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Ordered separation axioms and the Wallman ordered compactification

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Abstract. Two constructions have been given previously of the Wallman ordered compactification w_0X of a T_1-ordered, convex ordered topological space $(X, \tau, \leq)$. Both of those papers note that w_0X is T_1, but need not be T_1-ordered. Using this as one motivation, we propose a new version of T_1-ordered, called T_1^K-ordered, which has the property that the Wallman ordered compactification of a T_1^K-ordered topological space is T_1^K-ordered. We also discuss the R_0-ordered (R_0^K-ordered) property, defined so that an ordered topological space is T_1-ordered (T_1^K-ordered) if and only if it is T_0-ordered and R_0-ordered (R_0^K-ordered).

1. Introduction

Given a set X with a topology τ and a partial order $\leq$, we recall the topology $\tau^\sharp$ on X as the collection of all τ-open $\leq$-increasing subsets of X and the topology $\tau^\flat$ on X as the corresponding collection of all τ-open $\leq$-decreasing subsets of X. Thus, we may consider the topological space (X, τ), the ordered topological space $(X, \tau, \leq)$, or the bitopological space $(X, \tau^\sharp, \tau^\flat)$.

As the study of ordered topological spaces and bitopological spaces developed, important topological properties, including separation axioms, were defined

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for these new categories. As a general unifying principle, it seems natural that the chain of implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) should hold for the statements

(1) $(X, \tau^\sharp, \tau^\flat)$ satisfies the bitopological (or *pairwise*) property P,

(2) $(X, \tau, \leq)$ satisfies the ordered property P, and

(3) (X, τ) satisfies the (topological) property P.

This scheme is borne out in particular by the complete regularity properties when certain other reasonable necessary conditions are assumed. However, as the ordered theory and bitopological theory were often developed independently, there are exceptions and anomalies, which we address here. In particular, the standard definition of the T_1-ordered property is seen to fall short of expectations in this regard, as well as in regard to its relation to the Wallman ordered compactification. In Section 2, we review the T_0-ordered and T_1-ordered properties and introduce the R_0-ordered property, noting their relation to the corresponding pairwise properties of $(X, \tau^\sharp, \tau^\flat)$. In Section 3, we introduce the T_1^K-ordered and R_0^K-ordered properties, and show that the T_1^K-ordered property does not have the shortcomings of the standard T_1-ordered property. Continuing in this theme, Section 4 shows that the Wallman ordered compactification behaves more nicely for T_1^K-ordered spaces than for T_1-ordered spaces.

Our notation is that of Nachbin [15]. If $(X, \tau, \leq)$ is a partially ordered topological space and $A \subseteq X$, then the *increasing hull of* A is $i(A) = \{x \in X : \exists\, a \in A$ with $a \leq x\}$. If $A = i(A)$, we say A is an *increasing set*. The *closed increasing hull of* A, denoted $I(A)$, is the smallest closed increasing set containing A. The decreasing hull $d(A)$, closed decreasing hull $D(A)$, and decreasing sets are defined dually. If $A = \{a\}$, we write $i(a)$ for $i(\{a\})$. Another useful hull operator, used in the construction of the Wallman ordered compactification, is $C(A) = I(A) \cap D(A)$. Following [5] and [6], $C(A)$ is the *c-set hull* of A, and if $A = C(A)$ we say A is a *c-set*. We say $(X, \tau, \leq)$ is *convex* if τ has a subbase of monotone (i.e., increasing or decreasing) open sets. In other terminology, note that $D(A) = \mathrm{cl}_{\tau^\sharp}(A)$ and $I(A) = \mathrm{cl}_{\tau^\flat}(A)$ where $\tau^\sharp = \{U \in \tau : U = i(U)\}$ and $\tau^\flat = \{U \in \tau : U = d(U)\}$.

The T_0 *(-ordered) reflection* of a (partially ordered) topological space X is the T_0(-ordered) quotient space Y of X such that for any continuous (and increasing) function f from X into any arbitrary T_0(-ordered) space Z, there exists a unique continuous (and increasing) function $h : Y \to Z$ with $f = h \circ q$, where $q : X \to Y$ is the quotient map. The construction of the T_0-ordered reflection, considered in Section 2, utilizes the equivalence relation defined by $x \approx y$ if and only if $I(x) = I(y)$ and $D(x) = D(y)$, or equivalently, if and only if $C(x) = C(y)$ (see [8]). Let $[x]$ denote the $\approx$-equivalence class of x. The closure operator $C(\cdot)$

defines a reflexive, transitive relation $C = \bigcup_{x\in X}\{x\} \times C(x)$. The inverse relation C^{-1} is given by $C^{-1}(x) = \bigcap\{U \in \tau^\sharp : x \in U\} \cap \bigcap\{V \in \tau^\flat : x \in V\}$. (Some authors would call this the intersection of the $\tau^\sharp$-kernel of $\{x\}$ and the $\tau^\flat$-kernel of $\{x\}$.) If $C^s = C \cap C^{-1}$ is the symmetrization of C, then $C^s(x) = [x]$. Note that if X is convex, we have $x \approx y$ if and only if $\mathrm{cl}\{x\} = \mathrm{cl}\{y\}$. Also if X is convex, $[x] = \mathrm{cl}\{x\} \cap \bigcap\{U \in \tau : x \in U\} \subseteq \mathrm{cl}\{x\} = C(x)$.

2. T_0, T_1, and R_0 separation properties

T_0 Properties

We recall the following definitions (see [17]).

Definition 1. Suppose X is a set, τ is a topology on X, and $\leq$ is a partial order on X.

(a) (X, τ) is T_0 if $\mathrm{cl}(x) = \mathrm{cl}(y)$ implies $x = y$.

(b) $(X, \tau, \leq)$ is T_0-ordered if for any two distinct points, there exists a monotone open neighborhood of one of the points which does not contain the other, or equivalently, if $C^s(x) = \{x\}$ for all $x \in X$.

(c) (X, τ_1, τ_2) is weak pairwise T_0 if for distinct points x and y, one of the points is not in one of the closures (τ_1 or τ_2) of the other.

(d) (X, τ_1, τ_2) is pairwise T_0 if for distinct points x and y, either $x \notin \mathrm{cl}_{\tau_1}(y)$ or $y \notin \mathrm{cl}_{\tau_2}(x)$.

