g))$.

Lemma 1.2. *Let $x \in Y = (X \backslash Per(f)) \sqcup \Lambda$. Then*

(1) *If $f(x) \in Y$, then $g(x) = f(x)$.*
(2) *If $f(x) \notin Y$, then $g(x) = a$ with $a \in \Lambda$ satisfying $\overline{\{f(x)\}}^X = \overline{\{a\}}^X$.*

Proof. Let x be in Y.

(1) If $f(x)$ belongs to Y, then two cases arise:
- $f(x)$ is not a periodic point and then, on the one hand x is not periodic too and on the other hand $\overline{\{f(x)\}}^X \neq \overline{\{a\}}^X$ for each $a \in \Lambda$. Hence, by the construction of g, $g(x) = f(x)$.

- $f(x) \in \Lambda$ and since x is in Λ too, then x is a fixed point which implies that $x = f(x) = g(x)$.

(2) If $f(x)$ is not in Y, then $f(x)$ is a periodic point not in Λ and in this case there exists $a \in \Lambda$ such that $\overline{\{f(x)\}}^X = \overline{\{a\}}^X$. Hence by the construction of g, we have $g(x) = a$.

$\square$

Lemma 1.3. *For each $x \in Y$, we have $\overline{\{x\}}^Y \subseteq \overline{\{x\}}^X$ and $\mathcal{V}_g(x) \subseteq \mathcal{V}_f(x)$.*

Proof. First, we prove the result about closures. Suppose $x \in Y$ is given. From the definition of g and Lemma 1.2, observe that $g(x)$ is either x, $f(x)$, or a, where $\overline{\{a\}}^X = \overline{\{f(x)\}}^X$, so $a = f^n(f(x))$ for some $n \in \mathbb{N}$. In all cases, $g(x) \in \overline{\{x\}}^X = \mathcal{O}_f(x)$. Repeating this argument with x replaced by $g(x)$ shows $g(g(x)) \in \overline{\{x\}}^X$. Iterating inductively, it follows that $\mathcal{O}_g(x) = \overline{\{x\}}^Y \subseteq \overline{\{x\}}^X$.

Now we will show that $\mathcal{V}_g(x) \subseteq \mathcal{V}_f(x)$ for every $x \in Y$. Indeed, for $y \in \mathcal{V}_g(x)$, there exists $n \in \mathbb{N}$ such that $x = g^n(y)$, which implies that $x \in \overline{\{y\}}^Y \subseteq \overline{\{y\}}^X$. Hence, $x = f^m(y)$ with $m \in \mathbb{N}$, and thus $y \in \mathcal{V}_f(x)$. $\square$

Lemma 1.4. *For $x \in Y$, x is a periodic point of g if and only if x is a fixed point of g. That is,*

$$Per(g) = Fix(g).$$

Proof. For $y \in Y = (X \backslash Per(f)) \sqcup \Lambda$, consider the three cases $(g_1), (g_2)$, and (g_3) defining $g(y)$. In case (g_1), y is a fixed point of g. In case (g_2), $y \notin \Lambda$ and $g(y) \in \Lambda$. Since every point of Λ is a fixed point, $g^n(y) = g(y) \in \Lambda$ can never equal $y \notin \Lambda$. In case (g_3), iteratively applying g, either $g^n(y) \in \Lambda$ for some $n \in \mathbb{N}$ and consequently $g^{n+m}(y) \in \Lambda$ for every $m \in \mathbb{N}$, in which case the orbit of y with respect to g stays in Λ and cannot return to y, or $g^n(y) = f^n(y) \in X \backslash Per(f)$ for all $n \in \mathbb{N}$, in which case y is neither periodic with respect to f nor g. Thus, the only periodic points of g are the fixed points $y \in \Lambda$. $\square$

Proposition 1.5. *$(Y, \mathcal{P}(g))$ is a T_0-space.*

Proof. By Proposition 2.5 of [9], it suffices to prove that $Per(g) = Fix(g)$, and this was shown in Lemma 1.4. $\square$

With X and Y as above, we define the map $\theta_X : X \longrightarrow Y$ by $\theta_X(x) = x$ if $x \in Y$, and $\theta_X(x) = a$ if $x \in X \setminus Y = Per(f) \setminus \Lambda$ with $\overline{\{x\}}^X = \overline{\{a\}}^X$ for some $a \in \Lambda$. From the definition, θ_X is clearly a surjective map.

Then, we have the following result.

Proposition 1.6. *$\theta_X : (X, \mathcal{P}(f)) \longrightarrow (Y, \mathcal{P}(g))$ is a closed continuous map.*

Proof. We know that θ_X is a closed continuous map if and only if $\theta_X(\overline{A}^X) = \overline{\theta_X(A)}^Y$ for every subset A of X. Now, since $(X, \mathcal{P}(f))$ and $(Y, \mathcal{P}(g))$ are Alexandroff spaces, it suffices to show that $\theta_X(\overline{\{x\}}^X) = \overline{\{\theta_X(x)\}}^Y$ for every $x \in X$. For this, suppose $x \in X$.

If $x \in Per(f) \setminus \Lambda$, then $\theta_X(x) = a$ with $\overline{\{x\}}^X = \overline{\{a\}}^X$ for some $a \in \Lambda$. Hence $\overline{\{\theta_X(x)\}}^Y = \overline{\{a\}}^Y = \{a\}$. And on the other hand, we have

$$\begin{aligned}\theta_X(\overline{\{x\}}^X) &= \theta_X(\overline{\{a\}}^X)\\ &= \theta_X(\{f^n(a); 0 \leq n \leq r-1\}) \text{ where } r \text{ is the period of } a\\ &= \{\theta_X(a)\} \cup \{\theta_X(f^n(a)); 1 \leq n \leq r-1\}.\end{aligned}$$

For $a \in \Lambda \subseteq Y$, we have $\theta_X(a) = a$. Furthermore, for each $1 \leq n \leq r-1$ we have $f^n(a) \in Per(f) \setminus \Lambda$ and $\overline{\{f^n(a)\}}^X = \overline{\{a\}}^X$, so $\theta_X(f^n(a)) = a$ for each $1 \leq n \leq r-1$. Therefore $\theta_X(\overline{\{x\}}^X) = \{a\} = \overline{\{\theta_X(x)\}}^Y$.

Now, if $x \in Y$ then two cases arise.

