Indian J. pure appl. Math., 21 (6) : 527–529, June 1990

REMARKS ON N-POINT ORDER COMPACTIFICATIONS

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(Received 1 February 1989; accepted 17 January 1990)

Let (X^*, τ^*, $\leq^*$) be an n-point T_2-ordered compactification of the $T_{3.5}$-ordered topological space (X, τ, $\leq$). Let $\mathcal{V}_*^{\uparrow}(a)$ be the filter on X^* having a base of $\leq^*$-increasing τ^*-open neighborhoods of $a \in X^*$, with $\mathcal{V}_*^{\downarrow}(a)$ defined dually. Both the topology τ^* and the order $\leq^*$ are determined by the collection $\mathcal{C}^* = \{\mathcal{V}_*^{\uparrow}(a), \mathcal{V}_*^{\downarrow}(a) : a \in X^*\}$. An intrinsic characterization on X for this collection is pointed out.

A partially ordered topological space (X, τ, $\leq$) is T_2-*ordered* if the graph of the order is closed in $X \times X$. By a *space* we shall mean a T_2-ordered topological space. An *order compactification* of (X, τ, $\leq$) is a topological compactification (X^*, τ^*) of (X, τ) together with a closed order $\leq^*$ that extends the order $\leq$. A space has an order compactification iff it is $T_{3.5}$-ordered (completely regularly ordered in [3]). A subset A of a poset (X, $\leq$) is *increasing* if $a \in A$ and $x \geq a$ imply $x \in A$. *Decreasing sets* are defined dually. For further information on ordered topological spaces, see [3], [1], and [4]. With the order $\mathcal{F} \leq \mathcal{G}$ iff $\mathcal{F} \subseteq \mathcal{G}$, the supremum $\mathcal{F} \vee \mathcal{G}$ of filters $\mathcal{F}$ and $\mathcal{G}$ exists iff $\emptyset \notin \mathcal{B} \equiv \{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}$, in which case $\mathcal{B}$ is a base for $\mathcal{F} \vee \mathcal{G}$.

Let $\{G_i\}_{i=1}^n$ be an n-star (see [2]) corresponding to (X^*, τ^*), where $X^* = X \cup \{\omega_i\}_{i=1}^n$ and (X^*, τ^*, $\leq^*$) is an n-point order compactification of the $T_{3.5}$-ordered topological space (X, τ, $\leq$). Let $\mathcal{C} = \{\mathcal{V}^{\uparrow}(a), \mathcal{V}^{\downarrow}(a) : a \in X^*\}$ where $\mathcal{V}^{\uparrow}(a)$ is the trace on X of $\mathcal{V}_*^{\uparrow}(a)$, with $\mathcal{V}^{\downarrow}(a)$ defined dually. For $x \in X$, let $\mathcal{V}(x)$ be the neighborhood filter at x. Let $K = X \setminus \bigcup_{i=1}^n G_i$, and let $\mathcal{V}(\omega_i)$ be the filter generated by $\{N \subseteq X : (K \cup G_i) \setminus N \text{ is compact}\}$.

Theorem 1. $\mathcal{C}$ is the unique collection of filters on (X, τ, $\leq$) satisfying the following conditions for any $a, b \in X^*$ and any $x, y \in X$.

(1) $\mathcal{V}^{\uparrow}(a)$ (respectively, $\mathcal{V}^{\downarrow}(a)$) has a filter base of τ-open $\leq$-increasing (respectively, $\leq$-decreasing) sets.

(2) $\mathcal{V}^{\uparrow}(a) \vee \mathcal{V}^{\downarrow}(a) = \mathcal{V}(a)$.
(3) $\mathcal{V}^{\uparrow}(x) \vee \mathcal{V}^{\downarrow}(y)$ exists iff $x \leq y$.
(4) $\mathcal{V}^{\uparrow}(a) \vee \mathcal{V}^{\downarrow}(b)$ exists $\Longrightarrow \mathcal{V}^{\uparrow}(a) \leq \mathcal{V}(b)$ and $\mathcal{V}^{\downarrow}(b) \leq \mathcal{V}(a)$.

Proof: It is easy to see that $\mathcal{C}$ satisfies these conditions. Suppose that $\{\mathcal{U}^{\uparrow}(a),\ \mathcal{U}^{\downarrow}(a) : \quad a \in$ X*$\}$ is any other family of filters satisfying these conditions.