Note that pairwise T_0 is defined by $x \neq y \Rightarrow ([x \notin \mathrm{cl}_{\tau_1}(y) \vee y \notin \mathrm{cl}_{\tau_2}(x)] \wedge [y \notin \mathrm{cl}_{\tau_1}(x) \vee x \notin \mathrm{cl}_{\tau_2}(y)])$, while weak pairwise T_0 replaces the $\wedge$ between the bracketed items by $\vee$. Clearly pairwise T_0 implies weak pairwise T_0 and, from the open set characterization, it is easily seen that $(X, \tau, \leq)$ being T_0-ordered implies (X, τ) is T_0. Now $(X, \tau^\sharp, \tau^\flat)$ is weak pairwise T_0 if and only if $x \neq y$ implies $[x \notin D(y)$ or $x \notin I(y)$ or $y \notin D(x)$ or $y \notin I(x)]$. Since the bracketed condition is equivalent to $C(x) \neq C(y)$, we see that $(X, \tau^\sharp, \tau^\flat)$ is weakly pairwise T_0 if and only if $(X, \tau, \leq)$ is T_0-ordered. Furthermore, if $(X, \tau, \leq)$ is convex, these two properties are also equivalent to (X, τ) being T_0.

T_1 Properties

The standard definition of a T_1-ordered space, given by NACHBIN [15], does not meet many of the expectations one would have for T_1-ordered spaces. For example, if $(X, \tau^\sharp, \tau^\flat)$ is pairwise T_1, it does not follow that $(X, \tau, \leq)$ is T_1-ordered. Also, the Wallman ordered compactification of a T_1-ordered topological space is

T_1, but need not be T_1-ordered. These deficiencies in the standard definition of T_1-ordered led us to define a new version in the next section. Here we present the standard definitions.

Definition 2. Suppose X is a set, τ is a topology on X, and $\leq$ is a partial order on X.

(a) (X, τ) is T_1 if $\mathrm{cl}(x) = \{x\}$ for all $x \in X$.

(b) $(X, \tau, \leq)$ is T_1-ordered if $d(x)$ and $i(x)$ are closed for all $x \in X$.

(c) (X, τ_1, τ_2) is pairwise T_1 (see [14]) if for any two distinct points x and y in X, each has either a τ_1-open or a τ_2-open neighborhood which excludes the other.

Our definition of pairwise T_1, given by Murdeshwar and Naimpally [14] is called MN-pairwise-T_1 by Reilly [17] and Mršević [12]. Several other versions of pairwise T_1 have been studied.

(1) (Reilly [17]) (X, τ_1, τ_2) is *Reilly pairwise* T_1 if for distinct points x and y in X, there exists a τ_1-open neighborhood of x excluding y and a τ_2-open neighborhood of y excluding x.

(2) (Mršević [12]) (X, τ_1, τ_2) is *middle pairwise* T_1 if for distinct points x and y in X, $\mathrm{cl}_{\tau_1}(x) \cap \mathrm{cl}_{\tau_2}(y) = \emptyset$ or $\mathrm{cl}_{\tau_1}(y) \cap \mathrm{cl}_{\tau_2}(x) = \emptyset$.

(3) (Swart [19]) (X, τ_1, τ_2) is *weak pairwise* T_1 if for distinct points x and y in X, there exists a τ_1-open neighborhood of x excluding y and a τ_2-open neighborhood of y excluding x, or there exists a τ_1-open neighborhood of y excluding x and a τ_2-open neighborhood of x excluding y.

We note that, given an ordered topological space $(X, \tau, \leq)$, the corresponding bitopological space $(X, \tau^\sharp, \tau^\flat)$ is Reilly pairwise T_1 only if the order $\leq$ is discrete (that is, equality), rendering this version not useful in the study of ordered topological spaces. The version due to Mršević is called "middle" since it is implied by Reilly pairwise T_1 and implies Swart's weak pairwise T_1. All of these versions of pairwise T_1 are stronger than our definition of (MN) pairwise T_1. Some elementary facts regarding these definitions are found in [4] and [18]. The fact that many versions of the pairwise T_1 property have been considered suggests that there may be more than one useful version of the T_1-ordered property.

The new version of T_1-ordered introduced below, called T_1^K-ordered, is implied by $(X, \tau^\sharp, \tau^\flat)$ being (MN) pairwise T_1, the weakest of these pairwise T_1 properties.

It is easily seen that $(X,\tau,\leq)$ being T_1-ordered implies (X,τ) is T_1, and $(X,\tau^\sharp,\tau^\flat)$ being pairwise T_1 implies (X,τ) is T_1. However, $(X,\tau^\sharp,\tau^\flat)$ being pairwise T_1 does not imply $(X,\tau,\leq)$ is T_1-ordered. Indeed, the example after Theorem 15 shows that $(X,\tau^\sharp,\tau^\flat)$ being middle pairwise T_1 (the strongest of these pairwise T_1 properties which can possibly be satisfied by ordered spaces other than antichains) does not imply $(X,\tau,\leq)$ is T_1-ordered. That example also shows that $(X,\tau^\sharp,\tau^\flat)$ being middle pairwise T_1 does not imply that $(X,\tau^\sharp,\tau^\flat)$ is pairwise R_0 (for the definition of this concept, see below).

The following characterization of the T_1-ordered property will be used often.

Lemma 3. *$(X,\tau,\leq)$ is T_1-ordered if and only if $i(x)=\bigcap\{U\in\tau^\sharp : x\in U\}$ and dually for all $x\in X$.*

Proof. If X is T_1-ordered, then $y\notin i(x)$ implies $x\notin d(y)=D(y)$, so $i(x)\subseteq X\setminus D(y)\equiv H_y$, and thus $i(x)=\bigcap\{H_y : y\notin i(x)\}\supseteq\bigcap\{U\in\tau^\sharp : x\in U\}$. The reverse inclusion is immediate. Conversely, if $y\notin i(x)$, then $x\notin d(y)=\bigcap\{U\in\tau^\flat : y\in U\}$, so there exists an open decreasing neighborhood of y disjoint from x and therefore from $i(x)$, so $y\notin \mathrm{cl}(i(x))$, and thus $i(x)$ is closed. □

R_0 Properties

One motivation for R_0 spaces, introduced by Davis [3], is that a space is T_1 if and only if it is T_0 and R_0. The analogous equivalences in the categories of ordered topological spaces and bitopological spaces are used to motivate definitions of the appropriate concepts of R_0 in those categories. Note however, as seen in the example mentioned before Lemma 3, that the weak concept of pairwise T_1 used in this article does not imply pairwise R_0 as defined below. It appears that R_0-ordered spaces have not been studied previously, so our discussion here will be more thorough. We start with some theorems providing equivalent definitions of the R_0 properties in the three categories in question.

