Every finite topology is generated by a partial pseudometric

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Abstract: Given any preorder $\preceq$ on a finite set X, we present an algorithm to construct a partial pseudometric p on X which generates $\preceq$ in the sense that $a \preceq b$ if and only if $p(a,b) \leq p(a,a)$. The specialization topology generated by $\preceq$ agrees with the topology generated by the partial pseudometric p-balls, and consequently any topology on X is generated by a partial pseudometric.

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In Euclidean geometry and the theory of metric spaces, a point x has no length or width, and the distance from x to x is zero. In practice, particularly in computer applications, we must use representations of points which are not exact, and one approximation (whether a single pixel on a screen, or a truncated decimal such as 3.14) may represent many different exact values. The distances between these exact values represented by a single approximation a suggest the consideration of metrics allowing nonzero distances from a to a. The self-distance $d(a,a)$ essentially gives a measure of the ambiguity of the point a. Matthews [5] quantified these notions by introducing *partial metrics* which relax the metric restriction that $d(x,x) = 0$ and adjust the triangle inequality accordingly.

Definition 1 *A* partial metric *on X is a function $p : X \times X \to [0,\infty)$ such that* *(a) $p(x,y) \geq p(x,x)$ for all $x, y \in X$. (Small self-distances)*

(b) $p(x,y) = p(y,x)$ for all $x, y \in X$. (Symmetry)

(c) $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$ for all $x, y, z \in X$. (Triangle inequality)

(d) $x = y$ if and only if $p(x,y) = p(x,x) = p(y,y)$.

In this note, we consider *partial pseudometrics* on X, defined by Heckmann [2] as functions $p : X \times X \to [0, \infty)$ satisfying conditions (a), (b), and (c) of Definition 1.

Any partial pseudometric p on X generates a metric d on X defined $d(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ and a preorder $\preceq$ on X defined by $a \preceq b$ if and only if $p(a,b) \leq p(a,a)$. The specialization topology on X generated by $\preceq$ consists of the $\preceq$-increasing sets (i.e., sets B for which $b \in B$ and $b \preceq a$ imply $a \in B$) and agrees with the topology τ generated by the basis of open balls $\{B(a,r) : a \in X, r \in (0,\infty)\}$ where $B(a,r) = \{z \in X : p(a,z) < p(a,a) + r\}$ (see [2]). Conversely, the specialization preorder $\preceq$, defined from τ by $a \preceq b$ if and only if $a \in cl\{b\}$, agrees with the partial pseudometric preorder $a \preceq b$ if and only if $p(a,b) \leq p(a,a)$.

A *pseudometric* on X is a partial pseudometric p on X which also satisfies $p(x,x) = 0$ for every $x \in X$. Thus, a pseudometric is a metric without the condition that $p(x,y) = 0$ implies $x = y$. Not every topology on a finite set is generated by a pseudometric. Erné and Stege note in [1] that the following are equivalent for a topology τ on a finite set X: (1) (X,τ) is pseudometrizable, (2) (X,τ) is regular, (3) (X,τ) is completely regular, (4) the specialization order $\preceq$ generated by τ is an equivalence relation.

Dropping the small self-distances condition (a) from the definition of a partial metric or partial pseudometric gives, respectively, a *weak partial metric* or *weak partial pseudometric*. Heckmann shows that any topology arising from a weak partial (pseudo)metric also arises from a partial (pseudo)metric.

A *partial quasimetric* is obtained by dropping the symmetry condition of a partial metric and replacing (a) in Definition 1 by (a′): $p(x,x) \leq p(x,y)$ and $p(x,x) \leq p(y,x)$ for all $x, y \in X$, and replacing (d) by (d′): $x = y$ if and only if $(p(x,x) = p(x,y)$ and $p(y,y) = p(y,x))$ for any $x, y \in X$. Topologies arising from partial quasimetrics are investigated in [4].

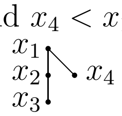
Because the range of a (partial-, pseudo-, quasi-) metric d is $[0,\infty)$ and because $0 < r < s$ implies $B(a,r) \subseteq B(a,s)$, the topology generated by

d-balls must be first countable. Consequently, there is no hope to represent every topology on an arbitrary set X by a partial metric with range $[0, \infty)$. However, it is shown in [3] that every topology is generated by a partial metric if we allow the range to be a value quantale.

