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POSETS OF ORDERED COMPACTIFICATIONS

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If $(X', \tau', \leq')$ is an ordered compactification of the partially ordered topological space $(X, \tau, \leq)$ such that $\leq'$ is the smallest order that renders $(X', \tau', \leq')$ a T_2-ordered compactification of X, then X' is called a *Nachbin-* (or *order-strict*) compactification of $(X, \tau, \leq)$. If $(X', \tau', \leq^*)$ is a finite-point ordered compactification of $(X, \tau, \leq)$, the Nachbin order $\leq'$ for (X', τ') is described in terms of $(X, \tau, \leq)$ and X'. When given the usual order relation between compactifications (ordered compactifications), posets of finite-point Nachbin compactifications are shown to have the same order structure as the poset of underlying topological compactifications. Though posets of arbitrary finite-point ordered compactifications are shown to be less well behaved, conditions for their good behavior are studied. These results are used to examine the lattice structure of the set of all ordered compactifications of ordered topological space $(X, \tau, \leq)$.

0. Introduction

It is well known that for a completely regular topological space X, the following are equivalent: (a) the poset $K(X)$ of compactifications of X is a complete lattice, (b) X has a smallest compactification, (c) X has a 1-point compactification, (d) X has a finite-point compactification, and (e) X is locally compact. The equivalence of (a) and (b) is purely lattice theoretic once it is known that $K(X)$ is a complete upper semi-lattice. Blatter [1] has noted that the poset $K_o(X)$ of ordered compactifications of an ordered topological space X is a complete upper semi-lattice, and thus the statement (a) $\iff$ (b) generalizes to the case of ordered topological spaces and compactifications. Along with the trivial implication (c) $\Longrightarrow$ (d), these are the only implications from above that generalize. Here, we pursue the question from [1] of determining which ordered spaces admit a smallest ordered compactification. Specifically, we focus on generalizing the implication (d) $\Longrightarrow$ (b) above.

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Preliminaries: The graphs of partial orders $\leq$ and $\leq^*$ on a set X with be denoted by θ and θ^*, respectively. If $(X, \leq)$ is a poset and $x \in X$, then $i_X(x) = \{y \in X : y \geq x\}$. For $B \subseteq X$, the *increasing hull* of B is $i_X(B) = \bigcup_{x \in B} i_X(x)$. Typically, $i_X(B)$ and $i_{X^*}(B)$ will be shortened to $i(B)$ and $i^*(B)$, respectively. B is an *increasing set* if $B = i(B)$. *Decreasing hulls* and *decreasing sets* are defined dually. A set B is *convex* if $B = i(B) \cap d(B)$, or equivalently, if $a, b \in B$ and $a \leq x \leq b$ imply $x \in B$. We will write $A < B$ if $a < b$ for every $a \in A, b \in B$.

A *partially ordered topological space*, or simply an *ordered space*, is a triple $(X, \tau, \leq)$ where τ is a topology on X and $\leq$ is a partial order on X. $(X^*, \tau^*, \leq^*)$ is an *ordered compactification* of ordered space $(X, \tau, \leq)$ if (X^*, τ^*) is a topological compactification of (X, τ) and $\leq^*$ extends $\leq$. A (quasi-) ordered space is T_2*-ordered* iff $x \not\leq y$ implies there exists an increasing neighborhood of x disjoint from a decreasing neighborhood of y. Equivalently, a (quasi-) ordered space (X, τ, θ), is T_2-ordered iff θ is closed in $X \times X$. Recall that a *quasi-order* is a transitive reflexive relation. An ordered space has a T_2-ordered compactification iff it is *completely regular ordered* as defined in [6]. We will only consider completely regular ordered spaces and T_2-ordered compactifications. For two ordered compactifications of an ordered space, we write $(X^*, \tau^*, \leq^*) \leq (X', \tau', \leq')$ if $(X^*, \tau^*) \leq (X', \tau')$ as topological compactifications and the canonical quotient map from X' to X^* is increasing.

If $(X, \tau, \leq)$ is an ordered space and (X^*, τ^*) is a topological compactification of (X, τ), we say (X^*, τ^*) is an *orderable compactification* if there exists some order $\leq^*$ on X^* that makes $(X^*, \tau^*, \leq^*)$ an ordered compactification of $(X, \tau, \leq)$. Such an order $\leq^*$ will be called a *compactification order* for (X^*, τ^*). The abundance of compactification orders for a given orderable compactification makes matters difficult. If θ^* and θ' are two compactification orders for (X^*, τ^*), then $(X^*, \tau^*, \theta^*) \leq (X^*, \tau^*, \theta')$ iff $\theta' \subseteq \theta^*$. Thus, if the collection $K_o(X)$ of ordered compactifications of ordered space X is partitioned into equivalence classes under the relation of topological equivalence of compactifications, then the largest member of an equivalence class is the member of that class with the smallest order. Since the intersection of compactification orders on a given orderable compactification is again a compactification order, each equivalence class has a largest member, called an *order-strict compactification*, or following [1], a *Nachbin compactification.* (Elsewhere, the term "Nachbin compactification" is sometimes used for the Stone-Čech ordered compactification). The *Nachbin order* of an orderable compactification (X^*, τ^*) is the intersection of all compactification orders on (X^*, τ^*). The equivalence class of all ordered compactifications of $(X, \tau, \leq)$ topologically equivalent to (X^*, τ^*) has a smallest member iff there is a largest compactification order on (X^*, τ^*), that is, iff the union of compactification orders on (X^*, τ^*) is also a compactification order.