Theorem 4. *The following are equivalent:*

(a) (X,τ) *is an* R_0*-space.*

(b) F *closed*, $x\notin F \Rightarrow \exists$ *open* U *with* $F\subseteq U$, $x\notin U$.

(c) U *open*, $x\in U \Rightarrow \mathrm{cl}\{x\}\subseteq U$.

(d) $\{\mathrm{cl}\{x\} : x\in X\}$ *is a partition of* X.

(e) τ *is lattice isomorphic to the topology of a* T_1*-space.*

(f) $\mathrm{cl}\{x\}\neq\mathrm{cl}\{y\}\Rightarrow\exists$ *neighborhood of* x *not containing* y.

(g) F *closed*, $\mathrm{cl}\{x\}\cap F\neq\emptyset\Rightarrow x\in F$.

(h) $\bigcap\{U\in\tau : x\in U\}=\mathrm{cl}\{x\}$ *for all* $x\in X$.

(i) *The* T_0*-reflection of* X *is* T_1.

(j) $\bigcap\{U \in \tau : A \subseteq U\} \subseteq \operatorname{cl} A$ *for all* $A \subseteq X$.

These equivalences may be found in [3] or [1], except for (j), which is easily shown. (In [1], a $T_{(\alpha,\beta)}$ space is one whose T_α reflection is already T_β, so there, R_0 spaces are called $T_{(0,1)}$ spaces.) Note that the containment in (j) cannot be strengthened to equality: Let $X = [0,1]$ with the discrete topology on $(0,1]$ and the usual neighborhoods of 0. Now (b) holds, but $A = (0,1] = \bigcap\{U \in \tau : A \subseteq U\} \neq \operatorname{cl} A = [0,1]$.

It will be interesting to note that many of the characterizations of R_0 given above have direct analogs in either the ordered topological setting or the bitopological setting, but not both.

The definition given below for an R_0-ordered space arises from the necessary and sufficient conditions in [8] for the T_0-ordered reflection of an ordered topological space to be T_1-ordered. For an ordered topological space $(X, \tau, \leq)$, we obtain the T_0-ordered reflection $(X/\approx, \tau^0, \leq^0)$ as an ordered quotient of X, modulo the equivalence relation $x \approx y$ if and only if $C(x) = C(y)$. The order $\leq^0$ on $X/\approx$ is the "finite step order" given by

$$[z_0] \leq^0 [z_n] \iff \begin{array}{l} \exists [z_1], [z_2], \ldots, [z_{n-1}] \text{ and } \exists z_i', z_i^* \in [z_i] \ (i = 0, 1, \ldots, n) \\ \text{with } z_i' \leq z_{i+1}^* \ \forall i = 0, 1, \ldots, n-1. \end{array}$$

Note that any closed or open monotone set S in X is $\approx$-saturated (that is, $x \in S$ implies $[x] \subseteq S$).

We note that Mršević [13] introduced a bitopological "quotient space" which, in the case of $(X, \tau^\sharp, \tau^\flat)$, is equivalent to the T_0-ordered reflection of $(X, \tau, \leq)$.

Lemma 5. *Suppose* $x, y \in X$, $F \subseteq X$, *and* $f : X \to X/\approx$ *is the quotient map from an ordered topological space* X *to its* T_0*-ordered reflection* $X/\approx$.

(a) *If* A *is closed and increasing in* X *then* $f(A)$ *is closed and increasing in* $X/\approx$, *and dually. If* A *is* $\approx$*-saturated, the converse holds.*

(b) B *is closed and increasing in* $X/\approx$ *if and only if* $f^{-1}(B)$ *is closed and increasing in* X, *and dually.*

(c) $f(I(x)) = I_{X/\approx}([x])$ *and* $f(D(x)) = D_{X/\approx}([x])$.

(d) $f^{-1}(I_{X/\approx}(f(x)) = f^{-1}(I_{X/\approx}([x])) = I(x)$ *and dually.*

(e) *If* $[y] \in f(D(F))$, *and* $[x] \leq [y]$ *in* $X/\approx$, *then* $x \in D(F)$, *and dually.*

Proof. (a), (b), and (c) were stated and justified in the paragraph before Theorem 3.1 of [8]. See also Corollary 2 and Proposition 5 of [13].

(d) $I(x) \subseteq f^{-1}(I_{X/\approx}([x]))$ since the latter set is a closed increasing set containing x and the former is the smallest such set. For the reverse inclusion, if $z \in f^{-1}(I_{X/\approx}([x]))$ but $z \notin I(x)$, then since $I(x)$ is saturated, applying f (and part (c)) gives $f(z) \notin f(I(x)) = I_{X/\approx}([x])$, contrary to $z \in f^{-1}(I_{X/\approx}([x]))$.

(e) Suppose $[y] \in f(D(F))$ with $[x] \leq [y]$ in $X/\approx$. Now $D(F)$ is closed and decreasing, and thus saturated, and contains y. Since $X/\approx$ carries the finite step order, it follows that $x \in D(F)$. □

Any of the equivalent statements below may be taken as the definition of an R_0-ordered space.

Theorem 6. *For an ordered topological space $(X, \tau, \leq)$ with T_0-ordered reflection $X/\approx$, the following are equivalent:*

(a) *$(X, \tau, \leq)$ is R_0-ordered.*

(b) *$\bigcap\{U \in \tau^\sharp : x \in U\} = i([x])$ and dually for all $x \in X$.*

(c) *The T_0-ordered reflection of X is T_1-ordered.*

(d) *$[x] \not\leq [y]$ in $X/\approx$ implies $x \notin D(y)$ and $y \notin I(x)$.*

(e) *$I(x) = f^{-1}(i_{X/\approx}([x]))$ and $D(x) = f^{-1}(d_{X/\approx}([x]))$ for all $x \in X$, where $f : X \to X/\approx$ is the natural ordered quotient map.*

Proof. The equivalence of (b) and (c) is Theorem 3.2 of [8], and was the impetus for taking these conditions to be the definition of R_0-ordered.

