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# Ordered Quotients and the Semilattice of Ordered Compactifications

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The ideas of quotient maps, quotient spaces, and upper semicontinuous decompositions are extended to the setting of ordered topological spaces. These tools are used to investigate the semilattice of ordered compactifications and to construct ordered compactifications with o-totally disconnected and o-zero-dimensional remainder.

## 1 Introduction

In the literature, there are several notions of ordered quotient spaces and maps. One natural definition is that an ordered quotient space is a topological quotient space which is ordered so that the quotient map is increasing. One limitation of this definition is that it does not tell how to order the blocks of a decomposition, rather it gives a method for determining whether a predefined order on the image is a quotient order. It is easy to see that if $X$ and $Y$ are ordered spaces and $f : X \to Y$ is both increasing and a quotient map, one can add order to $Y$ and $f$ will still be increasing and a quotient map. There is, then, no sense of *the* ordered quotient space associated with a decomposition (only *an* ordered quotient space). Another definition for an ordered quotient space was introduced by McCartan [5] who gave a constructive definition for the order on the blocks of a decomposition. He defined $[x] \leq [y]$ iff for all $x^* \in [x]$ and $y^* \in [y]$, $x^* \leq y^*$. This definition is examined below, but the class of spaces to which it can be applied is found to be very limited. Priestley [14] and Miwa [7] [8] [9] [10] propose an order on the blocks of a decomposition by defining $[x] \leq [y]$ iff there exists $x^* \in [x]$ and $y^* \in [y]$ with $x^* \leq y^*$. While this definition includes more spaces than McCartan's definition, it still is limited.

This paper proposes a definition of a quotient order which extends the quotient orders of Priestley, Miwa, and McCartan, but which does not consider all images of increasing quotient maps to be ordered quotient spaces. It is

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constructive, giving a method for defining an ordered decomposition space, and, consequently, an ordered quotient space. Also presented is a definition for an ordered quotient map which fits neatly into the rest of the theory.

We begin by reviewing pertinent notions from the theory of ordered topological spaces. These ideas are discussed in greater detail in Nachbin [11]. If $A$ is a set, $i(A)$ will denote the set of all points greater than or equal to some element of $A$ and is called the *increasing hull* of $A$. The *decreasing hull* of $A$, denoted by $d(A)$, is defined dually. A set $A$ is *increasing* if $A = i(A)$, *decreasing* if $A = d(A)$, and *convex* if $A = i(A) \cap d(A)$. The *closed increasing hull* of $A$, denoted by $I(A)$, is the intersection of all closed increasing sets containing $A$. The *closed decreasing hull* of $A$ is defined similarly and denoted by $D(A)$. An ordered topological space is a triple $(X, \tau, \leq)$ consisting of a set, a topology, and a partial order. It is standard to further assume that the topology is *convex* which means that the increasing open sets and the decreasing open sets form a subbasis. Ordered versions of the separation axioms can be defined; we review the definitions of importance to this paper. A space $X$ is *$T_2$-ordered* if for any two points $x \not\leq y$, there exist an increasing neighborhood (not necessarily open) of $x$ and a decreasing neighborhood of $y$ which are disjoint. A $\mathrm{T}_2$-order is sometimes called a *closed order* for its definition is equivalent to the order relation $\leq$ being a closed subset of the space $X \times X$. A space $X$ is *Tychonoff-ordered* if 1) for each $x \in X$ and $V$, a neighborhood of $x$, there exist two continuous real valued functions $f$ and $g$ on $X$ where $f$ is increasing and $g$ is decreasing such that $0 \leq f \leq 1$, $f(x) = 1$, $0 \leq g \leq 1$, $g(x) = 1$, and $\inf\{f(y), g(y)\} = 0$ if $y \in X \setminus V$; and 2) If $x, y \in X$ and $x \not\leq y$, then there exists a continuous increasing real valued function $f$ on $X$ such that $f(x) > f(y)$. Finally, a space is *normally-ordered* if for any two disjoint closed sets $H$ and $K$ with $H$ increasing and $K$ decreasing, there are disjoint open sets $U$ and $V$ with $U$ increasing and $V$ decreasing such that $H \subset U$ and $K \subset V$. An ordered compactification of a space $X$ is a compact $\mathrm{T}_2$-ordered space which contains $X$ as a dense ordered subspace. A noncompact ordered space $X$ has an ordered compactification if and only if it is Tychonoff-ordered. If an ordered space is compact $\mathrm{T}_2$-ordered, then it is normally-ordered. If $X$ is a compact $\mathrm{T}_2$-ordered space and $A$ is a closed subset of $X$, then $i(A) = I(A)$ and $d(A) = D(A)$.

If $X$ is a Tychonoff-ordered space, the collection of ordered compactifications of $X$ will be denoted by $K_o(X)$. We define the *projection order* on $K_o(X)$ by $\alpha X \geq \gamma X$, if there is a continuous increasing function (called a *projection function*) $f : \alpha X \to \gamma X$ such that $f|_X = id_X$. With this order, $K_o(X)$ forms a complete upper semilattice. For more information on the semilattice of ordered compactifications see Richmond [15]. For more on the topological theory of compactifications see Chandler [1] or Porter and Woods [13].