Case 1: If $f^n(x) \in Y$ for each $n \in \mathbb{N}$, then we have

$$\begin{aligned}\theta_X(\overline{\{x\}}^X) &= \{\theta_X(f^n(x)); n \in \mathbb{N}\}\\ &= \{f^n(x); n \in \mathbb{N}\}\\ &= \{g^n(x); n \in \mathbb{N}\} \quad \text{(by Lemma 1.2(1))}\\ &= \overline{\{x\}}^Y\\ &= \overline{\{\theta_X(x)\}}^Y.\end{aligned}$$

Case 2: If there exists $n_0 \in \mathbb{N}$ such that $f^n(x) \in Y$ for each $n \leq n_0$ and $f^{n_0+1}(x) \in Per(f) \setminus \Lambda$ with $\overline{\{f^{n_0+1}(x)\}}^X = \overline{\{a\}}^X$ for some $a \in \Lambda$. Then

$$\begin{aligned}\theta_X(\overline{\{x\}}^X) &= \{\theta_X(x), \theta_X(f(x)), \cdots, \theta_X(f^{n_0}(x))\} \cup \theta_X(\overline{\{a\}}^X)\\ &= \{x, f(x), \cdots, f^{n_0}(x)\} \cup \{a\}\\ &= \{x, g(x), \cdots, g^{n_0}(x)\} \cup \{a\} \quad \text{(by Lemma 1.2(1))}\\ &= \{x, g(x), \cdots, g^{n_0}(x)\} \cup \overline{\{g^{n_0+1}(x)\}}^Y\\ &= \overline{\{x\}}^Y\\ &= \overline{\{\theta_X(x)\}}^Y.\end{aligned}$$

□

Proposition 1.7. *Let U be a subset of Y. Then U is open in $(Y, \mathcal{P}(g))$ if and only if $\theta_X^{-1}(U)$ is open in $(X, \mathcal{P}(f))$.*

Proof. If U is open in $(Y, \mathcal{P}(g))$, then $\theta_X^{-1}(U)$ is open in $(X, \mathcal{P}(f))$ by Proposition 1.6.

Conversely, let U be a subset of Y such that $\theta_X^{-1}(U)$ is open in $(X, \mathcal{P}(f))$. By Lemma 1.3, for each $x \in Y$, $\mathcal{V}_g(x) \subseteq \mathcal{V}_f(x)$. Now, let $x \in U \subseteq Y$ then $\theta_X(x) = x$. Hence, $x \in \theta_X^{-1}(U)$. Since $\theta_X^{-1}(U)$ is open in X, then $\mathcal{V}_f(x) \subseteq \theta_X^{-1}(U)$ which implies that $\mathcal{V}_g(x) \subseteq \theta_X^{-1}(U)$. Thus $\mathcal{V}_g(x) = \theta_X(\mathcal{V}_g(x)) \subseteq \theta_X(\theta_X^{-1}(U)) = U$. Therefore, U is open in Y. □

Since θ_X is a surjection, Proposition 1.7 shows that θ_X is a quotient map, giving the following result.

Corollary 1.8. $(Y, \mathcal{P}(g))$ is a quotient space of $(X, \mathcal{P}(f))$.

Proposition 1.7 involves subspaces of a primal space. Below, we will use the fundamental fact that a subspace of a primal space is primal. Specifically, Echi [9, Examples 2.7(6)] showed that if Y is a subset of a primal space $(X, \mathcal{P}(f))$, the subspace topology on Y is $\mathcal{P}(h)$ for the map $h : Y \longrightarrow Y$ defined as follows: Let $y \in Y$. If $S_y := \{n \in \mathbb{N}^* : f^n(y) \in Y\} = \emptyset$, then we set $h(y) := y$. If $S_y \neq \emptyset$, then we define $h(y) := f^p(y)$, where p is the least element of the set S_y. It follows that the smallest open subset of Y containing y is of the form $U \cap Y$ where U is an open subset of X, so $\mathcal{V}_h(y) = \mathcal{V}_f(y) \cap Y$.

Proposition 1.9. *Let (X, f) be a flow in* **Set***, Z be a T_0-space, and h be a continuous map from the primal space $(X, \mathcal{P}(f))$ to Z. Then, there exists a unique continuous map $\widetilde{h}$ from $(Y, \mathcal{P}(g))$ to Z such that $\widetilde{h} \circ \theta_X = h$.*

Proof. For any $x \in X$, let us prove that $\widetilde{h}(\theta_X(x)) = h(x)$ defines $\widetilde{h}$ as a continuous map from $(Y, \mathcal{P}(g))$ to Z. Hence, it is sufficient to show that for any $x, a \in X$ such that $\theta_X(x) = \theta_X(a)$, we have $h(x) = h(a)$. In this situation, we only need to consider the case of $x \in Per(f) \setminus \Lambda$ and $a \in \Lambda$ such that $\overline{\{x\}}^X = \overline{\{a\}}^X$.

Indeed, let U be an open set of Z containing $h(a)$. Since h is continuous, then $h^{-1}(U)$ is an open set in X that contains a. Thus, $\overline{\{x\}}^X = \overline{\{a\}}^X$ implies that $x \in h^{-1}(U)$ and so $h(x) \in U$.
We conclude that $\overline{\{h(a)\}}^Z = \overline{\{h(x)\}}^Z$. Since Z is T_0, we have $h(a) = h(x)$. □

Now, we are in a position to give the main result of this section.

Theorem 1.10. $(Y, \mathcal{P}(g))$ *is homeomorphic to* $\boldsymbol{T}_0(X, \mathcal{P}(f))$.

Proof. Using Proposition 1.9 and the characterization given by MacLane in [17, page 89] one can easily see that $(Y, \mathcal{P}(g))$ is homeomorphic to $\mathbf{T}_0(X, \mathcal{P}(f))$. □

The following example illustrates Theorem 1.10. Note that it includes points which illustrate each of the three defining conditions $(g_1), (g_2)$, and (g_3) of the function g.

Example 1.11. Let $(X, \mathcal{P}(f))$ be a primal space with $X = \mathbb{Z}^2$ and $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ defined by:

$$f(n,m) = \begin{cases} (n, m+1) & \text{if } m < 0 \\ (n, -1) & \text{if } m = 0 \\ (n+1, 1) & \text{if } (n,m) \in 2\mathbb{Z} \times \{1\} \\ (n, 2) & \text{if } (n,m) \in 2\mathbb{Z}+1 \times \{1\} \\ (n-1, 2) & \text{if } (n,m) \in 2\mathbb{Z}+1 \times \{2\} \\ (n, 1) & \text{if } (n,m) \in 2\mathbb{Z} \times \{2\} \\ (n+1, m) & \text{if } m > 2. \end{cases}$$

This function is suggested by the left sketch of Figure 3.