(c) $\Rightarrow$ (d). The proof of this implication is modeled on the proof of the corresponding non-ordered case (Theorem 3.5 (i) $\Rightarrow$ (ii) in [1]). Suppose (c) holds and $[x] \not\leq [y]$ in $X/\approx$. Then $[y] \notin i_{X/\approx}([x]) = I_{X/\approx}([x])$. Applying f^{-1}, where $f : X \to X/\approx$ is the (ordered) quotient map, we have $y \notin f^{-1}(I_{X/\approx}([x])) = f^{-1}(f(I(x)) \supseteq I(x)$, so $y \notin I(x)$. Similarly, $[x] \notin d_{X/\approx}([y])$ implies $x \notin D(y)$.

(d) $\Rightarrow$ (c). Suppose (d) holds and $[y] \notin i_{X/\approx}([x])$. Then $x \notin D(y)$ implies that $X \setminus D(y)$ is an open increasing (and thus, saturated) neighborhood of x which does not include y, so $[y] \notin I_{X/\approx}([x])$. Similarly, $y \notin I(x)$ shows $X \setminus I(x)$ is an open decreasing saturated neighborhood of y not containing x.

(c) $\Rightarrow$ (e). Suppose (c) holds. Then $i_{X/\approx}([x])$ is closed and increasing, so $f^{-1}(i_{X/\approx}([x]))$ is closed, increasing, and contains x, so $I(x) \subseteq f^{-1}(i_{X/\approx}([x]))$. For the reverse inclusion, if $z \in f^{-1}(i_{X/\approx}([x]))$, then $[z] = f(z) \in i_{X/\approx}([x]) = I_{X/\approx}([x]) = f(I(x))$, and thus $z \in I(x)$ by Lemma 5 (e). Thus, we have $I(x) = f^{-1}(i_{X/\approx}([x]))$. The dual argument completes this implication.

(e) $\Rightarrow$ (c). Suppose (e). Then by Lemma 5 (d), we have $f^{-1}(I_{X/\approx}([x])) = f^{-1}(i_{X/\approx}([x]))$ and $f^{-1}(D(_{X/\approx}[x])) = f^{-1}(d_{X/\approx}([x]))$ for all $x \in X$, and applying f, which is onto, shows $I_{X/\approx}([x]) = i_{X/\approx}([x])$ and $D_{X/\approx}([x]) = d_{X/\approx}([x])$ for all $[x] \in X/\approx$, so $X/\approx$ is T_1-ordered. □

We note that items (b), (c), and (d) of Theorem 6 are direct analogs of items (h), (i), and (f), respectively, of Theorem 4.

Theorem 7. *An ordered topological space $(X, \tau, \leq)$ is T_1-ordered if and only if it is T_0-ordered and R_0-ordered.*

Proof. Suppose $(X, \tau, \leq)$ is T_0-ordered and R_0-ordered. The former condition implies $[x] = \{x\}$ and the latter condition then implies $i(x) = \bigcap\{U \in \tau^\sharp : x \in U\}$ and dually, so $(X, \tau, \leq)$ is T_1-ordered by Lemma 3.

Conversely, suppose $(X, \tau, \leq)$ is T_1-ordered. Then $[x] = C^s(x) = D(x) \cap I(x) \cap D^{-1}(x) \cap I^{-1}(x) = d(x) \cap i(x) \cap d^{-1}(x) \cap i^{-1}(x) = \{x\}$, so X is T_0-ordered. Thus, $i([x]) = i(C^s(x)) = i(x) = \bigcap\{U \in \tau^\sharp : x \in U\}$, by Lemma 3. With the dual argument, Theorem 6 (b) shows that X is R_0-ordered. □

Misra and Dube [11] give several characterizations of the pairwise R_0 property, including the following. The equivalence of (a) and (b) is taken to be the definition of the pairwise R_0 property.

Theorem 8 ([11])**.** *For a bitopological space (X, τ_1, τ_2), the following are equivalent:*

(a) *(X, τ_1, τ_2) is pairwise R_0.*

(b) *For every $x \in X$, if U is a τ_i-open neighborhood of x, then $\mathrm{cl}_{\tau_j}(x) \subseteq U$, where $\{i, j\} = \{1, 2\}$.*

(c) *$y \in \mathrm{cl}_{\tau_i}(x) \iff x \in \mathrm{cl}_{\tau_j}(y)$ for $\{i, j\} = \{1, 2\}$.*

(d) *If F is τ_i-closed and $x \notin F$, then there exists a τ_j-open set U with $F \subseteq U$ and $x \notin U$, for $\{i, j\} = \{1, 2\}$.*

(e) *$\bigcap\{U \in \tau_i : F \subseteq U\} = F$ for any τ_j-closed $F \subseteq X$, for $\{i, j\} = \{1, 2\}$.*

(f) *$\mathrm{cl}_{\tau_i}(x) \cap \mathrm{cl}_{\tau_j}(F) \neq \emptyset \Rightarrow x \in \mathrm{cl}_{\tau_j}(F)$ for all $x \in X$, $F \subseteq X$, and for $\{i, j\} = \{1, 2\}$.*

We observe that items (b), (d), and (f) of Theorem 8 are pairwise versions of items (c), (b), and (g), respectively, of Theorem 4. Of the standard R_0 defining items from Theorem 4, we found none which had direct analogs to both the ordered setting and the bitopological setting. This fact has also prompted our reconsideration of the standard definition of T_1-ordered, and subsequently, of R_0-ordered. We note that (X, τ_1, τ_2) being pairwise R_0 implies $(X, \tau_1 \vee \tau_2)$ is R_0 (Proposition 9 of [13]).

3. T_1^K-ordered and R_0^K-ordered spaces

Since the usual definitions of T_1-ordered, and hence R_0-ordered have some deficiencies as noted above, we propose the following new versions, denoted T_1^K-ordered and R_0^K-ordered.

Definition 9. An ordered topological space $(X, \tau, \leq)$ is T_1^K-ordered if $C(x) = \{x\}$ for all $x \in X$.

We make the following observations.

Theorem 10. *Suppose $(X, \tau, \leq)$ is an ordered topological space.*

(a) $(X, \tau, \leq)$ *is T_1^K-ordered if and only if* $\{x\} = \bigcap\{U \in \tau^\sharp \cup \tau^\flat : x \in U\}$ *for every* $x \in X$.