## 2 Ordered Quotient Maps, Ordered Quotient Spaces, and Ordered Upper Semicontinuous Decompositions

This section defines ordered upper semicontinuous decompositions, two types of ordered quotient maps, a quotient order, and an ordered quotient space. The issue of when an ordered quotient space or the image of an ordered quotient map is $T_2$-ordered is also addressed.

There are two results from the theory of topological quotients which are important in studying compact Hausdorff spaces in general and compactifications in particular.

**Proposition 1** *Let $f : X \to Y$ be a quotient map. Then $f$ is closed if and only if $\{f^{-1}(y)|y \in Y\}$ is an upper semicontinuous decomposition of $X$.*

**Proposition 2** *Let $f : X \to Y$ be a quotient map. If $X$ is a compact Hausdorff space, then $Y$ is Hausdorff if and only if $\{f^{-1}(y)|y \in Y\}$ is an upper semicontinuous decomposition of $X$.*

The second proposition implies the existence of an order reversing isomorphism between the "lattice" of compactifications of a Tychonoff space $X$ and the upper semicontinuous decompositions of $\beta X \setminus X$. (See Porter and Woods [13].)

Upper semicontinuous decompositions are necessary for having quotient spaces which are Hausdorff. We begin our development of ordered quotient spaces by defining ordered upper semicontinuous decompositions.

**Definition 3** A decomposition $\mathcal{D}$ of an ordered space $X$ is an *ordered upper semicontinuous decomposition* if

**a.** For each $[x] \in \mathcal{D}$ and each increasing open set $U$ containing $[x]$, there exists a saturated increasing open set containing $[x]$ which is contained in $U$;

**b.** Dually, for each $[x] \in \mathcal{D}$ and each decreasing open set $V$ containing $[x]$, there exists a saturated decreasing open set containing $[x]$ which is contained in $V$.

Next we turn to the question of finding an appropriate definition for an ordered quotient map. In some sense, topology and order are oppositely directed properties. For example, the morphisms associated with ordered topological spaces are increasing continuous functions. If $f : X \to Y$ is an increasing continuous function, then it remains continuous and increasing if one either weakens the topology on $Y$ or strengthens the order on $Y$. Dual statements apply to $X$. Based on this, the following seems a natural candidate for an ordered quotient map.

**Definition 4** A continuous increasing function $f : X \to Y$ is a *Type I ordered quotient map* if

**a.** $f^{-1}(U)$ is open implies $U$ is open;

**b.** $A$ is increasing implies $f(A)$ is increasing.

Note that the continuity of $f$ implies the converse of the first implication, hence a Type I ordered quotient map is a topological quotient map with some order restrictions.

With this definition, analogous versions of Propositions 1 and 2 can be proven using analogous arguments. However, there are order conditions on $Y$ which are forced by $f$ which may not be met by any ordering on $Y$.

**Example 5** *Let $X = \{a, b, c, d\}$ with the discrete topology and with order $a < b$ and $c < d$. Let $Y = \{a, \{b, c\}, d\}$. Define $f : X \to Y$ to be the function which is the identity map on $a$ and $d$ and which takes both $b$ and $c$ to $\{b, c\}$. Since $f$ is to be an increasing function, we have $a < \{b, c\} < d$. Now $A = \{a, b\}$ is increasing, but $f(A)$ is not. Thus $f$ is not a Type I ordered quotient.*

This example illustrates that Type I ordered quotient maps are of limited use since there are simple increasing quotient maps which are not of Type I, but which should be considered as ordered quotient maps in any reasonable definition. We are forced then to sacrifice some of the nice properties of an ordered quotient map (namely, analogous versions of Propositions 1 and 2) in exchange for a definition which will work with identifications such as in Example 5.

**Definition 6** A continuous increasing function $f : X \to Y$ is a *Type II ordered quotient map* (or simply an *ordered quotient map*) if

**a.** $f^{-1}(U)$ is open implies $U$ is open and

**b.** $f^{-1}(A)$ is increasing implies $A$ is increasing.

Note that $f$ being continuous and increasing implies the converse of both implications. Again this is a topological quotient map with order restrictions. Further observe that Condition b also implies that if $f^{-1}(A)$ is decreasing then $A$ is decreasing.

Before exploring quotient maps further, we need to define an order on $Y$ or, equivalently, on the decomposition space of $X$. If $f$ is to be increasing and if $x \leq y$ in $X$, then we must have $[x] \leq [y]$. This condition alone need not form an order. For example, if $x \leq y$ and $z \leq w$ and $[y] = [z]$, then $[x] \leq [w]$ must be true even if no point in $[x]$ is less than any point in $[w]$.

**Definition 7** The *finite step order* on $\mathcal{D}$, a decomposition of $X$, is the transitive closure of the relation defined by $x \leq y \to [x] \leq [y]$. Note that in general the transitive closure need not be antisymmetric. A decomposition will be called *admissible* if the transitive closure is indeed an order (i.e., is antisymmetric). All decompositions that we will consider will be admissible.

