An Arithmetic Triangle Arising From a One-Way Street Grid

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After the Euclidean metric, one of the most familiar distance functions on the plane is the Manhattan metric, where the distance between two points is the sum of the north-south distance between them and the east-west distance between them. This may accurately model walking in Manhattan, where one must stay on the street grid. The Manhattan metric is also called the taxicab metric [**2, 4**], but this disregards the fact that many of the Manhattan streets are one-way. We will consider distances between points on a square street grid where the even numbered streets run East, the odd numbered streets run West, the even numbered avenues run North, and the odd numbered avenues run South.

The first thing to notice is that these more realistic taxicab distances are not symmetric: on a 1 unit square block corresponding to rectangle $ABCD$ (oriented as written), it is a one-block trip from A to B but the return trip (B to C to D to A) is 3 blocks. Thus, our distance function is not a metric, but is a quasi-metric.

Foundations

A *quasi-metric* on a set X is a distance function $q : X \times X \to \mathbb{R}$ satisfying

(a) $q(x, y) \geq 0$ for all $x, y \in X$ (positivity)
(b) $x = y$ if and only if $q(x, y) + q(y, x) = 0$ (discernibility)
(c) $q(x, y) + q(y, z) \geq q(x, z)$ for all $x, y, z \in X$ (triangle inequality).

A quasi-metric q on X is a *metric* if it also satisfies the symmetry condition $q(x, y) = q(y, x)$ for all $x, y \in X$. See [**4**] for further information on quasi-metrics.

On $\mathbb{R}^2$, the Euclidean metric $d_\mathcal{E}$ and the Manhattan metric d_M are defined by

$$d_\mathcal{E}(\boldsymbol{x}, \boldsymbol{y}) = d_\mathcal{E}(\,(x_1, x_2), (y_1, y_2)\,) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \text{ and}$$

$$d_M(\boldsymbol{x}, \boldsymbol{y}) = d_M(\,(x_1, x_2), (y_1, y_2)\,) = |x_1 - y_1| + |x_2 - y_2|.$$

We define the taxicab quasi-metric q on $\mathbb{Z}^2$ taking $q(\boldsymbol{x}, \boldsymbol{y})$ to be the shortest distance from $\boldsymbol{x}$ to $\boldsymbol{y}$ along the one-way street grid shown in Figure 1. Starting from the origin, $\mathbf{0}$, you may drive north or east, into the first quadrant. For those lattice points $\boldsymbol{x}$ in the first quadrant which can be reached traveling only north and east, $q(\mathbf{0}, \boldsymbol{x}) = d_M(\mathbf{0}, \boldsymbol{x})$. Some lattice points $\boldsymbol{y}$ in the first quadrant cannot be reached from $\mathbf{0}$ traveling only north or east. Such points will require "overshooting" and looping around to reach them. Such points $\boldsymbol{y}$ in the first quadrant are marked with a square in Figure 1, and they satisfy $q(\mathbf{0}, \boldsymbol{y}) = d_M(\mathbf{0}, \boldsymbol{y}) + 2$. Similarly, points $\boldsymbol{y}$ in the other quadrants with $q(\mathbf{0}, \boldsymbol{y}) = d_M(\mathbf{0}, \boldsymbol{y}) + 2$ are marked with a square. Since the streets leaving $\mathbf{0}$ go north and east, there are points in the second and fourth quadrants (which are, respectively, north and east of $\mathbf{0}$) for which $q(\mathbf{0}, \boldsymbol{x}) = d_M(\mathbf{0}, \boldsymbol{x})$. However, to get from $\mathbf{0}$ to the third quadrant—south and west of $\mathbf{0}$—both initial directions from $\mathbf{0}$ must be reversed, so the quasi-metric distances from $\mathbf{0}$ to points $\boldsymbol{z}$ in the third quadrant are two or four blocks longer than $d_M(\mathbf{0}, \boldsymbol{z})$. Points $\boldsymbol{z}$ with $q(\mathbf{0}, \boldsymbol{z}) = d_M(\mathbf{0}, \boldsymbol{z}) + 4$ are marked by hexagons in Figure 1.