(b) $(X, \tau^\sharp, \tau^\flat)$ *is pairwise* $T_1 \iff (X, \tau, \leq)$ *is T_1^K-ordered* $\Rightarrow (X, \tau)$ *is* T_1, *and* $(X, \tau, \leq)$ *is T_1-ordered* $\Rightarrow (X, \tau, \leq)$ *is T_1^K-ordered.* $(X, \tau, \leq)$ *is T_1^K-ordered and convex* $\iff (X, \tau)$ *is* T_1 *and* $(X, \tau, \leq)$ *is convex.*

(c) *For a linearly ordered space $(X, \tau, \leq)$, T_1-ordered and T_1^K-ordered are equivalent.*

Proof. (a) is proved analogously to the corresponding characterization of T_1-ordered given in Lemma 3. The proof of (b) follows immediately from (a), Definitions 2 and 9, and the last sentence of Section 1. For (c), if X is linearly ordered and T_1^K-ordered and $y \notin i(x)$, then $y \in d(x) \subseteq D(x)$, but $y \notin \{x\} = C(x) = I(x) \cap D(x)$, so $y \notin I(x)$. Thus, $i(x) = I(x)$, and with the dual argument, X is T_1-ordered. $\square$

The example below shows that T_1^K-ordered is not equivalent to the middle- or weak- pairwise T_1 properties applied to $(X, \tau^\sharp, \tau^\flat)$.

Example 11. Let $X = \mathbb{N} \cup \{e, \pi\}$, where $\mathbb{N}$ is the set of natural numbers. Take all points in $\mathbb{N}$ to be topologically isolated and take the neighborhoods of $x \in \{e, \pi\}$ to be cofinite sets in X containing x. Now X is a T_1 space. Let $\leq = \Delta \cup \{(2n, e) : n \in \mathbb{N}\} \cup \{(2n-1, \pi) : n \in \mathbb{N}\}$. This is a partial order. Note that e and π are the only nonisolated points in X and for each $x \in X$, $d(x)$ and $i(x)$ are finite and thus closed, except for $d(\pi)$ and $d(e)$. Observe that $\pi \in D(e)$

and $e \in D(\pi)$. It immediately follows that $C(x) = \{x\}$ for every $x \in X$. Thus, X is a T_1^K-ordered space.

Consider e and π. There exists no increasing (open) neighborhood of e that does not contain π and no increasing (open) neighborhood of π that does not contain e. Thus, the associated bispace $(X, \tau^\sharp, \tau^\flat)$ is not a weak pairwise T_1 space, and hence is not middle pairwise T_1. Noting that $\pi \in \bigcap\{U \in \tau^\sharp : e \in U\} \neq i(e) = \{e\}$, Lemma 3 shows that X is not T_1-ordered. This is also easily seen since $d(\pi)$ is not closed.

Now we define R_0^K-ordered spaces and subsequently list some of their properties.

Definition 12. An ordered topological space $(X, \tau, \leq)$ is R_0^K-ordered if, for all $x, y \in X$, we have $y \in C(x)$ implies $x \in C(y)$.

Theorem 13. *Suppose $(X, \tau, \leq)$ is an ordered topological space.*

(a) *The following are equivalent:*
  - (i) *$(X, \tau, \leq)$ is R_0^K-ordered.*
  - (ii) *$C = C^{-1}$.*
  - (iii) *C is an equivalence relation.*
  - (iv) *$x \in C(y)$ if and only if $C(x) = C(y)$ for all $x, y \in X$.*
  - (v) *$x \in C(y)$ if and only if $[x] = [y]$ for all $x, y \in X$.*
  - (vi) *$\{C(x) : x \in X\}$ is a partition of X.*

(b) *$(X, \tau^\sharp, \tau^\flat)$ is pairwise $R_0 \Rightarrow (X, \tau, \leq)$ is R_0^K-ordered. $(X, \tau, \leq)$ is R_0-ordered $\Rightarrow (X, \tau, \leq)$ is R_0^K-ordered. $(X, \tau, \leq)$ is R_0^K-ordered and convex $\iff (X, \tau)$ is R_0 and $(X, \tau, \leq)$ is convex.*

(c) *If $(X, \tau, \leq)$ is a linearly ordered space, then R_0-ordered and R_0^K-ordered are equivalent properties.*

Proof. (a) is immediate.

(b) The first implication follows from a comparison of part (iv) of (a) above and Theorem 8 (c). X is R_0-ordered if and only if $X/\approx$ is T_1-ordered, which implies $X/\approx$ is T_1^K-ordered, and hence (by Theorem 15 below) X is R_0^K-ordered.

Suppose X is R_0^K-ordered and convex. We will show that $\{\mathrm{cl}(x) : x \in X\}$ partitions X. Suppose $z \in \mathrm{cl}(x) \cap \mathrm{cl}(y)$. Now $z \in \mathrm{cl}(x) \subseteq C(x)$ implies, by part (v) of (a), that $[x] = [z]$, and similarly $[y] = [z] = [x]$. Recalling that convexity implies $[w] \subseteq \mathrm{cl}(w)$ for any $w \in X$, we have $x \in [x] = [y] \subseteq \mathrm{cl}(y)$ and $y \in [y] = [x] \subseteq \mathrm{cl}(x)$. Applying the closure operator now shows $\mathrm{cl}(x) \subseteq \mathrm{cl}(y) \subseteq \mathrm{cl}(x)$, so $\mathrm{cl}(x) = \mathrm{cl}(y)$, and thus (X, τ) is R_0 by Theorem 4 (d). Conversely, suppose $(X, \tau, \leq)$ is convex and (X, τ) is R_0. We will show that $z \in C(x)$ implies $C(z) = C(x)$, from which it easily follows that $\{C(x) : x \in X\}$ partitions X and thus X is R_0^K-ordered. If $z \in C(x)$ and $x \in C(z)$, applying the closure operator C shows $C(x) = C(z)$. Thus, suppose $z \in C(x)$ and $x \notin C(z)$. Then $x \notin \mathrm{cl}(z)$, so $\mathrm{cl}(x) \neq \mathrm{cl}(z)$. Theorem 4 (f) implies the existence of a neighborhood N of z which does not contain x, and by convexity, we may assume N is monotone open. If N is increasing, then $X \setminus N$ is a closed decreasing set containing x and excluding z. This shows $z \notin D(x)$, giving the contradiction that $z \notin C(x)$. The dual argument applies if N is decreasing.