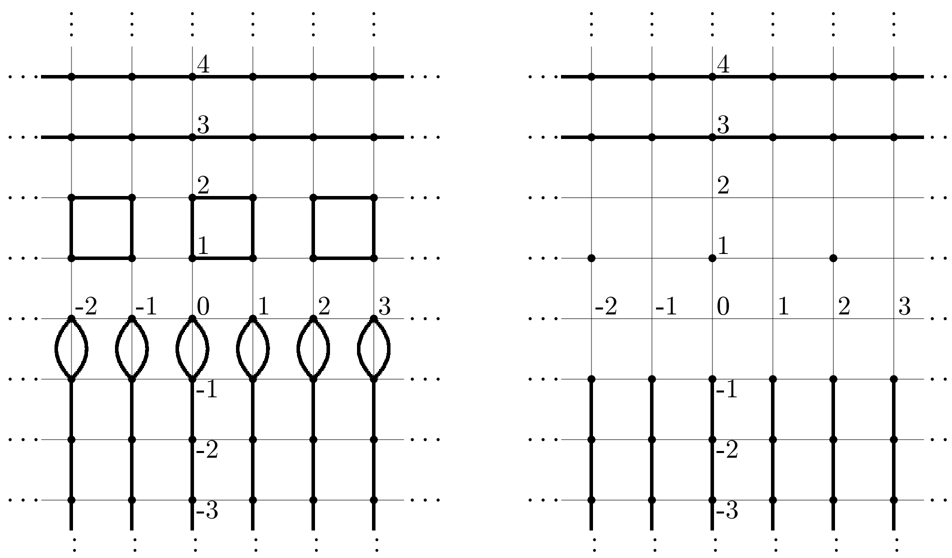


Figure 3.

In this case, we can choose $Y = (\mathbb{Z} \times \mathbb{Z}^*_-) \cup (2\mathbb{Z} \times \{1\}) \cup (\mathbb{Z} \times \{3, 4, 5, ...\})$, where $\mathbb{Z}^*_- = \{-1, -2, -3, \ldots\}$. Now $g : Y \to Y$ is given by:

$$g(n,m) = \begin{cases} (n, m+1) & \text{if } m < -2 & \text{by } (g_3) \\ (n, m+1) & \text{if } m = -2 & \text{by } (g_2) \\ (n, m) & \text{if } m = -1 & \text{by } (g_1) \\ (n, m) & \text{if } (n,m) \in 2\mathbb{Z} \times \{1\} & \text{by } (g_1) \\ (n+1, m) & \text{if } m > 2 & \text{by } (g_3). \end{cases}$$

This function is suggested by the right sketch of Figure 3.

In this case, the map θ_X is defined as follow:

$$\theta_X : (X, \mathcal{P}(f)) \longrightarrow (Y, \mathcal{P}(g))$$
$$\theta_X(n,m) = \begin{cases} (n,m) & \text{if } (n,m) \in Y \\ (n,-1) & \text{if } m = 0 \\ (n,1) & \text{if } (n,m) \in 2\mathbb{Z} \times \{1,2\} \\ (n-1,1) & \text{if } (n,m) \in 2\mathbb{Z}+1 \times \{1,2\} \end{cases}$$

Looking at the previous example, it is easy to see that the choice of Y is not unique. The following remark gives the number of possibilities of $(Y, \mathcal{P}(g))$ when $(X, \mathcal{P}(f))$ is fixed.

Remark 1.12. Recall that $Y = (X \backslash Per(f)) \sqcup \Lambda \subseteq X$, where Λ contains one representative from each equivalence class in $Per(f)$. Since there are many ways to choose the representatives, there are many possibilities for the choice of $Y \subseteq X$. If α denotes the number of possibilities of Y, then

$$\alpha = \prod_{x \in \Lambda} p(x) = \prod_{x \in \Lambda \backslash Fix(f)} p(x),$$

where $p(x)$ designates the period of x, and a product over an empty index set is 1.

In the example below, we find the number α of choices for $Y = (X \backslash Per(f)) \sqcup \Lambda \subseteq X$ for some particular spaces.

Examples 1.13.

(1) If $id_X : X \to X$ is the identity function $id_X(x) = x$, then $\mathcal{P}(id_X)$ is the discrete topology on X. Then $\Lambda = Fix(f) = X$, (so $\Lambda \setminus Fix(f) = \emptyset$) and there is only one possibility $Y = X$ for the set Y, so $\alpha = 1$.

(2) If τ is the indiscrete topology on a finite set $X = \{a_1, a_2, \ldots, a_n\}$, then (X, τ) is a primal space. Indeed, $\tau = \mathcal{P}(f)$ for the function $f : X \to X$ defined by $f(a_i) = a_{i+1}$ for $i \in \{1, 2, \cdots, n-1\}$ and $f(a_n) = a_1$. Now $\tau = \mathcal{P}(f)$ has only one nonempty closed set, so the closure (that is, the orbit) of any point $a_i \in X$ is X, so Λ is a singleton $\{a_i\}$, and there n choices for a_i. Thus there are $\alpha = p(a_i) = n$ possibilities for the set Y.

(3) Let $(X, \mathcal{P}(f))$ be a primal T_0-space. Then the product defining α has empty index set $\Lambda \setminus Fix(f)$, and thus there is only $\alpha = 1$ possibility $Y = X$ for the set Y.

(4) Let (X, τ) be the primal space with $X = \mathbb{Z}$ and $\tau = \mathcal{P}(f)$ where $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined by $f(n) = |n| - 1$. Then we have $\Lambda = I = \{a\}$ with $a \in \{-1, 0\}$. Thus $\alpha = p(-1) = p(0) = 2$ so there are two possibilities $Y = X \setminus \{-1\}$ or $Y = X \setminus \{0\}$ for the set Y.

(5) Let $(X, \mathcal{P}(f_k))$ be the primal space with $X = \mathbb{Z}^*$ and, for $k \in \mathbb{N}^*$, $f_k : \mathbb{Z}^* \longrightarrow \mathbb{Z}^*$ is the map defined by

$$f_k(n) = \begin{cases} n + \frac{|n|}{n} & \text{if } |n| < k \\ \frac{|n|}{n} & \text{if } |n| = k \\ k\frac{|n|}{n} & \text{if } |n| > k. \end{cases}$$

Then we have $Fix(f) = \emptyset$ and $\Lambda = \{a, b\}$ with $1 \leq a \leq k$ and $-k \leq b \leq -1$. Thus $\alpha = p(a) \times p(b) = k \times k = k^2$ so there are k^2 possibilities for the set Y.

