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How to Recognize a Power Function

M. B. Richmond and T. A. Richmond
Department of Mathematics
Western Kentucky University
Bowling Green, KY 42101

A power function is a function of the form $f(x) = x^n$. We will consider positive multiples of power functions with positive powers, that is, functions of the form $f(x) = kx^n$ $(k, n > 0)$ over the domain $\{x : x \geq 0\}$. For $r > 0$, let $R_1(r)$ be the first quadrant region under the curve $y = f(x)$ over an interval $[0, r]$, and let $R_2(r)$ be the first quadrant region to the left of $R_1(r)$, as shown in Figure 1 below. Revolve these regions $R_1(r)$ and $R_2(r)$ around the y-axis to get solids of revolution with volumes $V_1(r)$ and $V_2(r)$, respectively. We will show that the ratio of the volumes $V_1(r)$ and $V_2(r)$ is constant, and that this property can be used to characterize these power functions.

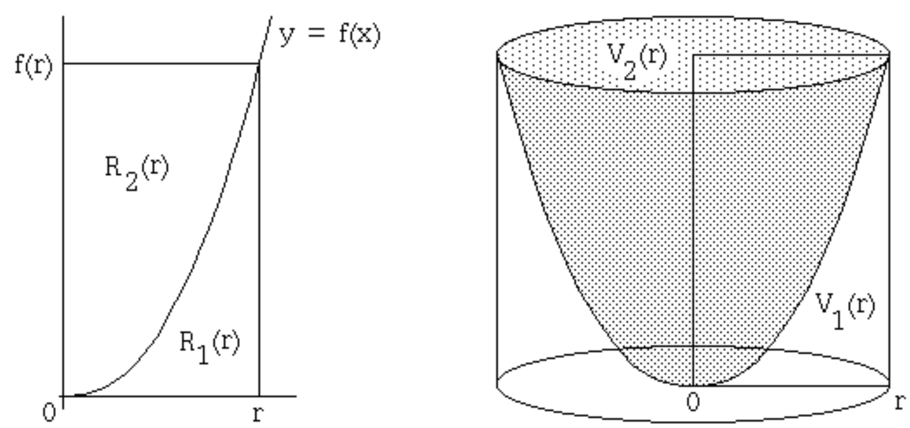

Figure 1

Using the method of cylindrical shells, we see that

$$\begin{aligned} V_1(r) &= 2\pi \int_0^r x f(x)\, dx \\ &= 2\pi \int_0^r k x^{n+1}\, dx \\ &= \frac{2\pi k r^{n+2}}{n+2}\ . \end{aligned}$$

The volume $V_2(r)$ can easily be calculated using cylindrical shells or disks. Alternately, by observing that $V_1(r)$ and $V_2(r)$ together give the volume of a cylinder of radius r and height $f(r)$, we see that

$$\begin{aligned} V_2(r) &= \pi r^2 f(r) - V_1(r) \\ &= \pi k r^{n+2} - \frac{2\pi k r^{n+2}}{n+2} \\ &= \frac{n\pi k r^{n+2}}{n+2}\ . \end{aligned}$$

The key to recognizing power functions with positive powers begins with the observation that $V_1(r)$ and $V_2(r)$ are quite similar. The ratio of $V_2(r)$ to $V_1(r)$ is

$$\frac{V_2(r)}{V_1(r)} = \frac{n}{2}.$$

Significantly, we note that this ratio is independent of r. The cases $n = 1$ and $n = 2$ are particularly interesting.

If $n = 1$, revolving the function $f(x) = kx^1$ gives a cone of volume $V_2(r)$ as the upper solid. From the ratio $\frac{V_2(r)}{V_1(r)} = \frac{n}{2}$, we have $V_1(r) = 2\,V_2(r)$. This verifies the well-known fact that a cone of height h and radius r has one-third the volume of a cylinder with the same height and radius, and suggests other natural ways to state that $\frac{V_2(r)}{V_1(r)} = \frac{n}{2}$. If $C(r)$ represents the volume of the cylinder composed of the solids with volumes $V_1(r)$ and $V_2(r)$, then it is a simple algebraic exercise to show that the following are equivalent:

(a) $\dfrac{V_2(r)}{V_1(r)} = \dfrac{n}{2}$ (b) $\dfrac{V_2(r)}{C(r)} = \dfrac{n}{n+2}$ (c) $\dfrac{V_1(r)}{C(r)} = \dfrac{2}{n+2}$.

Thus, any of these three statements could be used in the statement of the result.

The ratio $\frac{V_2(r)}{V_1(r)} = \frac{n}{2}$ equals 1 when $n = 2$, that is, when $f(x) = kx^2$. In this case, the volume inside the paraboloid determined by revolving $f(x) = kx^2$ over the inverval $[0, r]$ around the y-axis is exactly the same as the volume under the paraboloid. The surface of a paraboloid bisects the volume of the circumscribing cylinder!

Functions of form $f(x) = kx^n$ $(k, n > 0)$ are continuous, nonnegative, second differentiable, and have positive first derivatives on $(0, b)$ for any positive b. We have seen that the ratio $\frac{V_2(r)}{V_1(r)}$ is constantly $\frac{n}{2}$ for all $r > 0$. These properties characterize the functions $f(x) = kx^n$ $(k, n > 0)$, as we will see below. First observe that if $f(x)$ is any nonnegative continuous increasing function on $[0, b)$, then volumes $V_1(r)$, $V_2(r)$, and $C(r) = V_1(r) + V_2(r)$ corresponding to f may be defined as above for $r \in (0, b)$. With this notation, we have the following result.