We will show that every finite topology arises from a partial pseudometric. We start by considering topologies on a finite set $X = \{x_1, \dots, x_n\}$. Recall that any such topology is characterized by its specialization preorder. We will present an algorithm which shows that any preorder on a finite set X is determined by an appropriately constructed partial pseudometric. Because the ball topology defined by the partial pseudometric agrees with the specialization topology of the preorder determined by the topology, this will give the desired result.

Before presenting the algorithm, we give an example which provides the motivation for the algorithm. For brevity, we will denote $p(x_i, x_j)$ by ij, and we will specify the distance function p by defining the entries on and below the diagonal of the symmetric matrix whose (i, j)-entry is $ij = p(x_i, x_j)$.

Example 2 Consider the preordered set $\{x_1, x_2, x_3, x_4\}$ with $x_3 < x_2 < x_1$ and $x_4 < x_1$, as shown.



We want $p(x_i, x_j) = ij \le ii = p(x_i, x_i)$ if and only if $x_i \preceq x_j$. Because each point x_j which is above x_i in the preorder provides a distance $ij \le ii$, to insure that all such distances ij can be fitted under ii using integer values, we will take ii to be the number of elements strictly greater than x_i. This gives the diagonal entries of the p matrix, as shown in Figure 1a.

Now in row i, we want $ij \le ii$ if and only if $x_i \preceq x_j$, so we will slide across the row making a notation "$\le ii$" in the ij slot if $x_i \preceq x_j$ and making a notation "$> ii$" in the ij slot if $x_i \not\preceq x_j$. To restrict our work to the lower left half of the symmetric matrix, we will reflect the notes in the ik entries around the diagonal if $k > i$. Thus, for example, as we slide across row 2, we note that $x_2 \preceq x_1$, so we want the (2,1)-entry to be less than or equal to the (2,2)-entry, which is 1, so we record "≤ 1" in the (2,1) position. Since $x_2 \not\preceq x_3$ and $x_2 \not\preceq x_4$, we want the (2,3) entry and the (2,4) entry to be "> 1" and we mark this information as shown, in the symmetric (3,2)- and (4,2)-entries below the diagonal, as shown in Figure 1a.

Repeating this for the other rows, we arrive at the matrix shown in Figure 1b. Note that each off-diagonal entry will have two notations as marked.

	x_1	x_2	x_3	x_4
x_1	0			
x_2	≤ 1	1		
x_3		> 1	2	
x_4		> 1		1

Figure 1a.

	x_1	x_2	x_3	x_4
x_1	0			
x_2	$(> 0, \leq 1)$	1		
x_3	$(> 0, \leq 2)$	$(> 1, \leq 2)$	2	
x_4	$(> 0, \leq 1)$	$(> 1, > 1)$	$(> 2, > 1)$	1

Figure 1b.

	x_1	x_2	x_3	x_4
x_1	0	1	2	1
x_2	1	1	2	2
x_3	2	2	2	3
x_4	1	2	3	1

Figure 1c.

Now we consider the notations in each entry below the diagonal. If the notation in an entry has form $(> a, > b)$, we take the entry to be $\max\{a, b\}+1$. Entries of form $(> a, \leq b)$ will occur, as we will see, only if $b > a$, so we will assign such an entry the value of b. Entries of form $(\leq a, \leq b)$ will occur only if $a = b$ (and the associated points x_a, x_b satisfy $x_a \preceq x_b, x_b \preceq x_a$), and we assign this value to the entry. These rules give the matrix shown in Figure 1c for our example. Note that the diagonal entry is the minimum entry of each row, so the function p satisfies the small self-distances condition.

It remains to verify that the distance function p so constructed satisfies the triangle inequality. We note that for small matrices, the triangle inequality can be confirmed manually by fixing an entry ij on or below the diagonal, then sliding along the diagonal entries kk, mentally forming a rectangle with opposite corners ij and kk, and ik and kj, and confirming that the sum of the opposite corners ik and kj is not less than the sum of the other opposite corners ij and kk. This must be repeated for each entry ij on or below the diagonal. The details of this example are left to the reader; the general case is shown below. ▪

We now state the general algorithm.

Theorem 3 *Any preorder $\preceq$ on a finite set $X = \{x_1, x_2, \ldots, x_n\}$ is given by $x_i \preceq x_j$ if and only if $p(x_i, x_j) \leq p(x_i, x_i)$ where $p : X \times X \to \{0, 1, 2, \ldots\}$ is the partial pseudometric $p(x_i, x_j) = ij$ defined as follows:*

For $i \in \{1, 2, \ldots, n\}$, take $ii = |\{a \in X : x_i \prec a\}|$.