2. Resolvable and n-Resolvable Primal Spaces.

Resolvable spaces were introduced by E. Hewitt in his paper [13]. We give the more general definition of n-resolvable spaces found in [11].

Definition 2.1. A topological space X is *resolvable* if it is the union of two disjoint dense subsets. A dense subset of X whose complement is also dense is said to be a *CD-set* on X. A topological space X is called *n-resolvable* for a cardinal n $(2 \leq n \leq \omega)$ if there is a family of n-many mutually disjoint dense subsets of X.

Note that resolvable spaces are precisely the 2-resolvable spaces. Hewitt included the redundant condition that X have no isolated points in his definition of resolvable spaces. We note the obvious fact that X contains n-many mutually disjoint dense subsets if and only if X is the union of n-many mutually disjoint dense subsets.

For $n \in \mathbb{N} \setminus \{0, 1\}$, $(n+1)$-resolvable clearly implies n-resolvable. Spaces which are n-resolvable but not $(n+1)$-resolvable are called *exactly n-resolvable* and are discussed in Section 4. If X is ω-resolvable, then clearly it is n-resolvable for every $n \in \mathbb{N}\setminus\{0, 1\}$. The converse also holds, and is given by A. Illanes in [14, Theorem 5].

Examples 2.2.

(1) Consider the map $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n + 1$, where $\mathbb{Z}$ is the set of all integers. It is clearly seen that $\mathcal{V}_f(n)$ is infinite for each $n \in \mathbb{Z}$. Now, consider the topological space $(\mathbb{Z}, \mathcal{P}(f))$. For any integer $n \geq 2$, set $A_k = \{j \in \mathbb{Z} : j \equiv k \mod n\}$. Then $A_0, A_1, \ldots, A_{n-1}$ are mutually disjoint dense subsets of $(\mathbb{Z}, \mathcal{P}(f))$, showing that this space is n-resolvable for every $n \geq 2$ and thus ω-resolvable.

(2) Consider the map $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n+1$, where $\mathbb{N}$ is the set of all natural numbers including 0. It is clearly seen that $|\mathcal{V}_f(0)| = 1$. Now, in the topological space $(\mathbb{N}, \mathcal{P}(f))$ it is easy to see that a subset A of X is dense if and only if $0 \in A$. Therefore, $(\mathbb{N}, \mathcal{P}(f))$ is not a resolvable space.

In this section, we use a deep result of A. H. Stone [21] to characterize Alexandroff spaces which are n-resolvable. We then present direct arguments to provide elementary proofs of the corresponding characterizations of n-resolvability in primal spaces.

Recall (see [18], for example) that there is a one-to-one correspondence between Alexandroff topologies on X and quasiorders on X defined by the specialization relation $x \preceq y$ if and only if $x \in \overline{\{y\}}$. In a quasiordered set $(X, \preceq)$, for $A \subseteq X$ we define $\uparrow A = \{x \in X : \exists a \in A$ with $a \preceq x\}$ and $\downarrow A = \{x \in X : \exists a \in A$ with $x \preceq a\}$. Now if the Alexandroff topological space (X, τ) arises from the specialization quasiorder $\preceq$, then we have $\overline{A} = \downarrow A$. In particular, for $x \in X$, $\downarrow\{x\} = \overline{\{x\}}$ and $\uparrow\{x\}$ is the smallest neighborhood of x. For a primal space $(X, \mathcal{P}(f))$, we have $\downarrow\{x\} = \mathcal{O}_f(x)$, the orbit of x by f, and $\uparrow\{x\} = \mathcal{V}_f(x)$.

We now present A. H. Stone's result.

Theorem 2.3 (A. H. Stone [21])**.** *A necessary and sufficient condition for a quasiordered set (indeed, a set with a transitive relation) to admit a partition into n mutually disjoint cofinal sets is that each element of X has at least n successors; that is, for every $x \in X$, $\uparrow\{x\}$ has at least n elements.*

Observe that the following are equivalent:
(a) A is dense in an Alexandroff space X
(b) $X = \overline{A} = \downarrow A$
(c) for every $x \in X$, there exists $a \in A$ with $x \preceq a$
(d) A is cofinal in $(X, \preceq)$.

This allows us to rephrase Stone's result as follows.

Theorem 2.4. *An Alexandroff topological space (X, τ) with specialization quasiorder $\preceq$ is n-resolvable if and only if for every $x \in X$, $\uparrow\{x\}$ contains at least n elements. In particular, (X, τ) is resolvable if and only if there are no maximal elements in $(X, \preceq)$, or equivalently, if and only if (X, τ) has no isolated points.*

Corollary 2.5. An Alexandroff space $(X, \tau) = (X, \preceq)$ is n-resolvable if and only if for every $x \in X$, $\uparrow\{x\}$ contains at least n distinct elements. That is, an Alexandroff space (X, τ) is n-resolvable if and only if every maximal element $[x]$ in the T_0-reflection $T_0(X)$ arises from a cycle $x = x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n = x$ containing at least n distinct elements x_i.

We now turn our attention to providing direct, concrete proofs of these results for primal spaces.

First, given a primal space $(X, \mathcal{P}(f))$, let us define an equivalence relation $\mathcal{R}$ on X by:

$$x\mathcal{R}y \text{ if and only if there exists } (r, s) \in \mathbb{N} \times \mathbb{N} \text{ such that } f^r(x) = f^s(y).$$

We denote by $[x]$ the equivalence class of x under $\mathcal{R}$. In particular if x is a p-periodic point of f, then $[x] = \{x, f(x), f^2(x), \dots, f^{p-1}(x)\}$ is called a *cycle*.

Remark 2.6. Let $(X, \mathcal{P}(f))$ be a primal space and $\mathcal{R}$ the equivalence relation defined above. Then for any $x \in X$ and any $y \in [x]$, we have $\mathcal{V}_f(y) \subseteq [x]$.