$\mathcal{U}^{\uparrow}(a) \leq \mathcal{V}^{\uparrow}(a)$: For any $U' \in \mathcal{U}^{\uparrow}(a)$, there exists τ-open $\leq$-increasing $U \in \mathcal{U}^{\uparrow}(a)$ such that (1) $U \subseteq U'$, and (2) $\omega \in$ X*\X and $\omega \not\geq a$ imply that there exists $N \in \mathcal{U}^{\downarrow}(\omega)$ with $N \cap U = \emptyset$. Let $U^* = U \cup \{\omega \in$ X*\X $: \omega \geq a\}$. Now U^* is τ^*-open since U was a neighborhood of each of its points, and for $\omega \geq a$, the existence of $\mathcal{U}^{\uparrow}(a) \vee \mathcal{U}^{\downarrow}(\omega)$ implies $\mathcal{U}^{\uparrow}(a) \leq \mathcal{U}(\omega)$, and thus U^* is a neighborhood of ω. To see that U^* is $\leq^*$-increasing, suppose $b \in U^*$ and $b \leq^* c$. Clearly $c \in U^*$ if $b,\ c \in X$ or if $b,\ c \in X^*\backslash X$. Suppose $b \in U^* \cap X$, $c = \omega \in X^*\backslash X$. If $\omega \notin U^*$, then there exists $N \in \mathcal{U}^{\downarrow}(\omega)$ such that $N \cap U = \emptyset$; but $b <^* \omega \Longrightarrow \mathcal{U}^{\downarrow}(\omega) \leq \mathcal{U}(b) \implies b \in N \cap U$, a contradiction. Finally, if $b = \omega \in U^*\backslash X$, $\omega <^* c$, and $c \in X$, then $a <^* c \Longrightarrow \mathcal{U}^{\uparrow}(a) \leq \mathcal{U}(c) \Longrightarrow c \in U$. Now $U = U^* \cap X \subseteq U'$, where U' was an arbitrary element of $\mathcal{U}^{\uparrow}(a)$ and U^* is the τ^*-open $\leq^*$-increasing neighborhood of a described above. It follows that $\mathcal{U}^{\uparrow}(a) \leq \mathcal{V}^{\uparrow}(a)$.

$\mathcal{V}^{\uparrow}(a) \leq \mathcal{U}^{\uparrow}(a)$: First notice that $a \not\leq^* c$ implies there exist $N \in \mathcal{U}^{\downarrow}(c)$ and $U \in \mathcal{U}^{\uparrow}(a)$ with $N \cap U = \emptyset$, and therefore $c \notin \mathrm{cl}_{\tau*}(U)$. Thus, $\bigcap \{\mathrm{cl}_{\tau*}(U) : U \in \mathcal{U}^{\uparrow}(a)\} = i_*(a) \equiv \{x \in X^* : a \leq^* x\}$. Now suppose $\mathcal{V}^{\uparrow}(a) \not\leq \mathcal{U}^{\uparrow}(a)$. Then there exists $V \in \mathcal{V}^{\uparrow}(a)$ such that for all $U \in \mathcal{U}^{\uparrow}(a)$, $U\backslash V \neq \emptyset$. Let V^* be a τ^*-open $\leq^*$-increasing neighborhood of a in X* such that $V^* \cap X \subseteq V$. Without loss of generality, $\omega \in V^*\backslash X$ iff $a \leq^* \omega$. Because $U\backslash V \neq \emptyset \Longrightarrow \mathrm{cl}_{\tau*}(U)\backslash V^* \neq \emptyset$ $(\forall U \in \mathcal{U}^{\uparrow}(a))$, we have for any finite collection $\{U_i \in \mathcal{U}^{\uparrow}(a) : i = 1, \ldots, n\}$, $\emptyset \neq [\mathrm{cl}_{\tau*}(\bigcap_{i=1}^{n} U_i)]\backslash V^* \subseteq [\bigcap_{i=1}^{n} \mathrm{cl}_{\tau*}U_i]\backslash V^* = \bigcap_{i=1}^{n} [\mathrm{cl}_{\tau*}U_i\backslash V^*]$. Thus, $\mathcal{C} = \{\mathrm{cl}_{\tau*}U\backslash V^* : U \in \mathcal{U}^{\uparrow}(a)\}$ is a collection of τ^*-closed sets in X* satisfying the finite intersection property, but $\bigcap \mathcal{C} = i_*(a)\backslash V^* = \emptyset$, contrary to the compactness of X*.

The dual arguments for $\mathcal{U}^{\downarrow}(a)$ and $\mathcal{V}^{\downarrow}(a)$ complete the proof. ∎

By [2], $\mathcal{V}(\omega_i)$ is the filter of punctured neighborhoods of ω_i. Now, in view of item (2) of the theorem, $\mathcal{C}$ defines the topology τ^*. The extension of item (3) to arbitrary points of X* defines the order $\leq^*$. Thus, the collection $\mathcal{C}_*$ is determined by $\mathcal{C}$, and Theorem 1 gives a characterization of $\mathcal{C}$ intrinsic to X.