For distinct $i, j \in \{1, 2, \dots, n\}$, take

$$ij = \begin{cases} ii & \text{if } x_i \preceq x_j \\ jj & \text{if } x_j \preceq x_i \\ \max\{ii, jj\} + 1 & \text{if } x_i \not\preceq x_j \text{ and } x_j \not\preceq x_i \ \ (\text{i.e., if } x_i \| x_j) \end{cases}$$

Note that this definition is well-defined and symmetric, for if $x_i \preceq x_j$ and $x_j \preceq x_i$, then $ij = ii = |\{a \in X : x_i \prec a\}| = |\{a \in X : x_j \prec a\}| = jj = ji$.

Furthermore, this definition meets the needed requirement that $ij \leq ii$ if and only if $x_i \preceq x_j$:
If $x_i \preceq x_j$, then $ij = ii$.
If $x_i \not\preceq x_j$ but $x_j \preceq x_i$, then $ij = ji = jj = |\{a \in X : x_j \prec a\}| > |\{a \in X : x_i \prec a\}| = ii$.
If $x_i \not\preceq x_j$ and $x_j \not\preceq x_i$, then $ij = \max\{ii, jj\} + 1 > ii$.

Also observe that the definition implies that $ii \leq ij$ for any $i, j \in \{1, 2, \dots, n\}$. That is, the function p satisfies the small self-distances condition of Definition 1.

Thus, the proof of Theorem 3 now depends only on the the verification that the function $p(x_i, x_j) = ij$ defined here satisfies the triangle inequality. First, we list the following immediate observations for future reference. In all that follows, we adopt the notation of Theorem 3.

Proposition 4 *(a)* $(x_i \preceq x_j \text{ and } x_j \preceq x_i) \Rightarrow ii = jj$, *and consequently,* $im = jm \ \forall m \in \{1, 2, \dots, n\}$.

(b) $(x_i \preceq x_j \text{ and } x_j \not\preceq x_i) \Rightarrow jj < ii = ij$.

(c) $(x_i \not\preceq x_j \text{ and } x_j \not\preceq x_i) \Rightarrow ii < ij \text{ and } jj < ij$.

(d) $x_i \preceq x_j \Rightarrow jj \leq ii = ij$.

(e) $x_i \not\preceq x_j \Rightarrow jj \leq ij$.

For any distance function d on X satisfying $d(x, x) = 0$ for all $x \in X$, the triangle inequality holds trivially for three points a, b, c if any two of them are equal. We present a similar result in Proposition 5 below. Recall that in a preordered set $(X, \preceq)$, we may have distinct points $a, b \in X$ with $a \preceq b$ and $b \preceq a$. Defining $a \sim b$ if and only if ($a \preceq b$ and $b \preceq a$) gives an equivalence relation $\sim$ on X, and we obtain a partial order $\leq$ on the collection $X/\!\sim$ of all $\sim$-equivalence classes by taking $[a] \leq [b]$ if and only if $a \preceq b$.

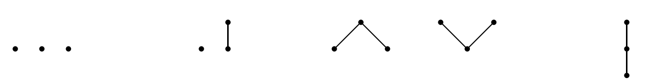
We note that if $x_i \sim x_j$, then the i^{th} and j^{th} rows of the matrix $[ij]$ are equal, as are the i^{th} and j^{th} columns.

Proposition 5 *Suppose p is as defined in Theorem 3. Then p satisfies the triangle inequality for three points $x_i, x_j, x_k \in X$ if any two of them are $\sim$-equivalent.*

Proof. If $x_i \sim x_k$, then by Proposition 4(a), $im = mi = km = mk = kk \ \forall m \in \{1, 2, \ldots, n\}$, and the desired inequality $ij + kk \leq ik + kj$ is equivalent to $kk + kk \leq kk + kk$, which is true. Replacing i by j proves the case $x_j \sim x_k$. Finally, suppose $x_i \sim x_j$. Then by Proposition 4(a), $im = jm = mj = ii \ \forall m \in \{1, 2, \ldots, n\}$, and the desired inequality $ij + kk \leq ik + kj$ is equivalent to $ii + kk \leq ii + ii$, or $kk \leq ii$. If $x_i \preceq x_k$, Proposition 4(d) gives $kk \leq ii$, as needed. If $x_i \not\preceq x_k$, Proposition 4(e) gives $kk \leq ik = ii$, as needed. ∎

We now prove that the function p defined in Theorem 3 satisfies the triangle inequality $ij + kk \leq ik + jk \ \forall i, j, k \in \{1, 2, \ldots, n\}$, assuming that no two of the points x_i, x_j, x_k are $\sim$-equivalent. The proof is divided into cases based on the relative position of x_i, x_j, and x_k in the preordered set X. By the previous proposition, the triangle inequality holds if antisymmetry fails among the preordered set $\{x_i, x_j, x_k\}$, so now we need only consider partially ordered configurations of the three points.