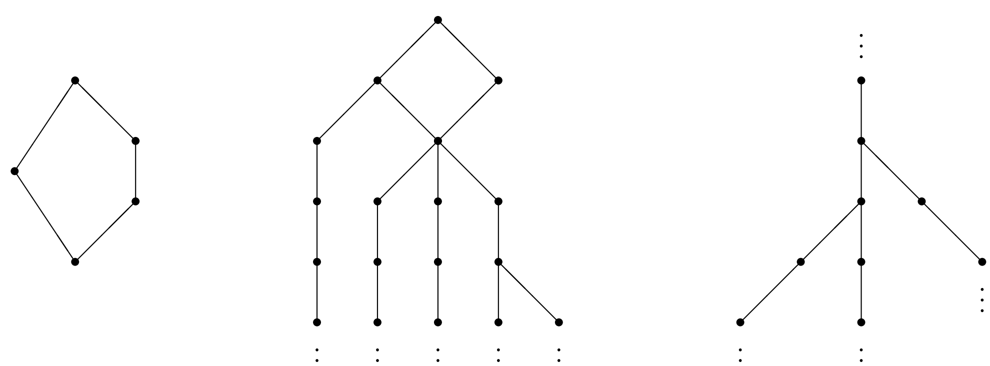
Lemma 2.7. *Suppose $(X, \mathcal{P}(f))$ is a primal space with $|\mathcal{V}_f(x)| \geq 2$ for all $x \in X$. Then $[x]$ is finite if and only if $[x]$ is a cycle $\{x, f(x), f^2(x), \ldots, f^n(x)\}$ for some $n \in \mathbb{N}^*$.*

Proof. First, we note that any equivalence class can have no more than one cycle. Now by Remark 2.6, for $y_1 \in [x]$, any infinite sequence $(y_i)_{i \in \mathbb{N}}$ of points with $y_i \in \mathcal{V}_f(y_{i-1}) \setminus \{y_{i-1}\}$ for $i \geq 2$ must be contained in $[x]$. If $[x]$ is finite, then the infinite sequence contains repeated entries and thus $[x]$ contains a cycle through y_1. Since this can be done for any $y_1 \in [x]$ and $[x]$ cannot contain more than one cycle, every point $y_1 \in [x]$ is in the cycle, so $[x]$ is a cycle. □

Suppose $(X, \mathcal{P}(f))$ is a primal space with $|\mathcal{V}_f(x)| \geq 2$ for all $x \in X$. Let $\{x_j : j \in J\}$ be a complete set of representatives of the equivalence classes of $\mathcal{R}$, so that $\{[x_j] : j \in J\}$ is a partition of X. By Remark 2.6, Lemma 2.7, and the convention that a fixed point is a 1-cycle, the collection $\{[x_j] : j \in J\}$ of equivalence classes may be written as $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, where

$$\begin{aligned} \mathcal{A} &= \{[x_j] : [x_j] \text{ is a cycle with finite cardinality}\} \\ \mathcal{B} &= \{[x_j] : [x_j] \text{ is infinite and contains periodic points belonging to a single cycle}\} \\ \mathcal{C} &= \{[x_j] : [x_j] \text{ is infinite and contains no periodic points}\}. \end{aligned}$$

These collections will be used below. A representative of each of these collections is shown in Figure 4.



$A \in \mathcal{A}$ $\qquad$ $B \in \mathcal{B}$ $\qquad$ $C \in \mathcal{C}$

Figure 4.

Theorem 2.8. *Let (X, f) be a flow in* **Set**. *For any given natural number $n \geq 2$, the following statements are equivalent.*

(i) *$(X, \mathcal{P}(f))$ is n-resolvable.*
(ii) *For each $x \in X$, $|\mathcal{V}_f(x)| \geq n$.*

Proof. $(i) \implies (ii)$ Suppose that $(X, \mathcal{P}(f))$ is n-resolvable. Then there exist n-many mutually disjoint dense subsets $\{A_i : 1 \leq i \leq n\}$ in X. For any $x \in X$ and for any $1 \leq i \leq n$, $\mathcal{V}_f(x) \cap A_i \neq \emptyset$, so consequently $\mathcal{V}_f(x)$ contains at least n points.

$(ii) \implies (i)$ To construct n-many mutually disjoint dense subsets of X, it suffices to find n-many mutually disjoint dense subsets of every $A \in \mathcal{A}$, of every $B \in \mathcal{B}$, and of every $C \in \mathcal{C}$: If $A_1, \ldots, A_n$ are mutually disjoint dense subsets of $A \in \mathcal{A}$, with $B_1, \ldots B_n$ and $C_1, \ldots, C_n$ defined similarly, then $X_i = \bigcup\{A_i : A \in \mathcal{A}\} \cup \bigcup\{B_i : B \in \mathcal{B}\} \cup \bigcup\{C_i : C \in \mathcal{C}\}$ for $i = 1, \ldots, n$ provide mutually disjoint dense subsets of X.

Since $\mathcal{V}_f(x) \subseteq [x]$, the condition that $|\mathcal{V}_f(x)| \geq n$, the collection $\mathcal{A}$ defined above has form

$$\mathcal{A} = \{[x_j] : [x_j] \text{ is a cycle with finite cardinality greater or equal to } n\}.$$

For $A \in \mathcal{A}$, there exist n distinct points $a_1, \ldots, a_n$ in the cycle $A = [x_j]$. With $A_i = \{a_i\}$ the sets $A_1, \ldots, A_n$ are mutually disjoint and dense in A.

For $B \in \mathcal{B}$, let x_0 be a fixed periodic point in B and for $k \in \{0, 1, \ldots, n-1\}$, put $B_k = \{x \in B \backslash Per(f) : \text{the least } r \in \mathbb{N} \text{ with } f^r(x) = x_0 \text{ satisfies } r \equiv k \mod n\}$. Now $B_0, B_1, \ldots, B_{n-1}$ is a partition of the set $B \backslash Per(f)$ of non-periodic points of B. We will show that each B_k is dense in B. Suppose $k \in \{0, \ldots, n-1\}$ and $x \in B$. If x is a periodic point, then $\mathcal{V}_f(x) = B$ and thus $\mathcal{V}_f(x) \cap B_k \neq \emptyset$. Therefore $x \in \overline{B_k}^B$. Now suppose $x \in B \backslash Per(f)$ and let r the be the least nonnegative integer such that $f^r(x) = x_0$. If $r \equiv j \mod n$, let m be the least nonnegative integer such that $r + m \equiv k \mod n$. That is, $m = k - j$ if $k \geq j$ and $m = n - (k - j)$ if $j > k$. Note that $m < n$, and since $|\mathcal{V}_f(x)| \geq n$, there exists $y \in B \backslash Per(f)$ such that $x = f^m(y)$. Now $f^{r+m}(y) = f^r(x) = x_0$, $r + m$ is the smallest nonnegative power with this property, and $r + m \equiv k \mod n$, so $y \in B_k$. Thus, $x \in \overline{\{y\}}^B \subseteq \overline{B_k}^B$, and this proves B_k is dense in B.