Observe that if $[x] \leq [y]$ then there exists a "finite chain" $[z_1], \cdots, [z_n] \in \mathcal{D}$ with points $z_i'$, $z_i^* \in [z_i]$ such that $x = z_1, y = z_n$, and $z_1' \leq z_2^*, \cdots, z_i' < z_{i+1}^*, \cdots, z_{n-1}' < z_n^*$. This justifies the name "finite step order." The finite step order is the minimum order on $Y$ which still allows $f$ to be increasing.

The condition that a decomposition is admissible implies that the decomposition blocks are convex. The converse, however, is not true.

**Example 8** *Let* $X = \{(0,0),(1,0),(1,1),(2,1),(2,2),(3,2)\}$ *with the discrete topology. Order* $X$ *with* $(i,j) < (i,j+1)$ *and with* $(3,2) < (0,0)$. *Partition* $X$ *so that the decomposition blocks are of the form* $\{(i,j),(i+1,j)\}$. *Each of these decomposition blocks is the intersection of an increasing and a decreasing set and is, therefore, convex, but the transitive closure of the finite step "order" is not antisymmetric.*

As noted before the definition of Type II ordered quotient maps, analogs of Propositions 1 and 2 do not go through without some additional restrictions on $X$ or $Y$. The following results, from Proposition 9 until Proposition 17, examine analogs of Proposition 1. Proposition 17 and Example 18 show that one direction of Proposition 2 has an analogous version, but the converse does not.

**Proposition 9** *If* $f : X \to Y$ *is an ordered quotient map and* $\mathcal{D} = \{f^{-1}(y) | y \in Y\}$ *is an ordered upper semicontinuous decomposition of* $X$, *then* $f$ *takes closed increasing or closed decreasing sets to closed sets. If, further,* $\mathcal{D}$ *has the finite step order, then* $f$ *takes closed increasing sets to closed increasing sets and closed decreasing sets to closed decreasing sets.*

*Proof.* Let $A$ be a closed increasing subset of $X$. If $y \notin f(A)$, then $f^{-1}(y) \subset X \setminus A$ which is open and decreasing. By ordered upper semicontinuity, there exists a $U$ which is saturated open satisfying $f^{-1}(y) \subset U \subset X \setminus A$. Thus $y \in f(U) \subset f[X \setminus A]$. Since $U$ is saturated, we have $f(U) \cap f(A) = \emptyset$ which implies that $f(A)$ is closed.

Now suppose $Y$ has the finite step order. If $y \notin f(A)$ but $y \geq z \in f(A)$, then $f^{-1}(y) \geq f^{-1}(z)$. Thus there exist $[z_1], \ldots, [z_n]$ as required by the definition of the finite step order. Then $f^{-1}(y) \subset X \setminus A$, which is open and decreasing. By ordered upper semicontinuity, there exists a saturated decreasing open set $U$ such that $f^{-1}(y) \subset U \subset X \setminus A$. Since $U$ is saturated, each $[z_i]$ is either completely in $U$ or completely outside of $U$. Since $[z_1] = f^{-1}(z)$ is outside of $U$ and $[z_n] = f^{-1}(y)$ is inside of $U$, there exists an $i$ such that $[z_i]$ is outside of $U$ and $[z_{i+1}]$ is inside $U$. But $z_i' \leq z_{i+1}^*$, so $U$ decreasing implies that $z_i' \in U$ which is a contradiction. □

**Proposition 10** *If $f : X \to Y$ is an ordered quotient map which takes closed increasing sets to closed increasing sets and dually, then $\{f^{-1}(y)|y \in Y\}$ is an ordered upper semicontinuous decomposition of $X$.*

*Proof.* Let $U$ be an open increasing set containing $[x]$. Then $X \setminus U$ is a closed decreasing set. So $f(X \setminus U)$ is a closed decreasing set, and hence $Y \setminus f(X \setminus U)$ is an open increasing set. Then $[x] \subset f^{-1}(Y \setminus f(X \setminus U)) \subset U$ where $f^{-1}(Y \setminus f(X \setminus U))$ is a saturated open increasing set. □

**Proposition 11** *If $X$ is a compact $T_2$-ordered space, $f : X \to Y$ is an ordered quotient map, and $\mathcal{D} = \{f^{-1}(y)|y \in Y\}$ is an ordered upper semicontinuous decomposition of $X$ ordered by making $f : \mathcal{D} \to Y$ an order isomorphism, then $\mathcal{D}$ has the finite step order.*