Proposition: Suppose $f(x)$ is a continuous nonnegative function on $[0, b)$ satisfying the following conditions:

1. $f'(x) > 0$ for $x \in (0, b)$,

2. $f''(x)$ exists for $x \in (0, b)$, and

3. the ratio $\frac{V_2(r)}{V_1(r)}$ is a constant $\frac{n}{2}$ for any $r \in (0, b)$.

Then $f(x) = kx^n$ over $[0, b)$ where k is some positive constant.

To verify this, assume the hypotheses of the proposition hold. We note that condition 3 is equivalent to $\frac{V_1(r)}{C(r)} = \frac{2}{n+2}$ for every $r \in (0, b)$. Differentiating this last equation with respect to r gives $\frac{d}{dr}\left[\frac{V_1(r)}{C(r)}\right] = 0$. Applying the quotient rule on the left and multiplying both sides of the equation by the resulting denominator gives

$$C(r)\ \frac{d}{dr}\left[V_1(r)\right] - V_1(r)\ \frac{d}{dr}\left[C(r)\right] = 0.$$

Substituting $C(r) = \pi r^2 f(r)$ and $V_1(r) = 2\pi \int_0^r x f(x)\, dx$, and applying the Fundamental Theorem of Calculus to find $\frac{d}{dr}V_1(r)$, we get

$$[\pi r^2 f(r)]\ [2\pi r f(r)] - [2\pi \int_0^r x f(x)\, dx]\ \ [\pi r^2 f'(r) + 2\pi r f(r)] = 0.$$

Dividing by $-2\pi^2 r \neq 0$ gives

$$[\ \int_0^r x f(x)\, dx\ ]\ [r f'(r) + 2f(r)] - r^2 (f(r))^2 = 0,$$

or

$$\int_0^r x f(x)\, dx = \frac{[r f(r)]^2}{r f'(r) + 2f(r)}.$$

Note that the hypotheses of the proposition imply that the denominator on the right is nonzero. Again differentiating with respect to r gives

$$r f(r) = \frac{[r f'(r) + 2f(r)] 2r f(r) [r f'(r) + f(r)] - [r f(r)]^2 [r f''(r) + 3f'(r)]}{(r f'(r) + 2f(r))^2}.$$

The existence of $f''(r)$, which appears on the right, is guaranteed by the hypotheses. After a bit of algebraic manipulation, the equation above becomes

$$r f'(r)^2 - f(r) f'(r) - r f(r) f''(r) = 0.$$

Changing to the customary notation of differential equations, we will let x represent the independent variable (instead of r) and let $y = f(x)$ represent the dependent variable (formerly $f(r)$). This gives the differential equation

$$xy'^{\,2} - yy' - xyy'' = 0.$$

The hypotheses that $f(0) \geq 0$ and $f'(x) > 0$ for $x \in (0, b)$ imply that $y = f(x) > 0$ on $(0, b)$. It follows that $-xyy' \neq 0$ for $x \in (0, b)$, and thus we may divide the last equation by this expression. Upon rearranging terms, this gives

$$\frac{y''}{y'} - \frac{y'}{y} = \frac{-1}{x} \qquad (\, x \in (0, b)\,).$$

Integrating with respect to x gives

$$\begin{aligned} \ln y' - \ln y &= c - \ln x \\ \text{or} \qquad \ln(\frac{y'}{y}) &= c - \ln x. \end{aligned}$$

Applying the exponential function to both sides of the equation gives

$$\frac{y'}{y} = \frac{a}{x}$$

where $a = e^c$ is a positive constant. Integrating again with respect to x gives $\ln y = a \ln x + K = \ln x^a + K$, and thus $y = f(x) = e^K x^a = kx^a$ where $k = e^K$ is a positive constant. The differential equation was valid for $x \in (0, b)$, so $f(x) = kx^a$ over $(0, b)$, and by the continuity of f, $f(x) = kx^a$ over $[0, b)$. From the discussion before the proposition, we see that, for this f, $\frac{V_2(r)}{V_1(r)} = \frac{a}{2}$. It follows that $n = a > 0$, so $f(x) = kx^n$ where k is some positive constant. This completes the proof of the proposition.

The proposition gives a theoretical test to determine if a first quadrant curve satisfying conditions 1 and 2 of the proposition is a positive multiple of a power function with positive power: Revolve the region between the y-axis and the curve around the y-axis to form a solid. For each $r > 0$, construct a cylinder of radius r and height $f(r)$. Fill it with water, and submerge the solid to the bottom. If the ratio of the volume of displaced water to the volume of water remaining is the same for every $r > 0$, then the original curve is of the form $f(x) = kx^n$ $(k, n > 0)$, and it is an easy matter to determine n and k. More practically, this test could be used to determine that a curve is not a positive multiple of a power function with positive power.

It should be noted that negative multiples of power functions with positive powers are simply reflections about the x-axis of positive multiples of power functions with positive powers, and thus are also recognizable using the proposition.