For $C \in \mathcal{C}$, fix a point $x_0 \in C$. For $x \in C$, there exists a unique pair $(r_x, s_x) \in \mathbb{N} \times \mathbb{N}$ with $f^{r_x}(x) = f^{s_x}(x_0)$. For $k \in \{0, 1, \ldots, n-1\}$, let $C_k = \{x \in C : r_x + s_x \equiv k \mod n\}$. Clearly the sets $C_0, C_1, \ldots, C_{n-1}$ are mutually disjoint. We will show each C_k is dense in C. Suppose $z \in C$. Let $j \in \{0, 1, \ldots, n-1\}$ be the integer such that $r_z + j + s_z \equiv k \mod n$. Since $|\mathcal{V}_f(z)| \geq n$, there exists $z' \in C$ with $f^j(z') = z$. Now $f^{r_z + j}(z') = f^{s_z}(x_0)$, so $z' \in C_k$, and $z \in \overline{\{z'\}}^C \subseteq \overline{C_k}^C$, so C_k is dense in C. □

Corollary 2.9. A primal space is resolvable if and only if it has no isolated points.

Proof. With $n = 2$, condition (ii) in Theorem 2.8 is equivalent to saying that $(X, \mathcal{P}(f))$ has no isolated points. □

Examples 2.10.

(1) As noted in Example 1.13(1), the discrete topology $(X, \tau) = (X, \mathcal{P}(id_X))$. Each point of X is a fixed point of $f = id_X$ with $|\mathcal{V}_f(x)| = 1$, so Corollary 2.9 shows that X is not resolvable.

(2) If $X = \{a_1, a_2, \dots, a_n\}$ with $n \geq 2$ and τ is the indiscrete topology on X, Example 1.13(2) showed that $\tau = \mathcal{P}(f)$, where f maps each point a_i to the next, cyclically. Now $|\mathcal{V}_f(x)| = n$ for every $x \in X$, and by Theorem 2.8, X is k-resolvable for every $k \in \{2, \dots, n\}$ and not k-resolvable for every $k \geq n+1$.

(3) Let $P = \{p \in \mathbb{Z} : |p| \text{ is a prime number}\}$ and let σ be any permutation on P having no cycles. That is, let σ be an element of the symmetric group on P with $\sigma^k(p) \neq p$ for every p in P and for every $k \in \mathbb{N}^*$.

Now, consider the primal space (X, τ) such that $X = \mathbb{Z} \setminus \{-1, 0, 1\}$ and $\tau = \mathcal{P}(f)$ for the function

$$f : X \longrightarrow X$$
$$f(n) = \begin{cases} \sigma(n) & \text{if } n \in P \\ \frac{n}{2} & \text{if } n \text{ is even and } n \notin P \cup \{0\} \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } n \notin P \cup \{-1, 1\}. \end{cases}$$

Part of the flow (X, f) is illustrated for the particular case when $\sigma(5) = 3$, $\sigma(3) = 7$, $\sigma(7) = 2$, $\sigma(2) = -3, \dots$ in Figure 5.

This space X has no periodic points, so $\mathcal{A} = \emptyset$ and $\mathcal{B} = \emptyset$. For every $x \in X$, there exists $r \in \mathbb{N}$ with $f^r(x) \in P$, and thus is easy to see that $x \sim y$ for every $x, y \in X$. It follows that X has one equivalence class C of type $\mathcal{C}$, and for every $x \in X$, $|\mathcal{V}_f(x)|$ is infinite. Now Theorem 2.8 shows that X is n-resolvable for every $n \geq 2$ and consequently ω-resolvable.

Given a resolvable primal space X, a natural question is, how many CD-sets does X admit? The answer to this question is given by the following proposition.

Proposition 2.11. *Let $(X, \mathcal{P}(f))$ be a resolvable primal space.*

(1) *If X is finite, then all $\mathcal{R}$-equivalence classes of X are cycles and X has exactly $\prod_{i=1}^{n}(2^{p_i} - 2)$ CD-sets, where n is the number of cycles and p_i is the length of i^{th} cycle.*

(2) *If X is infinite, then X has an infinitely many CD-sets.*

Proof. (1) If X is finite, then clearly $\mathcal{B} \cup \mathcal{C} = \emptyset$, and by Lemma 2.7, every $\mathcal{R}$-equivalence class is a finite cycle in $\mathcal{A} = \{A_i : i \in I\}$, where the index set $I = \{1, 2, \dots, n\}$ is finite. Since X is resolvable, there are no fixed points. For $A_i \in \mathcal{A}$, let $p_i \geq 2$ be the period of the cycle A_i. To construct a CD-set in A_i, we have to choose at least one point of the cycle A_i and at most $p_i - 1$. Thus, there are $2^{p_i} - 2$ possibilities for this choice. Therefore, there are $\prod_{i \in I}(2^{p_i} - 2)$ possibilities for CD-sets in X.

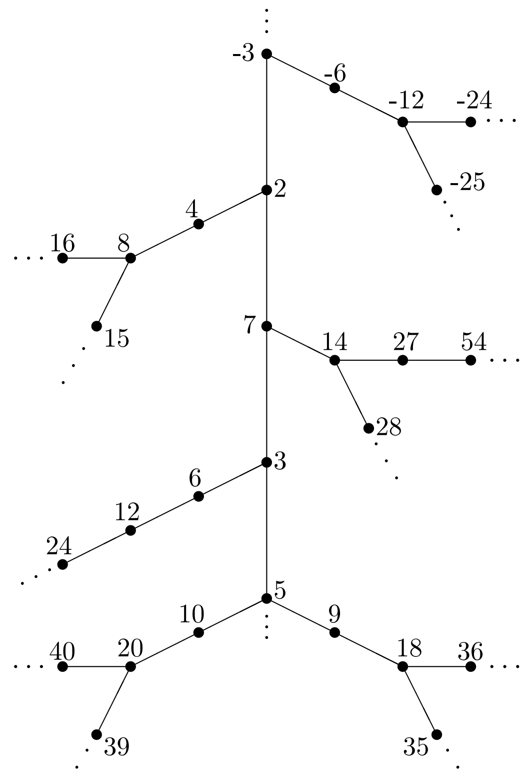


FIGURE 5.

(2) Two cases are to be considered, based on the types of $\mathcal{R}$-equivalence classes defined above Theorem 2.8.