1. Nachbin compactifications

Let (X', τ', θ^*) be a finite-point ordered compactification of completely regularly ordered space (X, τ, θ). We wish to characterize the Nachbin order θ' for (X', τ'). Define a partial order θ' on X' by $a \leq' b$ iff there exist $c_0, c_1, \ldots, c_n \in X'$ with $c_0 = a$ and $c_n = b$, such that for $i = 1, \ldots, n$, there exists a net (x_λ, y_λ) in θ with $(x_\lambda, y_\lambda) \to (c_{i-1}, c_i)$. The set $\{c_0, \ldots, c_n\}$ will be called a *trail* from a to b with $n+1$ *nodes* and *length* n. It is easy to check that θ' is a partial order that extends θ, and $\theta' \subseteq \theta''$ for any θ'' such that (X', τ', θ'') is an ordered compactification of X.

Observe that if $\{c_0, \ldots, c_n\}$ is a trail of length n from c_0 to c_n and c_{i-1}, c_i, $c_{i+1} \in X$ for some i, then the constant net (c_{i-1}, c_{i+1}) guarantees that $\{c_0, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n\}$ is a trail of length $n-1$ from c_0 to c_n. Since there are only a finite number of compactification points to appear as nodes, and since no trail ever needs more than two consecutive nodes from X, we see that there is an upper bound on the number of nodes needed in any trail. We define a *minimal trail* from a to b to be a trail from a to b with minimal length.

Theorem 1.1. The order θ' described above is the Nachbin order for (X', τ').

Proof: It only remains to show that the order θ' is closed. Let S_k be the statement that if (A, B) is any member of $X' \times X'$ such that there exists a net $(A_\gamma, B_\gamma)_{\gamma \in \Gamma}$ in θ' converging to (A, B) with $\max_{\gamma \in \Gamma}$ {length of a minimal trail from A_γ to B_γ} $\leq k$, then $(A, B) \in \theta'$. Suppose $(A, B) \in X' \times X'$ and (A_γ, B_γ) is a net in θ' converging to (A, B) such that $\forall \gamma$, there is a trail of length 1 (or 0) from A_γ to B_γ. Then for each γ, there is a net $(a_\lambda^\gamma, b_\lambda^\gamma)$ in θ converging to (A_γ, B_γ). Now for any open neighborhood N of (A, B) in $X' \times X'$, there exists γ_N such that $(A_{\gamma_N}, B_{\gamma_N}) \in N \cap \theta'$, and thus there exists λ_N such that $(a_{\lambda_N}^{\gamma_N}, b_{\lambda_N}^{\gamma_N}) \equiv (a_{\lambda_N}, b_{\lambda_N}) \in N \cap \theta$. Letting $\mathcal{U}$ be the directed set of open neighborhoods of (A, B), we have a net $(a_{\lambda_N}, b_{\lambda_N})_{N \in \mathcal{U}}$ in θ converging to (A, B), and thus $(A, B) \in \theta'$. This proves S_1. Suppose S_k is true, and suppose (A, B) is the limit of a net $(A_\gamma, B_\gamma)_{\gamma \in \Gamma}$ in θ' such that $\forall \gamma \in \Gamma$ there is a trail $\{C_0^\gamma, C_1^\gamma, \ldots, C_{n_\gamma}^\gamma\}$ of length less than or equal to $k+1$ from A_γ to B_γ. The net $(C_1^\gamma)_{\gamma \in \Gamma}$ of second nodes from these trails must have a convergent subnet $(C_1^{\sigma(\gamma)})_{\gamma \in \Gamma} \to C \in X'$. Now the induction hypothesis applies to the nets $(A_{\sigma(\gamma)}, C_1^{\sigma(\gamma)})$ and $(C_1^{\sigma(\gamma)}, B_{\sigma(\gamma)})$ to imply $(A, C) \in \theta'$ and $(C, B) \in \theta'$, whence $(A, B) \in \theta'$, and thus S_{k+1} is true. ∎

Theorem 1.2. If X_N^* and $X_N^{'}$ are finite-point Nachbin compactifications of X and X^* and X' are ordered compactifications of X topologically equivalent to X_N^* and X_N', respectively, then

a. $X' \leq X^*$ implies $X_N' \leq X_N^*$, and

b. $(X'_N, \tau', \theta'_N) \leq (X^*_N, \tau^*, \theta^*_N)$ iff $(X'_N, \tau') \leq (X^*_N, \tau^*)$.

Proof: For b., suppose $(X'_N, \tau') \leq (X^*_N, \tau^*)$, with continuous function $f : X^*_N \to X'_N$ such that $f\mid_X = id_X$. We will show f is increasing. Suppose $(a, b) \in \theta^*$. Then for a trail $\{c_o, \ldots, c_n\}$ in X^*_N between a and b, there exists nets $(x^i_\lambda, y^i_\lambda)$ in θ converging to (c_{i-1}, c_i) for each $i = 1, \ldots, n$. The nets $(x^i_\lambda, y^i_\lambda)$ must converge in X'_N, say to (d_{i-1}, d_i). Now $\{d_o, \ldots, d_n\}$ forms a trail of length $\leq n$ in X'_N from $f(a)$ to $f(b)$.

The converse is trivial. The proof of part a. is similar. ▮

The following result is immediate.

Corollary 1.3. A set of finite-point Nachbin compactifications has the same poset structure as the set of underlying topological compactifications.

Corollary 1.4. If X admits a 1-point ordered compactification, then the finite-point Nachbin compactifications of X form a lattice that is a complete lower semi-lattice.

Proof: If $\{X'_\gamma\}_{\gamma\in\Gamma}$ is a collection of finite-point Nachbin compactifications of X, $\inf\{X'_\gamma\}_{\gamma\in\Gamma} = \sup\{X^* : X^* \text{ is a lower bound of } \{X'_\gamma\}\}$. Note that this supremum is a finite-point Nachbin compactification since the Nachbin compactifications form a complete upper semilattice (see [1]) and this supremum is bounded above by a finite-point compactification. ▮

2. Unions of orders

Let (X', τ') be a fixed finite-point orderable compactification of $(X, \tau, \leq)$. Among the ordered compactifications topologically equivalent to (X', τ'), there is a smallest iff the union of compactification orders on (X', τ') is a compactification order. We write $\alpha \sqsubset \beta$ and say α is optionally less that β if $\alpha <^* \beta$ in some compactification order $\leq^*$ on (X', τ'). Thus, the graph of $\sqsubseteq$ is the union of the graphs of all compactification orders on (X', τ'). In general, $\sqsubseteq$ is not a partial order. With the goal of determining when $\sqsubseteq$ is a compactification order for (X', τ'), we first consider adding order to a compactification order.