*Proof.* Since $f$ is increasing, $\mathcal{D}$ must contain the finite step order. Suppose, by way of contradiction, that $\mathcal{D}$'s order strictly contains the finite step order. Then there exists $a$ and $b$ in $Y$ with $a \leq b$ and with no finite chain between $[f^{-1}(a)]$ and $[f^{-1}(b)]$. Then $i([f^{-1}(a)]) \cap d([f^{-1}(b)]) = \emptyset$. (If $x \in i([f^{-1}(a)]) \cap d([f^{-1}(b)])$ then $x \geq a_i$ and $x \leq b_j$ for some $a_i \in [f^{-1}(a)]$ and $b_j \in [f^{-1}(b)]$ so $[f^{-1}(a)] \leq [x] \leq [f^{-1}(b)]$ which is a finite chain.) Since $X$ is compact $T_2$-ordered, $i([f^{-1}(a)]) = I([f^{-1}(a)])$ and $d([f^{-1}(b)]) = D([f^{-1}(b)])$, and hence both are closed (see Nachbin [11]). So $X \setminus D([f^{-1}(b)])$ is an increasing open set containing $[f^{-1}(a)]$. By ordered upper semicontinuity, there exists a saturated open increasing set $f^{-1}(U)$ with $[f^{-1}(a)] \subset f^{-1}(U) \subset X \setminus D([f^{-1}(b)])$. Now $f^{-1}(U)$ open and increasing implies that $U$ is open and increasing. Thus $a \in U$ and $a \leq b$ which implies that $b \in U$. But $f^{-1}(U) \subset X \setminus D([f^{-1}(b)]) \subset X \setminus [f^{-1}(b)]$ implies $b \notin U$, which is a contradiction. □

Based on Proposition 11, we conclude that the finite step order should be considered to be *the* quotient order.

**Definition 12** Let $X$ and $Y$ be ordered topological spaces and $f : X \to Y$ be an increasing quotient map. Order $\mathcal{D} = \{f^{-1}(y)|y \in Y\}$ by making $f : \mathcal{D} \to Y$ an order isomorphism. The order on $Y$ is the *quotient order* if the order on $\mathcal{D}$ is the finite step order.

This allows us to define an ordered quotient space.

**Definition 13** Let $X$ and $Y$ be ordered topological spaces and $f : X \to Y$ be a map. Then $Y$ is an *ordered quotient space* if it has the quotient topology and the quotient order.

This definition of an ordered quotient space is further justified by the following evident proposition.

**Proposition 14** *Let $X$ and $Y$ be ordered topological spaces and $f : X \to Y$ a function. The space $Y$ is an ordered quotient space if it has the strongest topology making $f$ continuous and the weakest order making $f$ increasing.*

These results directly imply the following two corollaries.

**Corollary 15** *Let $X$ and $Y$ be ordered topological spaces and $f : X \to Y$ be an ordered quotient map. If $Y$ has the quotient order, then the following are equivalent:*

**a.** *$f$ takes closed increasing sets to closed increasing sets and dually; and*

**b.** *$\{f^{-1}(y) | y \in Y\}$ is an ordered upper semicontinuous decomposition of $X$.*

**Corollary 16** *Let $X$ and $Y$ be ordered topological spaces and $f : X \to Y$ be an ordered quotient map. If $X$ is compact $T_2$-ordered, then the following are equivalent:*

**a.** *$f$ takes closed increasing sets to closed increasing sets and dually; and*

**b.** *$\{f^{-1}(y) | y \in Y\}$ is an ordered upper semicontinuous decomposition of $X$.*

**Proposition 17** *If $X$ is compact $T_2$-ordered, and $\mathcal{D}$ is an ordered upper semicontinuous decomposition of $X$ into closed sets, then the ordered quotient space $Y = X/\mathcal{D}$ is a compact $T_2$-ordered space.*

*Proof.* Let $f$ be the implied ordered quotient map. Compactness follows from the continuity of $f$. By Nachbin [11], any $T_2$-ordered compact space is necessarily convex.

We show $T_2$-ordered. Suppose $[x] \not\leq [y]$. Then $i([x]) \cap d([y]) = \emptyset$. Since $X$ is compact $T_2$-ordered, $i([x]) = I([x])$ and $d([y]) = D([y])$. Since compact $T_2$-ordered implies normally-ordered, there exist disjoint open sets $U$ and $V$ such that $U$ is increasing, $V$ is decreasing, $D([y]) \subset V$, and $I([x]) \subset U$. By ordered upper semicontinuity, there exists a saturated open increasing set $S$ and a saturated decreasing open set $T$ such that $[x] \subset S \subset U$ and $[y] \subset T \subset V$. Then $f([x]) \in f(S)$ which is increasing and open and $f([y]) \in f(T)$ which is decreasing and open and $f(S) \cap f(T) = \emptyset$. $\square$

Example 5 demonstrates that the converse is not true, for $Y$ has the finite step order, is $\mathrm{T}_2$-ordered and compact, but the decomposition $\{f^{-1}(y)|y \in Y\}$ is not ordered upper semicontinuous.

The next example further illustrates the role of the finite step order.