Lemma 2. If (Y, τ, $\leq$) is a compact T_2-ordered lattice, with $\mathcal{V}^{\uparrow}(y)$ representing the filter generated by the τ-open $\leq$-increasing neighborhoods of y, then $\mathcal{V}^{\uparrow}(x) \vee \mathcal{V}^{\uparrow}(y) = \mathcal{V}^{\uparrow}(x \vee y)$.

Proof: Clearly $\mathcal{V}^{\uparrow}(x) \vee \mathcal{V}^{\uparrow}(y) \leq \mathcal{V}^{\uparrow}(x \vee y)$. Conversely, let N be an open increasing element of $\mathcal{V}^{\uparrow}(x \vee y)$. For $c \in Y\backslash N$, c is not an upper bound of x and y, so either $x \not\leq c$ or $y \not\leq c$. Thus, there exists an open decreasing neighborhood N_c of c disjoint from some open increasing neighborhood

M_c of x or y. Now $\{N_c : c \in Y\backslash N\}$ is an open cover of the compact $Y\backslash N$. If $\{N_c : c \in F\}$ is a finite subcover, then $\bigcap_{c\in F} M_c \subseteq N$ and $\bigcap_{c\in F} M_c \in \mathcal{V}^{\uparrow}(x) \vee \mathcal{V}^{\uparrow}(y)$. ∎

Theorem 3. Let X* be an n-point order compactification of X, and $\mathcal{C} = \{\mathcal{V}^{\uparrow}(a), \mathcal{V}^{\downarrow}(a) : a \in X^*\}$ be the associated collection of trace filters on X. Then X* is a lattice iff both $\mathcal{C}^{\uparrow} \equiv \{\mathcal{V}^{\uparrow}(a) : a \in X^*\}$ and $\mathcal{C}^{\downarrow} \equiv \{\mathcal{V}^{\downarrow}(a) : a \in X^*\}$ form upper semi-lattices.

Proof: That $\mathcal{C}^{\uparrow}$ and $\mathcal{C}^{\downarrow}$ from upper semi-lattices when X is a lattice follows from Lemma 2 and its dual. Conversely, if $\mathcal{C}^{\uparrow}$ is an upper semi-lattice, then for any $a, b \in X^*$, $\mathcal{V}^{\uparrow}(a) \vee \mathcal{V}^{\uparrow}(b)$ exists and is equal to an element $\mathcal{V}^{\uparrow}(c) \in \mathcal{C}^{\uparrow}$ for some $c \in X^*$. Now $\mathcal{V}^{\uparrow}(a) \leq \mathcal{V}^{\uparrow}(c) \leq \mathcal{V}(c)$ implies $\mathcal{V}^{\uparrow}(a) \vee \mathcal{V}^{\downarrow}(c)$ exists, whence $a \leq^* c$. Similarly, $b \leq^* c$, so that c is an upper bound of a and b. If d is another upper bound of a and b, then the existence of $\mathcal{V}^{\uparrow}(a) \vee \mathcal{V}^{\downarrow}(d)$ implies $\mathcal{V}^{\uparrow}(a) \leq \mathcal{V}(d)$ and $\mathcal{V}^{\uparrow}(b) \leq \mathcal{V}(d)$. Thus, $\mathcal{V}^{\uparrow}(c) = \mathcal{V}^{\uparrow}(a) \vee \mathcal{V}^{\uparrow}(b) \leq \mathcal{V}(d)$ so that $\mathcal{V}^{\uparrow}(c) \vee \mathcal{V}^{\downarrow}(d)$ exists, that is, $c \leq^* d$.

A similar argument shows that the existence of $a \wedge b$ follows from $\mathcal{C}^{\downarrow}$ being an upper semi-lattice. ∎

Let X be a bounded $T_{3.5}$-ordered poset and X* be any T_2-order compactification of X. Then X* must be bounded. Furthermore, any pair $a, b \in X^*$ must have a minimal upper bound since the upper bounds of a and b are given by $i(a) \cap i(b)$, which is closed since X* is T_1-ordered, and a theorem of Wallace (Theorem 1 in [5]) guarantees that a non-empty compact T_2-ordered space must have a minimal element. Thus, X* fails to be a lattice iff there exist $a, b \in X^*$ such that a and b have two minimal upper bounds, or two maximal lower bounds.

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NOTE: *This is a fascimilie of the original article. Typographical errors have been corrected and titles have been included in citations.*

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