Theorem 2.1. Let (X', θ') be a poset, and define

$$\theta^* = \theta' \cup \{(a, b) : \; a \leq' \alpha, \beta \leq' b\}.$$

a. If $\beta \not\leq' \alpha$, then θ^* is a partial order on X.

b. If (X', τ', θ') is a finite-point ordered compactification of (X, τ, θ) with $\beta \not\leq' \alpha$ and if $x < y$ for every pair $x, y \in X$ such that $x \leq' \alpha$ and $\beta \leq' y$, then (X', τ', θ^*) is a T_2-ordered compactification of X with $\alpha <^* \beta$.

b′. If (X', τ', θ') is a finite-point ordered compactification of (X, τ, θ) with $\beta \not\leq' \alpha$ and if $x \leq y$ for every pair $x, y \in X$ such that $x \leq' \alpha$ and $\beta \leq' y$, then (X', τ', θ^*) is a T_2-ordered compactification of X with $\alpha \leq^* \beta$.

Proof: a. is straightforward. For b., note that the condition $x < y$ when $x \in d'(\alpha) \cap X$ and $y \in i'(\beta) \cap X$ guarantees that θ^* introduces no new order on the underlying poset X, i.e., that $\theta^* \cap X^2 = \theta$. To see that (X', τ', θ^*) is T_2-ordered, suppose $x \not\leq^* y$. Then $x \not\leq' y$, so there exists a θ'-increasing neighborhood N of x disjoint from a θ'-decreasing neighborhood M of y. Because θ' is T_2-ordered and X' is a finite-point compactification, we may assume, without loss of generality, that $\mu \in (N \backslash X) \cup \{\alpha, \beta\}$ iff $x \leq' \mu$, and $\mu \in (M \backslash X) \cup \{\alpha, \beta\}$ iff $\mu \leq' y$. Now $i^*(N)$ and $d^*(M)$ are θ^*-monotone neighborhoods of x and y, respectively. Suppose $z \in i^*(N) \cap d^*(M)$. Then there exist $a \in N$ and $b \in M$ with $a \leq^* z \leq^* b$. This implies either (1) $a \leq' z$ or (2) $a \leq' \alpha, \beta \leq' z$; and either (a) $z \leq' b$ or (b) $z \leq' \alpha, \beta \leq' b$. Case (1a) implies $a \leq' b$, contrary to $M \cap N = \emptyset$. In case (2b), $a \in N$ and $a \leq' \alpha$ imply $\alpha \in N$, and thus $x \leq' \alpha$ by our choice by N. Similarly, we have $\beta \leq' y$. But $x \leq' \alpha$ and $\beta \leq' y$ imply $x \leq^* y$, a contradiction. The remaining two cases imply $\alpha \in N$ and $\beta \in M$, whence $x \leq' \alpha, \beta \leq' y$, contrary to $x \not\leq^* y$. Thus, $i^*(N)$ and $d^*(M)$ are disjoint θ^*-monotone neighborhoods separating x and y, so θ^* is T_2-ordered. ▮

Theorem 1.1 gave an intrinsic characterization of the Nachbin order on a finite-point orderable compactification. We now utilize this Nachbin order to characterize the relation $\sqsubset$.

Theorem 2.2. Let (X', τ', θ') be a finite-point Nachbin compactification of (X, τ, θ). Then $\alpha \sqsubset \beta$ iff $\beta \not\leq' \alpha$ and $x < y$ for every pair $x, y \in X$ such that $x \leq' \alpha$ and $\beta \leq' y$. Also, $\alpha \sqsubseteq \beta$ iff $\beta \not<' \alpha$ and $x \leq y$ for every pair $x, y \in X$ such that $x \leq' \alpha$ and $\beta \leq' y$.

Proof: If $\alpha \sqsubset \beta$, then $\alpha <^* \beta$ in some compactification order θ^* on (X', τ'). Now $\beta \leq' \alpha$ would lead to the contradiction that $\alpha <^* \beta \leq^* \alpha$. Also if $x \in d'(\alpha) \cap X$ and $y \in i'(\beta) \cap X$, then because $\theta' \subseteq \theta^*$, we have $x \leq^* \alpha, \beta \leq^* y$. Now $\alpha <^* \beta$ implies $x <^* y$, and since $\theta^* \cap X^2 = \theta$, we have $x < y$.

The converse follows from 2.1.b. ▮

In particular, if $x \in X$ and $\omega \in X' \backslash X$, then $x \sqsubset \omega$ iff $x < y$ for every $y \in X$ with $\omega <' y$, and $\omega \sqsubset x$ iff $y < x$ for every $y \in X$ with $y <' \omega$. Or, letting $A^{\uparrow}$ represent the set of strict upper bounds of A in X, and defining $A^{\downarrow}$ dually, we have $x \sqsubset \omega$ iff $x \in [i'(\omega) \cap X]^{\downarrow}$ and $\omega \sqsubset x$ iff $x \in [d'(\omega) \cap X]^{\uparrow}$. A direct consequence of Theorem 2.2 is the result below.

Corollary 2.3. If $X' = X \cup \{\omega\}$ is a one-point ordered compactification, then $\sqsubseteq$ is a partial order iff $[i'(\omega) \cap X]^{\downarrow} < [d'(\omega) \cap X]^{\uparrow}$, that is, iff $[i'(\omega) \cap X]^{\downarrow} \times [d'(\omega) \cap X]^{\uparrow} \subseteq \theta \backslash \triangle$.