Case 1: $\mathcal{B} \cup \mathcal{C} \neq \emptyset$. Then X has at least one infinite equivalence class $B \in \mathcal{B}$ or $C \in \mathcal{C}$. Suppose X has an equivalence class $B \in \mathcal{B}$. The construction in the proof of Theorem 2.8 showed that B has disjoint, dense, infinite subsets B_1 and B_2 with the property that for $i \in \{1, 2\}$, $B_i \setminus F$ is dense for any finite set F. Now B_1 and $B_2 = X \setminus B_1$ are complementary, and thus are CD-sets in B. But for every $b_2 \in B_2$, $B_2 \setminus \{b_2\}$ and $B_1 \cup \{b_2\}$ are also a CD-sets in B, providing an infinite number of CD-sets in B. By the same argument, any $C \in \mathcal{C}$ produces an infinite number of CD-sets in C. Following the construction of CD-sets for X based on those from $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $C \in \mathcal{C}$, it follows that X has an infinitely many CD-sets.

Case 2: $\mathcal{B} \cup \mathcal{C} = \emptyset$. Then all equivalence classes are in $\mathcal{A}$ and are finite, so by Lemma 2.7, each is a cycle. Since X is infinite, $\mathcal{A} = \{A_i : i \in I\}$ where the index set I is infinite. Now as in the proof of (1), there are $\prod_{i \in I} (2^{p_i} - 2)$ possibilities for CD-sets in X, which, as infinite product of numbers $2^{p_i} - 2 \geq 2$, is infinite.

$\square$

3. $\mathbf{T}_0$-Resolvablility. Let $\mathbf{D}$ be a reflective subcategory in **Top** with reflector $\mathbf{F}$, and P a topological property. It is a natural question to ask when $\mathbf{F}(X)$ satisfies P. In [3], Belaid, Echi and Lazaar study this concept for several separation axioms, and defined a topological space X to be $T_{(\mathbf{F},P)}$ if $\mathbf{F}(X)$ satisfies P. $T_{(\mathbf{F},P)}$ spaces for particular choices of $\mathbf{D}$, $\mathbf{F}$, and P are investigated in [3], [7] and [8]. In this section, we study another such case. In the reflective subcategory $\mathbf{D} = \mathbf{Top}_0$ of T_0-spaces in **Top**, with reflector $\mathbf{F} = \mathbf{T}_0$, we classify those spaces X for which $\mathbf{T}_0(X)$ is resolvable.

Definition 3.1. A topological space is called $\boldsymbol{T}_0$-*resolvable* if its $\mathbf{T}_0$-reflection is resolvable.

The following consequence of Theorem 2.8 characterizes $\mathbf{T}_0$-resolvable spaces for the class of primal spaces.

Theorem 3.2. *Let (X, f) be a flow in* **Set**. *Then the following statements are equivalent:*

(i) $(X, \mathcal{P}(f))$ *is a* $\boldsymbol{T}_0$*-resolvable space.*

(ii) $\mathcal{V}_f(x)$ *is infinite for each* $x \in X$.

Proof. Suppose that $\mathbf{T}_0(X)$ is resolvable. Now $T_0(X)$ is primal, with $T_0(X) = (Y, \mathcal{P}(g))$ as in Corollary 1.10, so Theorem 2.8 applies to $(Y, \mathcal{P}(g))$. Since $T_0(X)$ is T_0, there are no cycles in $(Y, \mathcal{P}(g))$, so each $\mathcal{R}$-equivalence class is an infinite set with no minimal points. So for every $y \in Y$, $\mathcal{V}_g(y)$ is infinite and consequently for any $x \in X$, there exists $y \in Y$, such that $\mathcal{V}_f(x) \supseteq \mathcal{V}_g(y)$ (see Lemma 1.2). Therefore, $\mathcal{V}_f(x)$ is infinite for every $x \in X$.

Conversely, suppose that for any $x \in X$, $\mathcal{V}_f(x)$ is infinite. Then the set $\mathcal{A}$ of $\mathcal{R}$-equivalence classes is empty and consequently, $\mathcal{V}_g(y)$ is infinite for every y in $(Y, \mathcal{P}(g)) = T_0(Y)$. Then Theorem 2.8 implies $\mathbf{T}_0((X, \mathcal{P}(f)))$ is resolvable. $\square$

Combining Theorem 2.8 and condition (ii) of Theorem 3.2 gives the following result.

Corollary 3.3. $\mathbf{T}_0$-resolvable implies resolvable. More precisely, if $\mathbf{T}_0((X, \mathcal{P}(f)))$ is resolvable, then $(X, \mathcal{P}(f))$ is n-resolvable for every $n \in \mathbb{N} \setminus \{0, 1\}$, and thus $(X, \mathcal{P}(f))$ is ω-resolvable.

The converse of Corollary 3.3 fails as seen by the following example.

Example 3.4. Let $X := \{0, 1\}$ and define f from X to itself by $f(0) = 1$ and $f(1) = 0$. Then $\mathcal{P}(f)$ is the indiscrete topology on X, so $\{0\}$ and $\{1\}$ are disjoint dense subsets of $(X, \mathcal{P}(f))$ and thus $(X, \mathcal{P}(f))$ is resolvable. But $\mathbf{T}_0((X, \mathcal{P}(f)))$ is a one-point space, which is not resolvable.

4. Exactly n-Resolvable Primal Spaces. Questions about resolvability of a topological space address basic foundational properties of the space, so it is not surprising that many variations of resolvability have been studied. In this section, we characterize the primal spaces which are exactly n-resolvable and strongly exactly n-resolvable. These variations of resolvability are discussed in [4, 5, 11]. We begin with the definitions.

Definition 4.1. A topological space X is called *exactly n-resolvable* for $n > 1$ if X is n-resolvable but not $(n+1)$-resolvable. Exactly 1-resolvable spaces are commonly called *irresolvable spaces.*

Definition 4.2. A topological space X is called *strongly exactly n-resolvable*, denoted by SE_nR, if X is n-resolvable and no nonempty subset of X is $(n+1)$-resolvable. *SE_1R-spaces* are commonly called *strongly irresolvable spaces.* Strongly irresolvable spaces (abbreviated as SI-spaces) are also called *hereditarily irresolvable spaces.*

Examples 4.3.

(1) It is clear that strongly exactly n-resolvable spaces are exactly n-resolvable spaces.

(2) Let X be a finite set with cardinality $n \geq 2$ equipped with the indiscrete topology. On the one hand it is clear that X is a k-resolvable space for every $2 \leq k \leq n$. Now every nonempty subset A of X satisfies $|A| < n+1$ and thus is not $(n+1)$-resolvable. Therefore X is strongly exactly n-resolvable. On the other hand X is not strongly exactly k-resolvable for every $2 \leq k \leq n-1$. Indeed X is k-resolvable and $(k+1)$-resolvable for every $2 \leq k \leq n-1$.