**Example 18** *Let*

$$X = \{(1 - \frac{1}{n}, 1 - \frac{1}{n})|n \in \mathbf{N}\} \cup \{(1 - \frac{1}{n+1}, 1 - \frac{1}{n})|n \in \mathbf{N}\} \cup \{(1, 1)\}$$

*with the topology inherited as a subspace of the usual real plane and the "up" order* $(x_1, y_1) \leq (x_2, y_2)$ *iff* $x_1 = x_2$ *and* $y_1 \leq y_2$. *Let*

$$\mathcal{D} = \{\{(1 - \frac{1}{n}, 1 - \frac{1}{n}), (1 - \frac{1}{n+1}, 1 - \frac{1}{n})\}|n \in \mathbf{N}\} \cup \{(1, 1)\}$$

*be a decomposition of* $X$. *Let*

$$Y = \{(1 - \frac{1}{n}, 1 - \frac{1}{n})|n \in \mathbf{N}\} \cup \{(1, 1)\}$$

*be linearly ordered by* $(1 - \frac{1}{n}, 1 - \frac{1}{n}) \leq (1 - \frac{1}{m}, 1 - \frac{1}{m})$ *if* $n \leq m$ *with* $(1, 1) \geq (1 - \frac{1}{n}, 1 - \frac{1}{n})$ *for all* $n$. *Finally, let* $f : X \to Y$ *be the map which takes* $\{(1 - \frac{1}{n}, 1 - \frac{1}{n}), (1 - \frac{1}{n+1}, 1 - \frac{1}{n})\}$ *to* $(1 - \frac{1}{n}, 1 - \frac{1}{n})$.

*Observe that* $Y$ *is a compact* $T_2$*-ordered space, the decomposition* $\{f^{-1}(y)| y \in Y\}$ *is not an ordered upper semicontinuous decomposition of* $X$, *and that* $Y$ *does not have the quotient order (a finite path does not exist between* $(0, 0)$ *and* $(1, 1)$*).*

*Thus this example shows that there exist compact* $T_2$*-ordered spaces* $X$ *and* $Y$ *and an increasing quotient map* $f : X \to Y$ *where* $\{f^{-1}(y)|y \in Y\}$ *is not an ordered upper semicontinuous decomposition of* $X$.

Further notice that in the preceeding example, $\{f^{-1}(y)|y \in Y\}$ is a decomposition of $X$ and we can form an ordered quotient space. The ordered quotient space associated with $X$ and $\mathcal{D}$ is the same as $Y$ but with $(1, 1)$ not related to any other point. This space has the finite step order and is not $\mathrm{T}_2$-ordered. That it is not $\mathrm{T}_2$-ordered is seen by the sequences $a_n = (1 - \frac{1}{n}, 1 - \frac{1}{n})$ and $b_n = (0, 0)$. We have $b_n \leq a_n$ for all $n$, but $\lim a_n = (1, 1)$ and $\lim b_n = (0, 0)$ and $(0, 0) \not\leq (1, 1)$ so the quotient is not $\mathrm{T}_2$-ordered. In the next section on ordered compactifications, we will observe that the space $X$ in this example can be realized as the remainder of an ordered compactification.

The decompositions in Examples 5 and 18 could be made into ordered upper semicontinuous decompositions by increasing the order on $X$. The following order was introduced by McCartan [5].

**Definition 19** If $X$ is an ordered space and $\mathcal{D}$ is an admissible ordered decomposition of $X$, the order on $X$ is a *McCartan order on $X$ with respect to $\mathcal{D}$* if

$$[x] \le [y] \to x^* \le y^* \, \forall x^* \in [x] \, \forall y^* \in [y].$$

Thus there is as much order between the decomposition blocks as is possible with an increasing function and there are specifications on the order within the decomposition blocks.

It is evident that a McCartan order is a finite step order where each of the finite steps has length one (i.e., a one step order).

**Proposition 20** *If $X$ has a McCartan order with respect to $\mathcal{D}$, then the Type I and Type II ordered quotient maps are the same.*

*Proof.* With a one step order, $A$ increasing implies $f(A)$ is increasing. To see this note that if $y \ge z \in f(A)$, then $[y] \ge [z]$ and there exists $z' \in [z]$ with $z' \in A$ and for all $y' \in [y]$, $y' \ge z'$. Thus for all $y' \in [y]$ we have $y' \in A$. Hence $[y] \subset A$ implies $y \in f(A)$. Therefore $f(A)$ is increasing. □

Given any ordered space $X$ and admissible ordered decomposition $\mathcal{D}$, order can be added to the order on $X$ until it has a McCartan order with respect to $\mathcal{D}$. This can be done by adding $x \le y$ if $[x] \le [y]$. This might indicate that we need only consider McCartan orders, since one can always add order to obtain one. There are serious limitations to this method. One can add order to an $T_2$-ordered space and lose the $T_2$-ordered property or, worse, lose the property of having a convex topology. Consider the following examples.

**Example 21** *Let $A = \{(0, -\frac{1}{n}) | n \in \mathbf{N}\} \cup \{(0,0)\}$ and let $B = \{(1, \frac{1}{n}) | n \in \mathbf{N}\} \cup \{(1,0)\}$ with $A$ and $B$ each ordered linearly, increasing in the up direction. Let $X = A \cup B$ with subspace topology from the real plane and ordered so that there is no order between any point of $A$ and any point of $B$. It is readily verified that $X$ is $T_2$-ordered and compact. Let $\mathcal{D} = \{(0,0),(1,0)\}$ together with the remaining points as one-point decomposition blocks. In other words, $(0,0)$ and $(1,0)$ are being identified. The ordered quotient space is (order homeomorphic to) the linearly ordered space $\{-\frac{1}{n} | n \in \mathbf{N}\} \cup \{0\} \cup \{\frac{1}{n} | n \in \mathbf{N}\}$ with the order and topology inherited from the real line. This defines an order on $\mathcal{D}$, and if the smallest McCartan order with respect to $\mathcal{D}$ is placed on $X$, the resulting space is not $T_2$-ordered. To see this, observe that $(0,0) \not\le (1,0)$, but every increasing neighborhood of $(0,0)$ meets every decreasing neighborhood of $(1,0)$.*