Now let us consider some strong restrictions on a Nachbin compactification $(X', \tau', \leq')$. For $\alpha \in X'$, we will say α is *accessible from above* if there exists a net a_λ in $i'(\alpha) \cap X$ converging to α. Accessibility from below is defined dually. Note that points from X are accessible from above and below. If $x \in X$ and $\alpha \in X' \backslash X$, then by Theorem 2.2, $x \sqsubseteq \alpha$ iff (a) $x \leq a \ \ \forall\, a \in X$ with $\alpha \leq' a$, and (b) $\alpha \not\leq' x$. If α is accessible from above, then (a) implies $x \leq' \alpha$, which in turn implies (b), and thus $x \sqsubseteq \alpha$ is equivalent to $x \leq' \alpha$. Similarly, if β is accessible from below, then $\beta \sqsubseteq y$ iff $\beta \leq' y$. If α is accessible from below and β from above, then $\alpha \sqsubseteq \beta$ is equivalent to $\beta \not\leq' \alpha$ and $x \leq y$ whenever $x, y \in X$ with $x \leq' \alpha, \beta \leq' y$. By the accessibility assumptions, this last condition implies $\alpha \leq' \beta$. Thus, when α is accessible from below and β from above, $\alpha \sqsubseteq \beta$ iff $\alpha \leq' \beta$. As a result, if every compactification point is accessible from above and from below, then $\sqsubseteq$ agrees with $\leq'$, so that X has a unique compactification order.

We mention two more special cases in which it can easily be determined whether $\sqsubseteq$ is a partial order or not. If $(X', \tau', \leq')$ has an order-isolated compactification point, i.e., a compactification point ω such that $\omega \not\leq' \alpha$ and $\alpha \not\leq' \omega$ for all $\alpha \in X'$, then $\sqsubseteq$ is not a partial order, for by Theorem 2.1.b. we could make ω a maximum or a minimum element. Also, if $\alpha \sqsubset \beta$ implies there exists $x \in X$ with $\alpha \leq' x \leq' \beta$, then $\sqsubseteq$ is a partial order.

Suppose (X^*, τ^*, θ^*) and (X', τ', θ') are finite-point Nachbin compactifications of (X, τ, θ) with $(X', \tau') \leq (X^*, \tau^*)$. By 1.2.b, the canonical quotient map $f : X^* \to X'$ is increasing and thus $i^*(\beta) \cap X \subseteq i'(f(\beta)) \cap X$ and $d^*(\alpha) \cap X \subseteq d'(f(\alpha)) \cap X$. From Theorem 2.2, it follows that $f(\alpha) \sqsubset' f(\beta)$ implies $\alpha \sqsubset^* \beta$, where $\sqsubseteq'$ is the union of compactification orders in (X', τ') and $\sqsubseteq^*$ is similarly defined.

Theorem 2.4. If S is the set of all finite-point Nachbin compactifications X^* of $(X, \tau, \leq)$ such that the union $\sqsubseteq^*$ of compactification orders on X^* is a partial order, the S is a decreasing set in the poset $K_{fpn}(X)$ of finite-point Nachbin compactifications of X.

Proof: Suppose X^* and X' are finite-point Nachbin compactifications of X with $X' \leq X^*$, and suppose $\sqsubseteq^*$ is a partial order. We will show $\sqsubseteq'$ is a partial order. Clearly $\sqsubseteq'$ is reflexive, and antisymmetry of $\sqsubseteq'$ follows from the transitivity of $\sqsubset'$, which we now show. Suppose $f(\alpha) \sqsubset' f(\beta)$ and $f(\beta) \sqsubset' f(\gamma)$. If $f(\alpha) \not\sqsubset' f(\gamma)$, then there exists $x, y \in X$ such that $x \not<' y$, yet $x <' f(\alpha)$ and $f(\gamma) <' y$. In particular, $x \sqsubset' f(\alpha)$, $f(\alpha) \sqsubset' f(\beta)$, $f(\beta) \sqsubset' f(\gamma)$, and $f(\gamma) \sqsubset' y$. Since $f(\omega) \sqsubset' f(\nu)$ implies $\omega \sqsubset^* \nu$ and since $\sqsubseteq^*$ is a partial order, we arrive at the contradiction $x < y$. ∎

The example below illustrates that the set S described in Theorem 2.4 need not be increasing in $K_{fpn}(X)$, and that S is, in general, a proper subset of $K_{fpn}(X)$.

Example 2.5. For $y \in \mathbf{R}$, define the following subsets of $\mathbf{R}^2$: $r(y) = (0,1]\times\{y\}$, $R(y) = [0,1]\times\{y\}$, $l(y) = [-1,0)\times\{y\}$, $L(y) = [-1,0]\times\{y\}$, and $ll(y) = [-2,-1) \times \{y\}$. Unless otherwise indicated, all ordered spaces and compactifications described with this notation carry the subspace topology inherited from $\mathbf{R}^2$ and the "due north" order $(a,b) \leq (c,d)$ iff $a = c$ and $b \leq d$.

Define $X = R(0) \cup r(1) \cup l(1) \cup R(3)$. Let $X' = X \cup \{(0,1)\}$ be the 1-point Nachbin compactification of X, and let X^* be the two-point Nachbin compactification $X^* = R(0) \cup r(1) \cup ll(1) \cup R(3) \cup \{\mu_1, \mu_2\}$ where $\mu_1 = (-1,1)$ and $\mu_2 = (0,1)$ are the compactification points. The order $\leq'$ on X' is the only compactification order for X', so $\sqsubseteq'$ is a partial order. As Nachbin compactifications, $X' \leq X^*$, but $\sqsubseteq^*$ is not a partial order: The order compactification $R(0) \cup r(1) \cup R(3) \cup l(4) \cup \{\mu_1, \mu_2\}$ where $\mu_1 = (0,1), \mu_2 = (0,4)$ and $x = (0,3)$, and the ordered compactification $R(0) \cup r(1) \cup l(2) \cup R(3) \cup \{\mu_1, \mu_2\}$ where $\mu_1 = (0,2), \mu_2 = (0,1)$, and $x = (0,3)$, are both topologically equivalent to X^* and illustrate that $\sqsubseteq^*$ is not transitive since $x \sqsubset^* \mu_1$ and $\mu_1 \sqsubset^* x$. ∎

If the union $\sqsubseteq$ of compactification orders on (X', τ') is a closed partial order, then there is a smallest ordered compactification topologically equivalent to (X', τ'), namely $(X', \tau', \sqsubseteq')$. The example below shows that the union $\sqsubseteq$ of compactification orders on (X', τ') need not be closed.