To insure that we cover all cases, let us itemize the possible partial orders on three elements x_i, x_j, x_k. We call a partial order on three elements a "Type n" partial order if there are n related pairs. Referring to Figure 2, we see that there is one Type 0 partial order on three elements, and six each of Types 1, 2, and 3.



Type 0 Type 1 Type 2 Type 3

Figure 2.

We present some cases below, and afterwards, we will show that each of the 19 configurations mentioned above has been verified.

Note that the triangle inequality $ij + kk \leq ik + kj$ is symmetric in i and j: interchanging i and j gives $ji + kk \leq jk + ki$, which is equivalent to $ij + kk \leq ik + kj$. The number of essentially different partially ordered configurations of x_i, x_j, x_k is reduced by this symmetry.

Case 1: (x_i is minimum). $x_i \preceq x_j$ and $x_i \preceq x_k$. From Proposition 4(b), $ij = ii = ik$. Now whether $x_j \preceq x_k$ or $x_j \not\preceq x_k$, Propositions 4(d) and (e) imply $kk \leq kj$. Thus, $ij + kk = ik + kk \leq ik + kj$, as needed.

Case 2: (x_k is minimum). $x_k \preceq x_i$ and $x_k \preceq x_j$. Proposition 4(b) gives $ik = kk > ii$, so $kk \geq ii + 1$, and $jk = kk > jj$, so $kk \geq jj + 1$. If $x_i \preceq x_j$, then $ij = ii < kk$. If $x_i \| x_j$, then $ij = \max\{ii + 1, jj + 1\} \leq kk$. If $x_j \preceq x_i$, then $ij = jj$, and we know $jj < kk$, and thus $ij \leq kk$ regardless of the relation between x_i and x_j. Now $ij + kk \leq kk + kk = ik + jk$, as needed.

Case 3: (x_i is maximum, $x_j \| x_k$). $x_j \preceq x_i$, $x_k \preceq x_i$, $x_j \| x_k$. By Proposition 4(b), $ij = jj$ and $ki = kk$, and by Proposition 4(c), $jk > jj$. Thus, $ij + kk = jj + ki < jk + ik$, as needed.

Case 4: (x_k is maximum, $x_i \| x_j$). $x_i \preceq x_k$, $x_j \preceq x_k$, $x_i \| x_j$. By Proposition 4(b), $ik = ii > kk$, so $1 + kk \leq ik$, and $jk = jj > kk$, so $1 + kk \leq kj$. Since $x_i \| x_j$, $ij = \max\{ii + 1, jj + 1\}$. If $ij = ii + 1$, then $ij + kk = ii + 1 + kk = ik + 1 + kk \leq ik + kj$, as needed. If $ij = jj + 1$, then $ij + kk = jj + 1 + kk = jk + 1 + kk \leq jk + ik$, as needed.

Case 5: (x_i unrelated to x_j and x_k; $x_j \preceq x_k$). $x_i \| x_j$, $x_i \| x_k$, $x_j \preceq x_k$. Since $x_j \preceq x_k$, we have $kk < jj = jk$. First suppose $ii \geq jj$. Then $ij = \max\{ii, jj\} + 1 = ii + 1$. Now $ik = \max\{ii, kk\} + 1$, and $ii \geq jj > kk$ implies $ik = ii + 1$. Thus, $ij + kk = ii + 1 + kk = ik + kk \leq ik + jk$, as needed. Next suppose $ii < jj$, so that $ii + 1 \leq jj$ and $ij = \max\{ii, jj\} + 1 = jj + 1$. Now $x_j \preceq x_k$ implies $kk + 1 \leq jj = jk$, and $x_i \| x_k$ implies $kk + 1 \leq ik$. Thus, $ij + kk = jj + 1 + kk = jk + 1 + kk \leq jk + ik$, as needed.