**Example 22** *Let $X = (\{0\} \times I) \oplus (\{1\} \times I)$, where each $I$ is a usual unit interval and $X$ has the direct sum topology and order (no order between the summands, usual order within the summands). Let $Y = I$ with the usual topology and order.*

*Define $f : X \to Y$ by $f((i, y)) = y$. The decomposition induced by $f$ has blocks consisting of the two points lying on a horizontal line. Let $X'$ be $X$ with the McCartan order induced by the decomposition. The set $((1, \frac{1}{4}), (1, \frac{3}{4}))$ is open in $X'$, but is not convex since any increasing and decreasing open sets will pick up common points in the other column.*

We conclude this section by examining the relationships between three notions of ordered quotient maps. Let $X$ be a compact ordered topological space with the discrete order, let $Y$ be a compact $T_2$-ordered space, and let $f : X \to Y$ be a topological quotient map, if one exists. Trivially, $f$ is an increasing quotient, though it may not be either a Type I or Type II ordered quotient map. If one increases the order on $X$ so as to maintain the property of $X$ being $T_2$-ordered, $f$ may become a Type I or Type II ordered quotient map. Let $\Theta$ be all the $T_2$-orders on $X$ which make $f$ increasing. Let $\Theta_{II}$ be all the $T_2$-orders on $X$ which make $f$ a Type II ordered quotient map. Finally, let $\Theta_I$ be all the $T_2$-ordered orders on $X$ which make $f$ a Type I ordered quotient map (and $Y$ an ordered quotient space in the sense of McCartan). Evidently, $\Theta_I \subset \Theta_{II} \subset \Theta$ with Examples 5 and 18 showing that spaces exist where these inclusions are proper. Further, the orders in $\Theta_{II}$ (and hence $\Theta_I$) force the decomposition induced by $f$ to be ordered upper semicontinuous. Some questions remain on this topic.

**Question 23** *(a) Is there another type of ordered quotient map which has an associated constructive definition for the quotient order which accounts for orders which lie between $\Theta_{II}$ and $\Theta$? (b) For which spaces are the inclusions $\Theta_I \subset \Theta_{II} \subset \Theta$ proper? (c) For which spaces do we have $\Theta_I = \Theta_{II} = \Theta$?*

# 3 Applications

## 3.1 Lattice of ordered compactifications

An important theorem (see 4Q of Porter and Woods [13]) in the theory of compactifications is that there is an order reversing bijection between the upper semicontinuous decompositions of a compactification $\alpha X$ of a Tychonoff space $X$ which contain $\{\{x\} | x \in X\}$ and the Hausdorff compactifications of $X$ which are projectively less than $\alpha X$. In particular if $f : \alpha X \to \gamma X$ is a projection map, then $\{f^{-1}(y) | y \in \gamma X\}$ is an upper semicontinuous decomposition of $\alpha X$. Conversely, any upper semicontinuous decomposition of $\alpha X$ into closed sets which contains $\{\{x\} | x \in X\}$ can be identified to give a Hausdorff compactification of $X$ which is projectively less than $\alpha X$. Further, all such decompositions of the Stone-Čech compactification give all the compactifications of $X$.

This subsection explores analogs for the setting of ordered compactifications. One direction of the ordered cases is identical to the unordered case.

**Proposition 24** *If $\mathcal{D}$ is an admissible ordered upper semicontinuous decomposition of an ordered compactification $\alpha X$ into closed sets which has the points of $X$ as decomposition blocks, then the ordered quotient space $\alpha X/\mathcal{D}$ is an $T_2$-ordered compactification of $X$.*

*Proof.* This is an immediate corollary of Proposition 17. □

The converse, however, is not true. Consider the following example.

**Example 25** *Let $X = \bigcup\{[0,1) \times \{\theta_i\} \times \{z_i\} \cup [0,1) \times \{\theta_{i+1}\} \times \{z_i\} | i \in \mathbf{N}\}$ $\cup [0,1) \times \{\frac{\pi}{2}\} \times \{1\}$ where $\theta_i = \frac{\pi}{2} - \frac{1}{i}$, and $z_i = 1 - \frac{1}{i}$. The topology on $X$ is inherited from considering this as a subset of $\mathbf{R}^3$ described in cylindrical coordinates. Give $X$ the "up" order $(r_1, \theta_1, z_1) \leq (r_2, \theta_2, z_2)$ iff $r_1 = r_2, \theta_1 = \theta_2$, and $z_1 \leq z_2$. This space can be visualized as an infinite spiral staircase, with the runners and risers getting smaller as one climbs. Let $\alpha X$ be the compactification of $X$ which puts one point on the end of each spoke with each compactification point smaller than the compactification points on higher levels (larger $z$ values) and unrelated to any point in the base space. Let $\gamma X$ be the compactification which identifies the compactification points which lie on the same horizontal level, that is the two compactification points associated with a given runner. Observe that the remainders of $\alpha X$ and $\gamma X$ are (order homeomorphic to) the spaces from Example 18 and that the projection map $f : \alpha X \to \gamma X$ restricted to the remainders is the map in Example 18. Thus $\{f^{-1}(y) | y \in \gamma X\}$ is not an ordered upper semicontinuous decomposition of $\alpha X$.*

While it might seem plausible to formulate a partial converse to Proposition 24 by putting the McCartan order on $\alpha X$, Examples 21 and 22 demonstrate that there is no guarantee that the resulting ordered compactification will be either $T_2$-ordered or have a convex topology.