(c) If X is a linearly ordered R_0^K-ordered space, by Theorem 15 below, $X/\approx$ is T_1^K-ordered. It is easy to see that the finite step order on $X/\approx$ is also linear, in which case $X/\approx$ is T_1-ordered, and therefore X is R_0-ordered. The converse follows from part (b). $\square$

Observing the appearance of convexity in (b) above, we note that R_0^K-ordered need not imply R_0 if the topology is not convex. For example, consider $X = \{\bot, a, b, \top\}$ where a and b are noncomparable and $\bot \leq x \leq \top$ for all $x \in X$. Give X the topology having $\{\{\top, \bot\}, \{a, b\}, \{a\}\}$ as base of closed sets. It is easy to check that $C(x) = X$ for each $x \in X$, so X is R_0^K-ordered. In fact, X is R_0-ordered since $i([x]) = i(X) = X = \bigcap\{U \in \tau^\sharp : x \in U\}$ and dually for each $x \in X$. However, $\{\mathrm{cl}(x) : x \in X\}$ does not partition X, so X is not R_0.

The next theorems show that the R_0^K-ordered and T_0^K-ordered properties interact as one would hope.

Theorem 14. *$(X, \tau, \leq)$ is T_1^K-ordered if and only if it is T_0-ordered and R_0^K-ordered.*

Proof. If X is T_0-ordered and R_0^K-ordered, then for all $x \in X$ we have $\{x\} = C^{-1}(x) \cap C(x) = C(x) \cap C(x) = C(x)$, where the first equality follows from the T_0-ordered property and the second equality from the R_0^K-ordered property. Thus, X is T_1^K-ordered. Conversely, if X is T_1^K-ordered, then $C(x) = \{x\}$ implies $C(x) \cap C^{-1}(x) = \{x\}$, so that X is T_0-ordered. Also, $y \in C(x) = \{x\}$ implies $x \in C(y)$, so X is R_0^K-ordered. $\square$

Theorem 15. *The T_0-ordered reflection $X/\approx$ of an ordered topological space X is T_1^K-ordered if and only if X is R_0^K-ordered.*

Proof. If $X/\approx$ is T_1^K-ordered, then $I_{X/\approx}([x]) \cap D_{X/\approx}([x]) = \{[x]\}$ for all $[x] \in X/\approx$. Applying f^{-1} as in Lemma 5 (d) gives $C(x) = I(x) \cap D(x) = [x]$. Now $y \in C(x) = [x] \iff C(x) = C(y)$, so X is R_0^K-ordered by Theorem 13 (a).

Conversely, suppose X is R_0^K-ordered and $[y] \in C_{X/\approx}([x]) = I_{X/\approx}([x]) \cap D_{X/\approx}([x])$. Applying f^{-1} as in Lemma 5 (d) gives $[y] \subseteq I(x) \cap D(x) = C(x)$, and now $y \in C(x)$ implies $[y] = [x]$. It follows that $C_{X/\approx}([x]) = \{[x]\}$ for any $[x] \in X/\approx$, so $X/\approx$ is T_1^K-ordered. This direction of the proof also follows from the bitopological quotient construction of Theorem 3.1 of [20], which also appears as Corollary 6 of [13]. $\square$

$(X, \tau^\sharp, \tau^\flat)$		$(X, \tau, \leq)$		(X, τ)
pairwise T_0	$\Rightarrow$	T_0-ordered	$\Rightarrow$	T_0
weak pairwise T_0	$\Longleftrightarrow$	T_0-ordered	$\Rightarrow$	T_0
		T_0-ordered + convex	$\Longleftrightarrow$	$\begin{cases} T_0\ + \\ (X, \tau, \leq) \text{ convex} \end{cases}$
pairwise R_0	$\not\Rightarrow$	R_0-ordered	$\not\Rightarrow$	R_0
		$\Downarrow$		
pairwise R_0	$\Rightarrow$	R_0^K-ordered	$\not\Rightarrow$	R_0
		R_0^K-ordered + convex	$\Longleftrightarrow$	$\begin{cases} R_0\ + \\ (X, \tau, \leq) \text{ convex} \end{cases}$
pairwise T_1	$\not\Rightarrow$	T_1-ordered	$\Rightarrow$	T_1
		$\Downarrow$		
pairwise T_1	$\Longleftrightarrow$	T_1^K-ordered	$\Rightarrow$	T_1
		T_1^K-ordered + convex	$\Longleftrightarrow$	$\begin{cases} T_1\ + \\ (X, \tau, \leq) \text{ convex} \end{cases}$
pairwise completely regular + $(X, \tau, \leq)$ convex + $(X, \tau, \leq)$ T_1-ordered	$\Rightarrow$	completely regularly ordered	$\Rightarrow$	completely regular

We note that T_1-ordered is a strictly stronger property than T_1^K-ordered. For example, consider the interval $[0, 1] \subseteq \mathbb{R}$ with the usual topology. Impose the usual order on $(0, 1]$, with 0 noncomparable to all other points. This ordered topological space is easily seen to be $(\tau^\sharp, \tau^\flat)$-pairwise T_1, and hence T_1^K-ordered, but since, for example, $d(1) = (0, 1]$ is not closed, it is not T_1-ordered. Furthermore, since T_1^K-ordered implies T_0-ordered, this space is its own T_0-ordered reflection. Now the characterizations in Theorem 6 and Theorem 15 show that this space is R_0^K-ordered but not R_0-ordered. Thus, R_0-ordered is a strictly stronger property than R_0^K-ordered.

The table above summarizes the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) suggested in the introduction. A bitopological space is *pairwise completely regular* (see [9]) if and only if it admits a compatible quasi-uniformity $\mathcal{U}$ in the sense that $\tau(\mathcal{U})$ is the first topology and $\tau(\mathcal{U}^{-1})$ is the second topology. An ordered topological space is *completely regularly ordered* (see [7], [16]) if and only if there is a quasi-uniformity $\mathcal{U}$ with $\bigcap\mathcal{U} =\leq$ and $\tau(\mathcal{U}) \vee \tau(\mathcal{U}^{-1}) = \tau$. Such spaces are always convex and T_1-ordered. Completely regular topological spaces are the uniformizable spaces.

The following result is analogous to Theorem 4 (e).