Example 2.6. With the notation of Example 2.5, let $X = r(0) \cup r(1)$. For $a \in (0,1]$, let $X_a = X \cup \{\alpha, \beta\}$ be the two point ordered compactification of X with $(a,0) < (a,1) <_a \beta <_a \alpha$. Now if $(a_\lambda, 1)$ is a net in $r(1)$ converging to $\alpha = (0,1)$, then $(a_\lambda, 1) \sqsubseteq \beta$ for each λ, yet $\alpha \not\sqsubseteq \beta$, and thus $\sqsubseteq$ is not closed. ∎

The following proposition shows that the situation in the previous example is essentially the only way $\sqsubseteq$ could fail to be closed.

Proposition 2.7. The union $\sqsubseteq$ of compactification orders on a finite-point Nachbin compactification $(X', \tau', \leq')$ is a closed relation on X' iff for $\beta \leq' \alpha$ in X', there is no net α_λ in $X' \backslash \{\alpha\}$ converging to α with $\alpha_\lambda \sqsubset \beta$ for each λ and no net β_λ in $X' \backslash \{\beta\}$ converging to β with $\alpha \sqsubset \beta_\lambda$, for each λ.

Proof: The former condition above clearly implies the latter, so assume the latter. Let (α, β) be the limit of a net $(\alpha_\lambda, \beta_\lambda)$ where $\alpha_\lambda \sqsubset \beta_\lambda$ for all λ. We will use Theorem 2.2 to show $\alpha \sqsubset \beta$. First, $\beta \not\leq' \alpha$, for otherwise there would exist a θ'-increasing τ'-open neighborhood N of α disjoint from a θ'-decreasing τ'-open neighborhood M of β. The assumed condition guarantees that there exists λ_0 such that $(\alpha_{\lambda_0}, \beta_{\lambda_0}) \in (N \times M) \cap \theta$, contrary to M and N being disjoint. Next, $x \leq y$ whenever $(x, y) \in [d'(\alpha) \times i'(\beta)] \cap X^2$, for otherwise there exists a θ'-increasing neighborhood N of x disjoint from a θ'-decreasing neighborhood M of y. But $(x, y) \in [d'(\alpha) \times i'(\beta)]$ implies $\alpha \in N$ and $\beta \in M$, and the argument above again contradicts $M \cap N = \emptyset$. ∎

3. DT Compactifications

Let $\sqsubseteq$ represent the union of compactification or ders on a finite-point Nachbin compactification $(X', \tau', \leq')$, and consider the following statements about $\alpha, \beta \in X'$.

3.1.a. $\alpha \sqsubseteq \beta$

3.1.b. Either $\alpha = \beta$ or $x < y$ for every $x, y \in X$ with $x \sqsubseteq \alpha, \beta \sqsubseteq y$.

3.1.c. $\beta \not\leq' \alpha$.

Using Theorem 2.2, it is easily shown that 3.1.b. and 3.1.c. together imply 3.1.a. We will say $\sqsubseteq$ is *determined by transitivity*, or *DT*, if for each $\alpha, \beta \in X'$, the following implications hold: 3.1.a. $\Longrightarrow$ 3.1.b. $\Longrightarrow$ 3.1.c. A finite-point ordered compactification $(X', \tau', \leq^*)$ will be called a *DT compactification* if the union $\sqsubseteq$ of compactification orders on (X', τ') is DT. Note that in a DT compactification, 3.1.a. is equivalent to 3.1.b. In this section, we show that $\sqsubseteq$ being DT is closely related to $\sqsubseteq$ being a partial order. But first, we have an example to illustrate that the DT compactifications form a proper subset of the finite-point ordered compactifications.

Example 3.2. (3.1.a. $\not\Longrightarrow$ 3.1.b): With the notation and conventions of Example 2.5, let $X = R(0) \cup R(1) \cup r(2) \cup l(3)$. From the two-point ordered compactification $X^* = X \cup \{\alpha, \beta\}$ where $\alpha = (0, 2)$ and $\beta = (0, 3)$, we see that $\alpha \sqsubset \beta$. However, if $x = (0, 1)$ and $y = (0, 0)$, the two-point ordered compactification X^{**} depicted by $R(0) \cup R(1) \cup r(2) \cup l(-1) \cup \{\alpha, \beta\}$ where $\alpha = (0, 2)$ and $\beta = (0, -1)$ is topologically equivalent to X^* and shows that $x \sqsubset \alpha, \beta \sqsubset y$ does not imply $x < y$. ∎

If $\sqsubseteq$ is a partial order, then clearly $\sqsubseteq$ is DT. The converse need not be true, as seen by this example. Let $X = \{(-1, y) \in \mathbf{R}^2 : y \in [0, 1)\} \cup \{(1, y) \in \mathbf{R}^2 : y \in [0, 1)\} \cup \{(0, 2)\}$ have the topology inherited from $\mathbf{R}^2$. Impose the 'due north' order on the two segments of X, and put $x \leq (0, 2) = g$ for every $x \in X$. The two-point Nachbin compactification of X is $X \cup \{\alpha, \beta\}$ where $\alpha = (-1, 1)$ is greater than each point of the left segment, $\beta = (1, 1)$ is greater than each point of the right segment, and g remains the greatest element. Topologically equivalent two-point ordered compactifications X^* and X^{**} of X can be obtained by additionally imposing $\alpha <^* \beta$ and $\beta <^{**} \alpha$.