### 3.2 Zero-dimensional and totally disconnected remainders

In this section, we look at decompositions of special types (components and o-components) and obtain ordered compactifications with special types of remainders (totally disconnected, o-totally disconnected, zero-dimensional, or o-zero-dimensional). We begin by reviewing some definitions. A space is *totally disconnected* if any two points can be separated by clopen sets, equivalently, if the connected components are all one point sets. A space is *o-totally disconnected* if for any $x \not\leq y$ there is a clopen increasing set containing $x$ but not $y$ (and equivalently a decreasing clopen set containing $y$ and not $x$). A space is *zero-dimensional* if it has a base of clopen sets. A space is *o-zero-dimensional* if it has a subbasis consisting of clopen increasing sets and clopen decreasing sets and $x \leq y$ in $X$ if and only if $y \in cl_{\mathcal{U}}(\{x\})$, where $\mathcal{U}$ is the topology generated by the clopen increasing subsets of $X$. In a compact ordered space, o-totally disconnected and o-zero-dimensional are equivalent. The ordered versions of these topological properties are discussed at length in the survey paper by Nailana [12]. The o-totally disconnected property is also discussed in detail in Davey and Priestley [2].

The following proposition and corollary may be found in Engelking [4]. A fiber is the inverse image of a point.

**Proposition 26** *If $f : X \to Y$ is a closed (or open) surjection with connected fibers, then for every connected subset $C$ of $Y$, the inverse image $f^{-1}(C)$ is connected.*

**Corollary 27** *If $f : X \to Y$ is a closed map and $\{f^{-1}(y)|y \in Y\}$ are components of $X$, then $Y$ is totally disconnected.*

This gives the following two applications (see Diamond [3]).

**Corollary 28** *If $\alpha X$ and $\gamma X$ are compactifications of $X$ and $f : \alpha X \to \gamma X$ is a projection map and if $\{f^{-1}(y)|y \in \gamma X\}$ partitions $\alpha X$ so that the remainder is partitioned into components, then $\gamma X$ has totally disconnected remainder. If, further, $X$ is locally compact, then $\gamma X$ has zero-dimensional remainder.*

**Corollary 29** *If $\alpha X$ is a compactification of $X$, $C_p$ is the connected component in $\alpha X \setminus X$ of $p \in \alpha X \setminus X$ and $\mathcal{D} = \{\{x\}|x \in X\} \cup \{C_p|p \in \alpha X \setminus X\}$ and $\mathcal{D}$ is upper semicontinuous, then $\alpha X/\mathcal{D}$ is a compactification with totally disconnected remainder. Further, if $X$ is locally compact, then $\alpha X/\mathcal{D}$ has zero-dimensional remainder.*

The last two results transfer readily to the ordered setting.

**Corollary 30** *If $\alpha X$ and $\gamma X$ are ordered compactifications of $X$ and $f : \alpha X \to \gamma X$ is a projection map and if $\{f^{-1}(y)|y \in \gamma X\}$ partitions $\alpha X$ so that the remainder is partitioned into components, then $\gamma X$ has totally disconnected remainder. If, further, $X$ is locally compact, then $\gamma X$ has zero-dimensional remainder.*

**Corollary 31** *Let $\alpha X$ be an ordered compactification of $X$, and $C_p$ be the connected component in $\alpha X \setminus X$ of $p \in \alpha X \setminus X$. If $\mathcal{D} = \{\{x\}|x \in X\} \cup \{C_p|p \in \alpha X \setminus X\}$ and $\mathcal{D}$ is an admissible ordered upper semicontinuous decomposition, then the ordered quotient space $\alpha X/\mathcal{D}$ is a $T_2$-ordered compactification with totally disconnected remainder. Further, if $X$ is locally compact, then $\alpha X/\mathcal{D}$ has zero-dimensional remainder.*

In the ordered setting, however, one can talk about both totally disconnected ordered spaces and o-totally disconnected ordered spaces. We next turn to translating the results of this section to the o-totally disconnected setting. Recall that an ordered space $X$ has an *o-disconnection* if there exist a nonempty increasing open set $X_1$ and a nonempty decreasing open set $X_2$ such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. A space is *o-connected* if it has no o-disconnection. An o-disconnection is equivalent to an ordered space having either a proper nonempty increasing clopen set or a proper nonempty decreasing clopen set. An *o-component* is a maximal o-connected set.