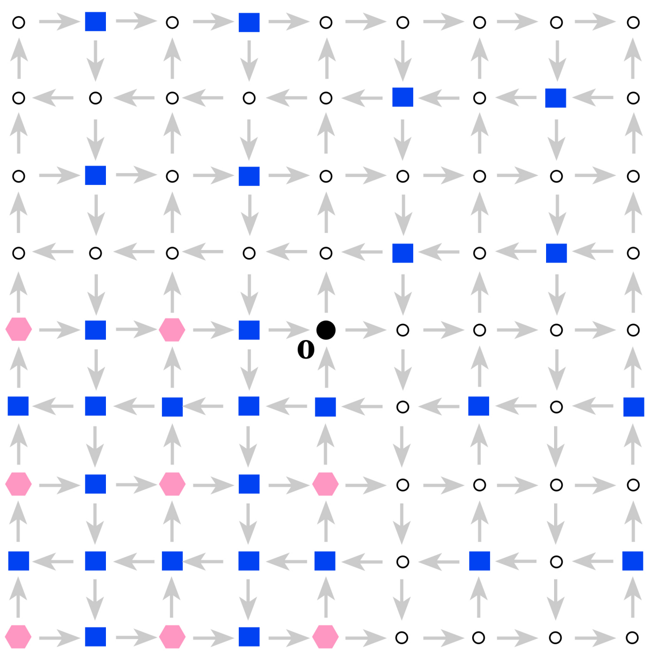

Figure 1. From the origin •, the taxicab quasi-metric to square (hexagonal) vertices is two (four) more than the Manhattan distance.

As with metrics, the quasi-metric *ball* centered at $\boldsymbol{x}$ with radius r is defined as

$$B(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{Z}^2 : q(\boldsymbol{x}, \boldsymbol{y}) < r\}.$$

For the Euclidean metric, balls are circular. For the Manhattan metric, balls are diamond-shaped. For the taxicab quasi-metric, balls have more irregular shapes. Figure 2 shows $B(\mathbf{0}, r)$ for $r = 2, 3, 4, 5$, and 6, relative to q. Balls centered at $\boldsymbol{x} \neq \mathbf{0}$ are translations of the balls centered at $\mathbf{0}$, rotated so that the L-shaped $B(\boldsymbol{x}, 2)$ points in the directions of the streets leaving $\boldsymbol{x}$. Figure 2 also shows $B((4, 3), 2)$ and $B((5, 5), 2)$.

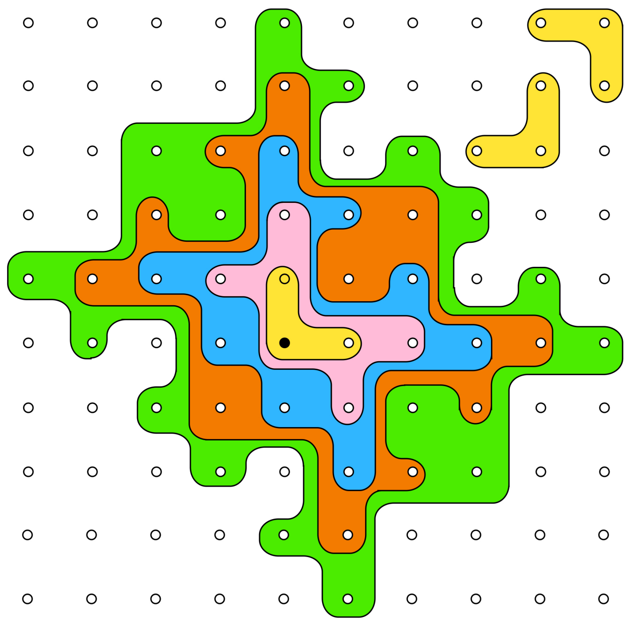

Figure 2. Balls of radius $2, 3, 4, 5$, and 6 centered at $\mathbf{0}$; Balls of radius 2 centered at $(4, 3)$ and $(5, 5)$.

In [**5**], a point x of a quasi-metric space X is called *symmetric* (*asymmetric*) if $q(x, y)$ is always (never) equal to $q(y, x)$ for all $y \in X - \{x\}$. The taxicab quasi-metric on $\mathbb{Z}^2$ has no symmetric points and no asymmetric points: Given any point A in $\mathbb{Z}^2$, there is a 1×1 oriented rectangle $ABCD$ with $q(A, B) = 1 \neq 3 = d(B, A)$, but $q(A, C) = q(C, A)$.

We leave it to the reader to confirm the following result, suggested in Figure 3.

Proposition. If q is the taxicab quasi-metric on $\mathbb{Z}^2$, then $q(\boldsymbol{a}, \boldsymbol{a'}) = q(\boldsymbol{a'}, \boldsymbol{a})$ if and only if $\boldsymbol{a}$ and $\boldsymbol{a'}$ lie n blocks apart on an oriented rectangular circuit of length $2n$ blocks ($n \geq 0$). Furthermore, if this occurs, $q(\boldsymbol{a}, \boldsymbol{a'}) = n$ if and only if $\boldsymbol{a}$ is one of the corners of the rectangular circuit or is one block away from a corner (and otherwise, $q(\boldsymbol{a}, \boldsymbol{a'}) < n$).