Thus, $\alpha \sqsubseteq \beta$ and $\beta \sqsubseteq \alpha$ even though α and β are distinct. Thus $\sqsubseteq$ fails to be antisymmetric even though $\sqsubseteq$ is DT.

Proposition 3.3. If $\sqsubseteq$ is DT and antisymmetric, then it is a partial order.

Proof: Since $\sqsubseteq$ is clearly reflexive, only transitivity remains to be checked. Suppose $\alpha \sqsubset \gamma$ and $\gamma \sqsubset \beta$. If all three of the points α, β, γ are from the space X, then clearly $\alpha \sqsubset \beta$. One can easily show that transitivity holds when the middle point γ is from X. If $x, y \in X$ and $x \sqsubset \gamma, \gamma \sqsubset y$, then $x < y$ follows from defining condition 3.1.a. $\Longrightarrow$ 3.1.b. of $\sqsubseteq$ being DT, applied to $\gamma \sqsubseteq \gamma$. If $x \sqsubset \gamma, \gamma \sqsubset \beta$, where $x \in X$, then to show $x \sqsubset \beta$, it suffices to show $x < y$ for any $y \in X$ such that $\beta <' y$. But $\beta <' y$ implies $\beta \sqsubset y$. Now $x \sqsubset \gamma, \beta \sqsubset y$, and the definition of DT applied to $\gamma \sqsubseteq \beta$ imply $x < y$. The dual argument also holds, leaving only the case $\alpha \sqsubseteq \gamma, \gamma \sqsubseteq \beta$ where α, β and γ are all compactification points. Recalling the equivalence of 3.1.a. and 3.1.b. in a DT compactification, it suffices to show 3.1.b, that is, for $x, y \in X, x < y$ whenever $x \sqsubseteq \alpha$ and $\beta \sqsubseteq y$. If $x, y \in X$ with $x \sqsubseteq \alpha$ and $\beta \sqsubseteq y$, by the preceding case, $x \sqsubseteq \alpha$ and $\alpha \sqsubseteq \gamma$ imply $x \sqsubseteq \gamma$, and dually, $\gamma \sqsubseteq y$. Applying the DT property 3.1.a. $\Longrightarrow$ 3.1.b. to $\gamma \sqsubseteq \gamma$ shows $x < y$. ▮

In determining whether $\sqsubseteq$ is a partial order, we see that if $\sqsubseteq$ is DT, transitivity would hold if antisymmetry does, but antisymmetry need not hold. Since $\sqsubseteq$ agrees with $\leq$ on X, antisymmetry cannot fail for two points from X. The DT property guarantees that antisymmetry holds for any pair $x \in X, \alpha \in X' \backslash X$, for $x \sqsubset \alpha$ implies $x < y$ whenever $\alpha \sqsubset y$ $(y \in X)$, and thus $x \sqsubset \alpha, \alpha \sqsubset x$ would lead to the contradiction $x < x$. Thus, antisymmetry of $\sqsubseteq$ in a DT compactification can only fail between two compactification points. We will show that *if $\sqsubseteq$ is DT but fails to be antisymmetric, then two offending compactification points can be identified*. The needed ordered quotient construction is provided below.

Let $(X, \tau, \leq)$ be an ordered topological space with a finite-point orderable compactification (X', τ'). Let $\leq'$ be a T_2-ordered quasi-order on X' that extends $\leq$. Suppose $\{\omega_i\}_{i \in A}$ is a (finite) set of compactification points from X'. Identify the set $\{\omega_i\}_{i \in A}$ into a single point ω, and define a relation $\leq''$ on this quotient space (X'', τ'') by

$a <'' \omega$ iff $a <' \omega_i$ for some $i \in A$ $(a \in X'' \backslash \{\omega\})$
$\omega <'' a$ iff $\omega_i <' a$ for some $i \in A$ $(a \in X'' \backslash \{\omega\})$
$a \leq'' b$ iff $a \leq' b$ OR $a \leq' \omega_i, \omega_j \leq' b$ for some $i, j \in A$ $(a, b \in X'' \backslash \{\omega\})$
$\omega \leq'' \omega$

Some of the properties of this "ordered quotient space" are summarized below.

Theorem 3.4.

a. The relation $\leq''$ is a T_2-ordered quasi-order on X''.

b. The quotient map from X' to X'' is increasing

c. If $\{\omega_i\}_{i\in A}$ is $\leq'$-convex, then $\leq''$ is antisymmetric whenever $\leq'$ is.

d. If $(X', \tau', \leq')$ is a DT compactification, then $\leq''$ introduces no new order on X.

Proof: We only show $\leq''$ is T_2-ordered; the other verifications are routine. Suppose $a \not\leq'' b$, and first consider the case $a, b \neq \omega$. Then $a \not\leq' b$, so there exists a $\leq'$-increasing neighborhood N of a disjoint from a $\leq'$-decreasing neighborhood M of b. Without loss of generality, $\omega_i \in N$ iff $a \leq' \omega_i$ and $\omega_i \in M$ iff $\omega_i \leq' b$. Now $N \cap \{\omega_i\}_{i\in A} = \emptyset$ or $M \cap \{\omega_i\}_{i\in A} = \emptyset$, for otherwise $a \leq' \omega_i, \omega_j \leq' b$ leads to the contradiction $a \leq'' b$. We will consider the case $N \cap \{\omega_i\}_{i\in A} = \emptyset$. Let $N'' = N$, and $M'' = [M \backslash \{\omega_i\}_{i\in A}] \cup \{\omega\}$. Now $i''(N'')$ and $d''(M'')$ are $\leq''$-monotone neighborhoods of a and b. We will show that they are disjoint. Suppose $c \in i''(N'') \cap d''(M'')$. Since $\omega_i \notin N$ for all i, we have $\omega \notin i''(N'')$. In particular $c \neq \omega$. Now by choice of c, there exist $n \in N'', m \in M''$ with $n \leq'' c$ and $c \leq'' m$. Since $c, n \neq \omega$, we have two cases: either $m = \omega$ or $m \neq \omega$. If $m = \omega$, we have $n \leq'' c, c \leq'' \omega$. Since $\leq''$ is transitive, this would give the contradiction $n \leq'' \omega$. If $m \neq \omega$, then $n \leq'' c, c \leq'' m$ imply either (1) $n \leq' c$ or (2) $n \leq' \omega_i, \omega_j \leq' c$ for some $i, j \in A$; and either (a) $c \leq' m$ or (b) $c \leq' \omega_k, \omega_l \leq' m$ for some $k, l \in A$. Case (1a) implies $n \leq' m$, contrary to our choice of M and N. In case (1b), $n \leq' c \leq' \omega_k, \omega_l \leq' m$ implies $\omega \in i''(N'')$, a contradiction. Cases (2a) and (2b) also lead to this contradiction.