Theorem 16. *An ordered topological space $(X, \tau, \leq)$ is R_0^K-ordered if and only if there is a lattice isomorphism φ between τ and the topology τ_1 of some T_1^K-ordered topological space $(X_1, \tau_1, \leq_1)$ such that the restrictions $\varphi|_{\tau^\sharp}$ and $\varphi|_{\tau^\flat}$ are lattice isomorphisms from $\tau^\sharp$ onto $\tau_1^\sharp$ and from $\tau^\flat$ onto $\tau_1^\flat$, respectively.*

Proof. Suppose $(X, \tau, \leq)$ is an ordered topological space, $(X_1, \tau_1, \leq_1)$ is a T_1^K-ordered topological space, and there exists a lattice isomorphism from τ to τ_1 which induces lattice isomorphisms from $\tau^\sharp$ to $\tau_1^\sharp$ and from $\tau^\flat$ to $\tau_1^\flat$. Now $\{x\} = C(x) = I(x) \cap D(x)$ for every $x \in X_1$, so every τ_1-closed set F is a disjoint union of minimal nonempty closed sets, each of which is the intersection of a closed increasing set and a closed decreasing set. Correspondingly, in X, each closed set, and in particular, X, is partitioned into minimal nonempty closed sets, each of which is the intersection of a closed increasing set and a closed decreasing set. For $x \in X$, let $\hat{x}$ be the member of this partition of X which contains x. Then $\hat{x} = C(x)$, so $\{C(x) : x \in X\}$ partitions X, and thus X is R_0^K-ordered.

For the converse, take $(X_1, \tau_1, \leq_1)$ to be the T_0-ordered reflection of X. The result follows from Lemma 5 (a) and (b), noting that closed or open monotone sets in X are $\approx$-saturated. $\square$

As additional evidence that the standard definition of T_1-ordered and the corresponding definition of R_0-ordered are not optimal, we show that Theorem 16 does not remain valid if R_0^K-ordered and T_1^K-ordered are replaced by R_0-ordered and T_1-ordered. Let $N = \{a, b, c, d\}$ with the order $\Delta_N \cup \{(a, b), (c, d), (c, b)\}$, and the topology $\tau = \tau^\sharp \cup \tau^\flat$ where $\tau^\sharp = \{\emptyset, \{a, b\}, N\}$ and $\tau^\flat = \{\emptyset, \{c, d\}, N\}$. Now $i([c]) = i(\{c, d\}) \neq N = \bigcap\{U \in \tau^\sharp : c \in U\}$. Thus N is not R_0-ordered. However, if $(Y, \tau_1, \leq_1)$ is the subspace $\{a, d\}$ of N, then Y is T_1-ordered and there is a natural lattice isomorphism between τ and τ_1 which induces a lattice isomorphism between $\tau^\sharp$ ($\tau^\flat$) and $\tau_1^\sharp$ ($\tau_1^\flat$). Further note that both N and Y are convex. One direction of Theorem 16, however, does remain valid if the K is dropped from R_0^K-ordered and T_1^K-ordered. The argument of the last paragraph of the proof still holds. (See also Proposition 6 of [13].)

We also note that for $(X, \tau, \leq)$ to be R_0^K-ordered, it is not sufficient to have $\tau^\sharp$ and $\tau^\flat$ individually lattice isomorphic to the upper and lower topologies $\tau_1^\sharp$ and $\tau_1^\flat$ of a T_1^K-ordered topological space, even if both spaces are convex. Consider

$X = \{a,b,c,d\}$ with the order $\Delta_X \cup \{(a,b),(c,d)\}$ and with subbase of open sets $\{\{b\},\{c\},X\}$. Note that $\tau^\sharp = \{\emptyset,\{b\},X\}$ is a 3-element nested chain, as is $\tau^\flat = \{\emptyset,\{c\},X\}$, and $(X,\tau,\leq)$ is convex. Now if $X_1 = \{1,2\}$ with the discrete topology τ_1 and usual order $\leq_1$, then it is easy to see that $(X_1,\tau_1,\leq_1)$ is T_1-ordered and thus T_1^K-ordered, $\tau_1^\sharp$ and $\tau_1^\flat$ are individually lattice isomorphic to $\tau^\sharp$ and $\tau^\flat$, yet $(X,\tau,\leq)$ is not R_0^K-ordered since, for example, $a \in C(b)$ but $b \notin C(a)$.

4. The Wallman ordered compactification

The Wallman ordered compactification of a T_1-ordered convex ordered topological space $(X,\tau,\leq)$ was first constructed by Choe and Park [2] using maximal bifilters, and later by Kent [5] using maximal c-filters. A c-filter is a filter having a base of c-sets. Let w_0X be the set of maximal c-filters on $(X,\tau,\leq)$. For $A \subseteq X$, let $A^* = \{\mathcal{F} \in w_0X : A \in \mathcal{F}\}$. Then $\{A^* : A \text{ is a } c\text{-set in } X\}$ is a closed subbase for a topology on w_0X. A partial order on w_0X is defined by $\mathcal{F} \leq \mathcal{G}$ if and only if $I(\mathcal{F}) \subseteq \mathcal{G}$ and $D(\mathcal{G}) \subseteq \mathcal{F}$, where $I(\mathcal{F})$ is the filter generated by $\{I(F) : F \in \mathcal{F}\}$ and $D(\mathcal{G})$ is defined similarly. Now w_0X with this topology and order is the Wallman ordered compactification. If H is a monotone open or closed set in X, then H^* has the same properties in w_0X. Choe and Park's earlier construction uses maximal bifilters $(\mathcal{F},\mathcal{G})$ where a bifilter is defined to be a pair of filters $(\mathcal{F},\mathcal{G})$ such that $\mathcal{F} \vee \mathcal{G}$ exists, $\mathcal{F}$ has a base of closed decreasing sets, and $\mathcal{G}$ has a base of closed increasing sets. The set of maximal bifilters is given the topology having a subbase of closed sets of form $\{(\mathcal{F},\mathcal{G}) \in w_0X : D(A) \in \mathcal{F} \text{ and } I(A) \in \mathcal{G}\}$ where A is any subset of X, and they order the set of maximal bifilters by $(\mathcal{F},\mathcal{G}) \leq (\mathcal{F}',\mathcal{G}')$ if and only if $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{G} \subseteq \mathcal{G}'$. The equivalence of the bifilter construction and the c-set construction of w_0X is seen by the bijection $(\mathcal{F},\mathcal{G}) \to \mathcal{F} \vee \mathcal{G}$ from the set of maximal bifilters on X to the set of maximal c-filters on X.