**Proposition 32** *If $f : X \to Y$ is an ordered quotient map and $Y$ has the quotient order, then for any convex subspace $L$ of $Y$, the restriction $f_L : f^{-1}(L) \to L$ is also an ordered quotient map.*

*Proof.* That $f_L$ is increasing is trivial, continuous is well known, and closed (quotient) is 2.1.4 of Engelking [4]. It remains to show that $f_L^{-1}(A)$ increasing implies that $A$ is increasing. By way of contradiction, suppose $A$ is not increasing. Then there exists $y \in L \setminus A$ and $a \in A$ such that $y \geq a$. Then $f^{-1}(y) \geq f^{-1}(a)$ in the decomposition $\mathcal{D}$ induced by $f$. By the finite step order there exists $z_1, \ldots, z_n \in X$ with $f^{-1}(a) = [z_1]$, $f^{-1}(y) = [z_n]$, and $z_i', z_i^* \in [z_i]$ such that $z_i \leq z_{i+1}^*$. Since $f^{-1}(A)$ is saturated, each $[z_i]$ is either in $f^{-1}(A)$ or disjoint from $f^{-1}(A)$. Since $L$ is convex, so is $f_L^{-1}(L)$. By convexity, each $[z_i] \subset f_L^{-1}(L)$. Since $[z_1] \subset f^{-1}(A)$ and $[z_n] \subset X \setminus f^{-1}(A)$, there must be an $i$ such that $[z_i] \subset f^{-1}(A)$ and $[z_{i+1}] \subset X \setminus f^{-1}(A)$ and $z_i' \leq z_{i+1}^*$. But this contradicts that $f_L^{-1}(A)$ is increasing. □

**Proposition 33** *If $X$ is an ordered space, $f : X \to Y$ is an order quotient map with o-connected fibers, and if $Y$ has the quotient order, then for every convex o-connected subset $C$ of $Y$, the inverse image $f^{-1}(C)$ is o-connected.*

*Proof.* By the preceding proposition, it suffices to show that if $Y$ is o-connected so is $X$. Let $X = X_1 \cup X_2$ where $X_1$ is open and increasing, $X_2$ is open and decreasing and $X_1 \cap X_2 = \emptyset$. Since $f$ has o-connected fibers, each $f^{-1}(y)$ must be completely contained in either $X_1$ or $X_2$. Therefore there exist $Y_1$ and $Y_2$ in $Y$ such that $f^{-1}(Y_1) = X_1$, $f^{-1}(Y_2) = X_2$, $Y_1 \cup Y_2 = Y$, and $Y_1 \cap Y_2 = \emptyset$. Since $f$ is a quotient map both $Y_1$ and $Y_2$ are open, and since $f$ is an ordered quotient map $Y_1$ is increasing and $Y_2$ is decreasing. Thus $Y_1$ and $Y_2$ form an o-separation of $Y$, unless one or the other is empty, hence unless $X_1$ or $X_2$ is empty. □

**Proposition 34** *The o-components of an ordered space are convex.*

*Proof.* Let $C$ be an o-component of an ordered space $X$. If $C$ contains only one point, it is trivially convex, so assume $C$ has at least two points. By way of contradiction, assume $C$ is not convex. Then there exists a point $b \notin C$ and points $a$ and $c$ in $C$ such that $a < b < c$. Since $C$ is an o-component, $C \cup \{b\}$ is not o-connected. There exists an increasing open set $U$ and a decreasing open set $V$ which are disjoint and whose union is $C \cup \{b\}$. The point $b$ is in either $U$ or $V$. Without loss of generality assume $b \in U$. Then $c \in U$. Therefore the sets $C \cap U$ and $C \cap V$ are nonempty and form an o-separation of $C$, contradicting the hypothesis that $C$ is an o-component. □

**Corollary 35** *If $X$ is an ordered space, $f : X \to Y$ is an ordered quotient map, and $\{f^{-1}(y)|y \in Y\}$ are o-components of $X$, then $Y$ is o-totally disconnected.*

*Proof.* If $Y$ has an o-component $A$ with more than one point, then $f^{-1}(A)$ is an o-connected set consisting of more than one component. □

Thus we have the following conclusions.

**Corollary 36** *If $\alpha X$ and $\gamma X$ are ordered compactifications of $X$ and $f : \alpha X \to \gamma X$ is a projection map and if $\{f^{-1}(y)|y \in \gamma X\}$ partitions $\alpha X$ so that the remainder is partitioned into o-components, then $\gamma X$ has o-totally disconnected remainder. If, further, $X$ is locally compact, then $\gamma X$ has o-zero-dimensional remainder.*

**Corollary 37** *Let $\alpha X$ be an ordered compactification of $X$ and $C_p$ be the o-component in $\alpha X \setminus X$ of $p \in \alpha X \setminus X$. If $\mathcal{D} = \{\{x\}|x \in X\} \cup \{C_p|p \in \alpha X \setminus X\}$ and $\mathcal{D}$ is an admissible ordered upper semicontinuous decompositon, then the ordered quotient space $\alpha X/\mathcal{D}$ is a $T_2$-ordered compactification with o-totally disconnected remainder. Further, if $X$ is locally compact, then $\alpha X/\mathcal{D}$ has o-zero-dimensional remainder.*

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