Now we return to the case $a \not\leq'' \omega$. Then $a \not\leq' \omega_i$ for all $i \in A$, so $\forall i \in A$, there exists a $\leq'$-increasing neighborhood N_i of a disjoint from a $\leq'$-decreasing neighborhood M_i of ω_i. Let $N = N'' = \bigcap_{i\in A} N_i$, $M = \bigcup_{i\in A} M_i$, and $M'' = [M \backslash \{\omega_i\}_{i\in A}] \cup \{\omega\}$. Notice that $i''(N'')$ and $d''(M'')$ are $\leq''$-monotone neighborhoods of a and ω respectively, and $\omega \notin i''(N'')$. The argument of the paragraph above shows that $i''(N'')$ and $d''(M'')$ are disjoint. The case $\omega \not\leq'' a$ is similar, and this completes the proof. ∎

Note that if $\leq''$ is antisymmetric and extends $\leq$, then $(X'', \tau'', \leq'')$ is a T_2-ordered compactification of X. Thus, from 3.4.c. and 3.4.d. we see that if we identify a convex set of compactification points from a DT compactification, this construction yields a smaller ordered compactification. We note a particular case of this in the corollary below.

Corollary 3.5. If $(X', \tau', \leq')$ is a finite-point Nachbin compactification of $(X, \tau, \leq)$ with $\omega_1, \omega_2 \in X' \backslash X$ such that $\omega_1 \sqsubset \omega_2$ and $\omega_2 \sqsubset \omega_1$, then the ordered quotient space $(X'', \tau'', \leq'')$ obtained by identifying ω_1 and ω_2 is a finite-point ordered compactification of $(X, \tau, \leq)$.

Since every finite-point orderable compactification X' has a Nachbin order, the corollary above guarantees that identifying two points at which antisymmetry of $\sqsubseteq$ fails always results in a smaller orderable compactification of $(X, \tau, \leq)$. Thus, we may speak of the union $\sqsubseteq''$ of compactification orders on $(X'', \tau'', \leq'')$, where X'' is the ordered quotient obtained by identifying a pair of non-antisymmetric compactification points from X'.

If X' is a DT compactification for which $\sqsubseteq$ is not a partial order, then there must be compactification points in X' at which $\sqsubseteq$ is not antisymmetric. Can these compactification points be identified in the equivalent Nachbin compactifications, two at a time, until we arrive at a smaller ordered compactification $X^{(n)}$ of X for which the union $\sqsubseteq^{(n)}$ of its compactification orders is antisymmetric, and therefore a partial order? The answer would be affirmative if our ordered quotient construction preserved the DT property, and could thereby the repeated. It can be shown that if $\omega_1 \sqsubset \omega_2$ and $\omega_2 \sqsubset \omega_1$ in X', if X'' is the ordered quotient formed by identifying ω_1 and ω_2, and if 3.1.b. implies 3.1.c. in X'', then $\alpha \sqsubseteq'' \beta$ in X'' iff $\alpha \sqsubseteq \beta$ in X', and in particular, X'' is DT.

In summary, Proposition 3.3 shows that $\sqsubseteq$ being DT is equivalent to $\sqsubseteq$ being a partial order if and only if $\sqsubseteq$ is antisymmetric. If X' is a DT compactification and in each successive ordered quotient 3.1.b. implies 3.1.c., then there exists an ordered compactification $X^{(n)}$ of X such that $X^{(n)} \leq X'$ and $\sqsubseteq^{(n)}$ is a partial order. In particular, if we define a *strong DT space* to be a DT space in which 3.1.b. implies 3.1.c. in all successive ordered quotients in which convex sets of compactification points are identified, then every strong DT space is larger than some ordered compactification $X^{<n>}$ which is smallest among the equivalence class of ordered compactifications topologically equivalent to it. An open question significant to this discussion is whether every DT space is strongly DT.

4. Lattice considerations

The existence of a smallest ordered compactification of $(X, \tau, \leq)$ would imply that the poset $K_o(X)$ of ordered compactifications of X forms a complete lattice. If the set $K_{fpo}(X)$ of finite-point ordered compactifications of X has a smallest member, then $K_{fpo}(X)$ is a lattice and a complete lower semilattice. If $(X', \tau', \leq')$ and $(X', \tau', \leq^*)$ are topologically equivalent ordered compactifications of X, then the larger ordered-compactification is the one with the smaller order. We can now apply some of our results on unions of compactification orders to the question of determining when X has a smallest finite-point ordered compactification. Existence of a one-point ordered compactification is neither necessary nor sufficient for the existence of a smallest ordered compactification, as seen by the intervals (0,1) and

[0,1) in $\mathbf{R}$, given the usual topology and order. The smallest ordered compactification of (0,1) is a two-point compactification, and by taking the compactification point of [0,1) to be maximum in one and minimum in another one-point ordered compactification, we see that [0,1) has no smallest ordered compactification. But first, we present some lattice results on the two extreme cases of partial orders: when all orders involved are equality, and when the order is total. In the former case, we are simply considering topological spaces.