(3) Consider the set $X = \{0, 1, 2, 3\}$ equipped with the topology defined by $\overline{\{0\}} = \{0, 1, 2, 3\}$, $\overline{\{1\}} = \{1, 2, 3\}$, $\overline{\{2\}} = \{2, 3\}$ and $\overline{\{3\}} = \{3\}$. Every dense subset of X must contain 0, so X is an irresolvable space. Furthermore, if A is a subset of X, then every dense subset of A must contain the smallest element of A and thus every subset A of X is irresolvable. Therefore X is hereditarily irresolvable.

As an immediate consequence of Theorem 2.8, we have the following result.

Corollary 4.4. Let (X, f) be a flow in **Set** and $n \in \mathbb{N}$ $(n \geq 2)$. Then the following statements are equivalent:

(i) $(X, \mathcal{P}(f))$ is exactly n-resolvable.
(ii) $(X, \mathcal{P}(f))$ is an n-resolvable space with at least one n-periodic point.

For general Alexandroff spaces, Theorem 2.4 provides the following immediate result.

Corollary 4.5. An Alexandroff space $(X, \preceq)$ is exactly n-resolvable if and only if it is n-resolvable and there exists a maximal element $[x]$ in $T_0(X)$ generated from a cycle of length exactly n.

In [14, Theorem 5], A. Illanes proved that if a topological space X is n-resolvable for each n, then X is ω-resolvable. Hence the following result is an immediate consequence of Theorem 2.8, Corollary 3.2 and [14, Theorem 5].

Corollary 4.6. Let (X, f) be a flow in **Set**. Then the following statements are equivalent:

(i) $(X, \mathcal{P}(f))$ is an ω-resolvable space.

(ii) For any $x \in X$, $\mathcal{V}_f(x)$ is infinite.

(iii) $(X, \mathcal{P}(f))$ is $\mathbf{T}_0$-resolvable space.

As another consequence of Theorem 2.8, we now characterize the primal spaces which are strongly exactly n-resolvable for $n \geq 2$.

Proposition 4.7. *Let (X, f) be a flow in* **Set** *and $1 < n \in \mathbb{N}$. Then the following statements are equivalent.*

(i) *$(X, \mathcal{P}(f))$ is a strongly exactly n-resolvable space.*

(ii) *For each $y \in X$, y is a periodic point with period n.*

Proof. $(i) \Longrightarrow (ii)$ Suppose $(X, \mathcal{P}(f))$ is a strongly exactly n-resolvable space. By Theorem 2.8, $|\mathcal{V}_f(y)| \geq 2$ for all $y \in X$.

First we will show that each point $y \in X$ is periodic. Suppose to the contrary that that there exists a non-periodic point $y \in X$. We will show that $\mathcal{V}_f(y)$ is $(n+1)$-resolvable, contrary to X being strongly exactly n-resolvable. Set $B = \{y\}$ and $A_0 = \bigcup_{k \in \mathbb{N}} f^{-kn}(B)$, $A_1 = f^{-1}(A_0) = \bigcup_{k \in \mathbb{N}} f^{-kn-1}(B)$, $\ldots$, $A_n = f^{-1}(A_{n-1}) = \bigcup_{k \in \mathbb{N}} f^{-kn-n}(B)$. It is clear that $\{A_j : 0 \leq j \leq n\}$ is a family of $n+1$ disjoint dense subsets of $\mathcal{V}_f(y)$, a contradiction. Therefore, for each $y \in X$, we have y is a periodic point.

Next, we will show that each $y \in X$ has a period $p = n$. Suppose that there exists $y \in X$ with period $p > n$. Then, one can easily see that the family $\{F_k : 0 \leq k \leq n\}$ where $F_0 = \{y\}$, $F_1 = \{f(y)\}$, $F_2 = \{f^2(y)\}, \ldots,$ $F_n = \{f^n(y)\}$, $F_{n+1}(y) = \{f^m(y) : n+1 \leq m \leq p\}$ is a family of $n+1$ disjoint dense subsets of $\overline{\{y\}}$, again contradicting that X has no $(n+1)$-resolvable subspaces.

$(ii) \Longrightarrow (i)$ Suppose that each $y \in X$ is a periodic point of period $p = n$. Then, $|\mathcal{V}_f(y)| = n$ for every $y \in X$ and thus, by Theorem 2.8, $(X, \mathcal{P}(f))$ is n-resolvable.

Now, let A be a nonempty subset of X and $\{A_i : i \in I\}$ be a family of disjoint dense subsets of A. Now for each $y \in X$, $\overline{\{y\}} = \mathcal{V}_f(y)$ is an open subset of X with n points, so $U = \overline{\{y\}} \cap A$ is an open subset of A with no more than n points. For $y \in A$, we have $A_i \cap \overline{\{y\}} \neq \emptyset$ for each $i \in I$. Since $\{A_i : i \in I\}$ is a family of disjoint subsets of A, then $|I| \leq n$. Thus, no nonempty subset of X is $(n+1)$-resolvable. $\square$

We note that Example 2.10(2) is exactly n-resolvable. Examples 2.2(1) and 2.10(3) are ω-resolvable.

Example 4.8. The space $(\mathbb{Z}^2, \mathcal{P}(f))$ given in Example 1.11 is exactly 4-resolvable. Indeed for every $x \in \mathbb{Z}^2$, we have $|\mathcal{V}_f(x)| \geq 4$ and all points of the squares are 4-periodic points. From Corollary 4.4, it follows that $(\mathbb{Z}^2, \mathcal{P}(f))$ is exactly 4-resolvable.

The subspace $B = \mathbb{Z} \times \{1, 2\}$ of $(\mathbb{Z}^2, \mathcal{P}(f))$ is strongly exactly 4-resolvable. Indeed B is the subset of $\mathbb{Z}^2$ (equipped naturally with induced topology, which is primal by Echi [9]) consisting of the squares in Figure 3. Every point in B is 4-periodic and consequently, by Proposition 4.7, B is strongly exactly 4-resolvable.

The subspace $C = \mathbb{Z} \times \{3, 4, 5, ...\}$ of $(\mathbb{Z}^2, \mathcal{P}(f))$ is ω-resolvable. This subspace is represented by the horizontal lines in Figure 3, and thus for every $x \in C$, $\mathcal{V}_f(x)$ is infinite. Corollary 4.6 shows C to be ω-resolvable.

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