Case 6: (x_i unrelated to x_j and x_k; $x_k \preceq x_j$). $x_i \| x_j$, $x_i \| x_k$, $x_k \preceq x_j$. From $x_k \preceq x_j$, we have $kk = jk > jj$, and from $x_i \| x_k$ we have $ii + 1 \leq ik$. First suppose $ii \geq jj$ so that $ij = \max\{ii, jj\} + 1 = ii + 1$. Then $ij + kk = ii + 1 + kk = ii + 1 + jk \leq ik + jk$, as needed. Next suppose $ii < jj$ so that $ij = jj + 1$ and $kk > jj > ii$, and thus $ik = \max\{ii, kk\} + 1 = kk + 1$. Then $ij + kk = jj + 1 + kk = jj + ik < jk + ik$, as needed.

Case 7: (x_k unrelated to x_i and x_j). $x_i \| x_k$, $x_j \| x_k$. Then $ik > ii$, $ik > kk$, $jk > jj$, and $jk > kk$. If $x_i \preceq x_j$, then $ij = ii$ so $ij + kk = ii + kk < ik + jk$, as needed. If $x_j \preceq x_i$, interchanging i and j in the previous sentence gives the desired inequality. Finally, suppose $x_i \| x_j$, so that $\{x_i, x_j, x_k\}$ is an antichain in $(X, \preceq)$. Then $ij = \max\{ii, jj\} + 1$. If $ij = ii + 1$, then $ij + kk = ii + 1 + kk < ik + 1 + kk \leq ik + kj$ since $kj = \max\{kk, jj\} + 1 \geq kk + 1$. If $ij = jj + 1$, then $ij + kk = jj + 1 + kk < jk + 1 + kk \leq jk + ik$ since $ik = \max\{ii, kk\} + 1 \geq kk + 1$.

Now we confirm that these cases cover all the situations. The one Type 0 partial order is covered by Case 7. The six Type 1 partial orders are covered by Cases 5, 6, and 7 and their (i, j)-symmetric forms. The three Type 2 partial orders having a maximum element are covered by Case 3, its (i, j)-symmetric form, and Case 4. The three Type 2 partial orders having a minimum element are covered by Case 1, its (i, j)-symmetric form, and Case 2. There are six Type 3 partial orders (chains). Case 1 and its (i, j)-symmetric form cover the four having x_i or x_j as minimum element, and Case 2 covers the two having x_k as minimum element. This completes the proof of Theorem 3.

We observe that, as expected, many partial pseudometrics may generate the same topology. For example, for a chain $x_1 > x_2 > \cdots > x_n$, we may assign the natural (asymmetric) signed distances $q(x_i, x_j) = j - i$ between points, then add constants to the rows of $[ij]$ to eliminate negative entries and create a symmetric matrix. This will give a partial pseudometric with $ij = d(x_i, x_j) = i + j - 2$.

We note that Theorem 3 remains valid if we redefine ii to be the number of elements strictly greater than $[x_i]$ in $(X/{\sim}, \leq)$, where $X/{\sim}$ is the set of equivalence classes determined by the equivalence relation $a \sim b$ if and only if $a \preceq b$ and $b \preceq a$, and $\leq$ is the partial order on $X/{\sim}$ defined by $[a] \leq [b]$ if and only if $a \preceq b$. With ii so redefined, the partial pseudometric p defined by Theorem 3 is minimal among the integer-valued partial pseudometrics generating the same topology: if d is an integer-valued partial pseudometric on X which generates the same topology as p, then $p(x_i, x_j) \leq d(x_i, x_j)$ for any $x_i, x_j \in X$.

Finally, if p is a partial pseudometric on X, $a \in X$, and if $B_a \cap X = \emptyset$, we may define a partial pseudometric on $X \cup B_a$ by $p'(x, y) = p(x, y)$, $p'(x, b) = p'(b, x) = p(x, a)$, and $p'(b, b) = p(a, a)$ for any $x, y \in X$ and any $b \in B_a$. Then in the preorder induced by p', we have $a \preceq b$ and $b \preceq a$ for any $b \in B_a$, so the topology on $X \cup B_a$ is simply the topology on X with the set B_a added to every neighborhood of a. Using this idea, the results for topologies on finite sets can be extended to any finite topology on an arbitrary set. Specifically, if τ is a finite topology on X, define $a \approx b$ if and only if $\bigcap\{U \in \tau : a \in U\} = \bigcap\{U \in \tau : b \in U\}$, find the partial pseudometric p generating the topology on the finite space $(X/{\approx}, \tau/{\approx})$, and append to each point $[a] \in X/{\approx}$ the set $B_a = [a] \setminus \{a\}$ as described above to get a partial pseudometric p' on X which generates τ.

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