Theorem 4.1. Let (X, τ) be a locally compact topological space, and let $K_{fp}(X)$ be the poset of finite-point compactifications of X. Then the following are equivalent.

a. X has a largest finite-point compactification X^*.

b. $K_{fp}(X)$ is a complete lattice.

c. X has no countable compactification (i.e., no compactification X' such that $X' \backslash X$ is countably infinite).

Proof: a. $\Longrightarrow$ b: Since $K(X)$ is a complete lattice, $K_{fp}(X)$ is closed under the formation of arbitrary infima. Also, $\inf\{X' \in K(X) : X'' \leq X' \text{ for all } X'' \in C\}$ exists and is a finite-point compactification necessarily less than or equal to X^*. This infimum is $\sup C$.

b. $\Longrightarrow$ c: If $K_{fp}(X)$ is a complete lattice, then there is no n-point compactification of X for $n > \mid \sup(K_{fp}(X)) \backslash X \mid$. By Theorem 2.1 of [3], X has no countable compactification.

c. $\Longrightarrow$ a: Suppose X has no countable compactification. McCartney has shown (2.1 and 2.4ff of [5]) that X has a maximum totally disconnected compactification X', and that $X' = \sup\{K_{fp}(X)\}$. By Theorem 2.1 of [3], every totally disconnected compactification of X is a finite-point compactification. Thus, X' is a finite-point compactification, and the largest one. ▮

Of course, the simple example of the real line shows that the largest finite-point compactification need not be βX, the largest compactification.

In case $(X, \tau, \leq)$ is a totally ordered topological space, the poset $K_o(X)$ of ordered compactifications of X is always a complete lattice, and either $K_{fpo}(X) = K_o(X)$ or $K_{fpo}(X) = \emptyset$. In the latter case, all members of $K_o(X)$ are of the same cardinality. Details of totally ordered compactifications can be found in [2].

We now return to the general partially ordered case.

Proposition 4.2. If X^* and X^{**} are finite-point ordered compactifications of X with $X^{**} \leq X^*$ and if $\sqsubseteq^*$ is a closed partial order, then $(X^*, \tau^*, \sqsubseteq^*)$ is a "strong order lifting" of X^{**}, in the sense that for any $a, b \in X^{**}$ with $a <^{**} b$, and for any $a^* \in \phi^{-1}(a)$, $b^* \in \phi^{-1}(b)$ where $\phi : X^* \to X^{**}$ is the canonical increasing quotient map, we have $a^* \sqsubset^* b^*$.

Proof: Suppose $a <^{**} b$ in X^{**}. Since $\phi : X^* \to X^{**}$ is increasing, $b^* \not\leq^* a^*$ for any $b^* \in \phi^{-1}(b), a^* \in \phi^{-1}(a)$. Thus, to see $a^* \sqsubseteq^* b^*$, we must show that $x \leq y$ for any $x, y \in X$ such that $x \leq^*_{\min} a^*, b^* \leq^*_{\min} y$, where $\leq^*_{\min}$ is the Nachbin order on X^*. Given such x and y, it follows from the hypothesis and the fact that ϕ is increasing that $x \leq^{**} a, a <^{**} b$, and $b \leq^{**} y$, whence $x < y$. Thus $a^* \sqsubseteq b^*$. Clearly $a^* \neq b^*$ since their images under ϕ are not equal. ∎

An already obvious result of this proposition is that if $(X^*, \tau^*, \leq^*)$ is the supremum of a collection of finite-point ordered compactifications of X and $\sqsubseteq^*$ is a closed partial order, then $\leq^*$ is $\sqsubseteq^*$. The proposition could be rephrased to say that, under the hypotheses, X^{**} is an 'ordered quotient space' of X^* in the sense of McCartan [4].

We conclude with some conditions sufficient to insure that the infimum of a collection of finite-point ordered compactifications exists.

Theorem 4.3. Suppose $\sup\{X_i^{'} : i \in A\} = X^*$, X^* is a finite-point strong DT ordered compactification, and the union $\sqsubseteq^*$ of compactification orders on X^* is a closed partial order. For each $i \in A$, let $\phi_i : X^* \to X_i^{'}$ be the canonical increasing quotient map, and let $\mathcal{F}_i = \{F \subseteq X^* : F = \phi_i^{-1}(\alpha)$ for some $\alpha \in X_i^{'}\}$. Define an equivalence relation $\sim_A$ on X^* by $\nu \sim_A \eta$ iff there exists a finite chain $\{\omega_j\}_{j=o}^{m}$ with $\omega_o = \nu$ and $\omega_m = \eta$ such that for each $j = 1, 2, \ldots, m$, there exists $F \in \bigcup_{i \in A} \mathcal{F}_i$ with $\{\omega_{j-1}, w_j\} \subseteq F$. If the $\sim_A$ - equivalence classes of X^* are convex relative to $\sqsubseteq^* = \leq^*$, then $\inf\{X_i^{'} : i \in A\}$ exists.

Proof: Note that $X^*/\sim_A$ is the topological compactification infimum of $\{X_i^{'} : i \in A\}$. Since X^* is a strong DT space and the $\sim_A$ - equivalence classes are convex, they can be identified one at a time using the ordered quotient construction described in Theorem 3.4. Thus, the topological compactification $X^*/\sim_A$ can be ordered to get an ordered compactification which we will call X^o. For each $i \in A$, clearly $X^o \leq X_i^{'}$ as topological compactifications, so it only remains to show that the quotient map $\psi_i : X_i^{'} \to X^o$ is increasing. Note that identification of the points of X^* to get to X^o can be performed in such a way that $X_i^{'}$ is one of the intermediate steps. By Theorems 2.4 and 3.4, the union $\sqsubseteq_i^{'}$ of compactification orders on $X_i^{'}$ is a closed partial order, and now Proposition 4.2 applied to X^* and $X_i^{'}$ implies ψ_i is increasing. ∎

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