Both constructions of the Wallman ordered compactification use the hypothesis that $(X,\tau,\leq)$ be T_1-ordered only to insure that X is embedded in w_0X, that is, only to insure that $\{\mathcal{S}(x)\}$ is a maximal c-filter or $(\mathcal{S}(d(x)),\mathcal{S}(i(x)))$ is a maximal bifilter, where $\mathcal{S}(A)$ represents the collection of supersets of A. Thus, the c-filter construction of w_0X only uses T_1-ordered to imply the T_1^K-ordered property, that $\{x\} = C(x)$ so that $\mathcal{S}(x)$ is a maximal c-filter. It follows that Kent's c-filter construction of w_0X remains valid for any T_1^K-ordered space.

If X is T_1-ordered, Choe and Park consider the points of $X \subseteq w_0X$ to be the maximal bifilters $(\mathcal{S}(d(x)),\mathcal{S}(i(x))) = (\mathcal{S}(D(x)),\mathcal{S}(I(x)))$. This equality need not hold if X is only assumed to be T_1^K-ordered, so to ensure that the bifilters have bases of closed sets, we must use $(\mathcal{S}(D(x)), \mathcal{S}(I(x)))$ for the construction of the Wallman ordered compactification of a T_1^K-ordered space. With this modification, the proof in [2] that $(\mathcal{S}(d(x)), \mathcal{S}(i(x)))$ is a maximal bifilter for all $x \in X$ does not show that the bifilter $(\mathcal{S}(D(x)), \mathcal{S}(I(x)))$ is maximal if X is T_1^K-ordered. We remedy this situation with the following Lemma.

Lemma 17. *If X is a convex T_1^K-ordered topological space and $x \in X$, then $(\mathcal{S}(D(x)), \mathcal{S}(I(x)))$ is a maximal bifilter.*

Proof. It is easy to see that $(\mathcal{S}(D(x)), \mathcal{S}(I(x)))$ is a bifilter. If it is not maximal, then there exists a bifilter $(\mathcal{F}, \mathcal{G}) \supseteq (\mathcal{S}(D(x)), \mathcal{S}(I(x)))$ with either $\mathcal{S}(D(x)) \neq \mathcal{F}$ or $\mathcal{S}(I(x)) \neq \mathcal{G}$. The cases are dual, so we will only consider the case $\mathcal{S}(D(x)) \neq \mathcal{F}$. Since $\mathcal{F}$ has a base of closed decreasing sets, there exists a closed decreasing set $F \in \mathcal{F}$ such that $F \notin \mathcal{S}(D(x))$, or equivalently, $D(x) \nsubseteq F$. Since $D(x) \in \mathcal{F}$, we have $A = D(x) \cap F \in \mathcal{F}$. Now $A \subseteq F$ implies $D(x) \nsubseteq A$, so $x \notin A$. We also have $I(x) \subseteq X \setminus A$, for if $y \in I(x) \cap A \subseteq I(x) \cap D(x)$, then we have the contradiction that $y = x \notin A$ since, by the T_1^K-ordered property, $I(x) \cap D(x) = \{x\}$. Thus, we have $X \setminus A \in \mathcal{S}(I(x)) \subseteq \mathcal{G}$ and $A \in \mathcal{F}$, contradicting that $\mathcal{F} \vee \mathcal{G}$ exists. This shows that the bifilter $(\mathcal{S}(D(x)), \mathcal{S}(I(x)))$ is maximal. □

The Wallman ordered compactification of a T_1-ordered space need not be T_1-ordered, but the theorem below shows the advantage of the T_1^K-ordered property.

Theorem 18. *If X is any convex T_1^K-ordered topological space, the Wallman ordered compactification w_0X is T_1^K-ordered.*

Proof. We will show that w_0X satisfies the characterization of T_1^K-ordered given in Theorem 10 (a). Suppose $\mathcal{F}$ and $\mathcal{G}$ are distinct maximal c-filters on X, that is, $\mathcal{F}$ and $\mathcal{G}$ are distinct points in w_0X. Now either $I(\mathcal{G}) \nsubseteq \mathcal{F}$ or $I(\mathcal{G}) \subseteq \mathcal{F}$. In case $I(\mathcal{G}) \nsubseteq \mathcal{F}$, there exists $I(G) \in I(\mathcal{G}) \subseteq \mathcal{G}$ with $I(G) \notin \mathcal{F}$. Since $\mathcal{G}$ is a filter and $\mathcal{F}$ is maximal, it follows that $X \setminus I(G) \notin \mathcal{G}$ and $X \setminus I(G) \in \mathcal{F}$, so that $\mathcal{G} \notin (X \setminus I(G))^*$ and $\mathcal{F} \in (X \setminus I(G))^*$. Now $(X \setminus I(G))^*$ is an open decreasing neighborhood of $\mathcal{F}$ in w_0X which excludes $\mathcal{G}$. In case $I(\mathcal{G}) \subseteq \mathcal{F}$, we have $D(\mathcal{G}) \nsubseteq \mathcal{F}$, for otherwise $I(\mathcal{G}) \vee D(\mathcal{G}) = C(\mathcal{G}) = \mathcal{G} \subseteq \mathcal{F}$, contradicting the maximality of $\mathcal{G}$. Now there exists $D(G) \in \mathcal{G}$ with $D(G) \notin \mathcal{F}$, and the dual argument of the previous case shows that $(X \setminus D(G))^*$ is an open increasing neighborhood of $\mathcal{F}$ in w_0X which excludes $\mathcal{G}$. In either case, we have found that $\mathcal{G} \neq \mathcal{F}$ implies $\mathcal{G} \notin \bigcap\{$ open monotone neighborhoods of $\mathcal{F}\}$, and it follows that w_0X is T_1^K-ordered. □

We mention that the expected properties of the Wallman ordered compactification remain valid even when the construction is applied to T_1^K-ordered topological spaces. For example, if X is convex and T_1^K-ordered, $\varphi : X \to w_0X$ is the natural embedding, and $f : X \to Y$ is a continuous increasing function from X into an arbitrary compact T_2-ordered space Y (i.e., Y is compact and the graph of its order is closed in $Y \times Y$), then there exist a unique continuous increasing function $\overline{f} : w_0X \to Y$ such that $\overline{f} \circ \varphi = f$.

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