Counting paths from A to B

In a square grid of 2-way streets, if B lies j blocks north and k blocks east of A, the number of shortest paths from A to B is $\binom{j+k}{j}$: of the $j + k$ blocks, we may chose any j of them to traverse to the north, with the remaining k blocks traversed to the east. Thus, counting the number of paths from $\mathbf{0}$ to points of the first quadrant gives

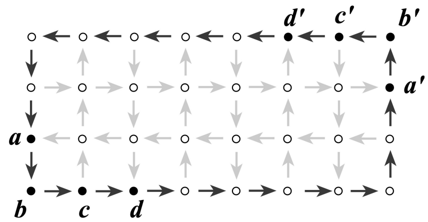

Figure 3. For $\boldsymbol{x} \in \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$, $q(\boldsymbol{x}, \boldsymbol{x}') = q(\boldsymbol{x}', \boldsymbol{x})$, and for $\boldsymbol{x} \in \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$, $q(\boldsymbol{x}, \boldsymbol{x}')$ is half the length of the rectangular circuit shown.

rise to Pascal's triangle (see [**3**], for example). In a similar manner, we may calculate the number of shortest routes from A to B in the 1-way street grid of Figure 1. As an initial case, we count the number of routes from $\mathbf{0}$ into the first quadrant, and we rotate the street grid of Figure 1 so the numbers appear in the usual triangular array, as seen in Figure 4. We call this triangular array of numbers the *fundamental triangle*. In Figure 4, if there is a street arriving at a corner B from the north, then all shortest routes from $\mathbf{0}$ arrive from the north, so the entry at B is the sum of the one or two numbers at corners from which B is one block southeast or southwest. If both streets arriving at corner B arrive from the south, then the entry at B is the sum of the entries one block southeast and one block southwest of B.

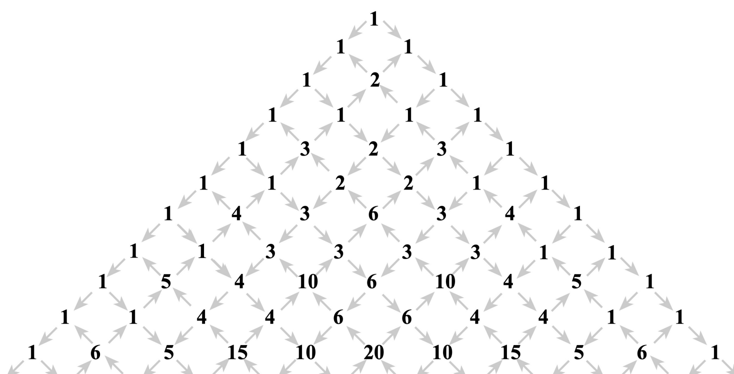

Figure 4. The fundamental triangle, showing the number of (shortest) routes from A to B.

If the initial point is not $\mathbf{0}$, or equivalently, if the initial point is $\mathbf{0}$ but we wish to count paths to points in a quadrant other than the first quadrant, the relevant number of paths are shown in Figure 5. Figure 5(a) (or its reflection around a vertical axis) shows the number of paths from the circled initial point if only one of the initial departing streets points toward the quadrant we wish to reach. Observe that this triangular array is identical to the fundamental triangle of Figure 4 with an additional diagonal of 1s added to the right. Figure 5(b) shows the number of paths from the initial circled point when both streets are going away from the quadrant we wish to reach. Notice that this array of numbers is identical to the fundamental triangle except that the two indicated entries have been transposed. Thus, we will focus on the fundamental triangle shown in Figure 4.

As with Pascal's triangle, we will count the rows and entries starting from zero. In Pascal's triangle, the k-th entry of the n-th row is $\binom{n}{k}$, which is sometimes de-

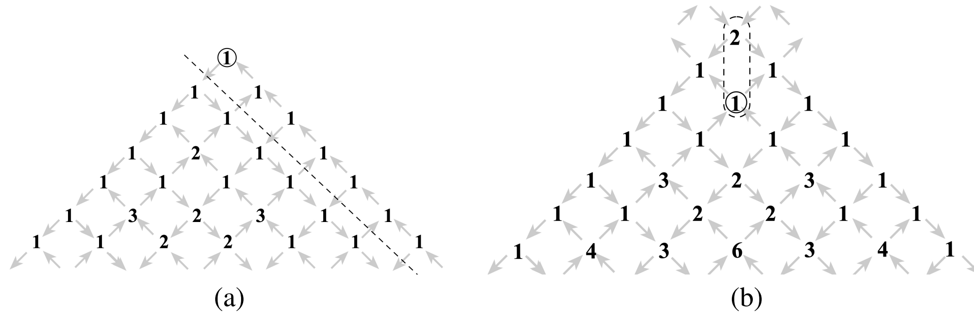

Figure 5. Number of (shortest) routes from the circled initial point.

noted $C(n, k)$. We denote the k-th entry of the n-th row of our fundamental triangle by $R(n, k)$. The letter R is used since it counts shortest routes. In the fundamental triangle, if we take only the (even, even) entries, then the shortest paths all follow southerly routes (so the quasi-metric q agrees with the Manhattan metric d_M) and the corresponding entries give a copy of Pascal's triangle embedded in the fundamental triangle. Every entry of the fundamental triangle is adjacent to or at a diagonally opposite corner of a block from an (even, even) point (see Figure 6).

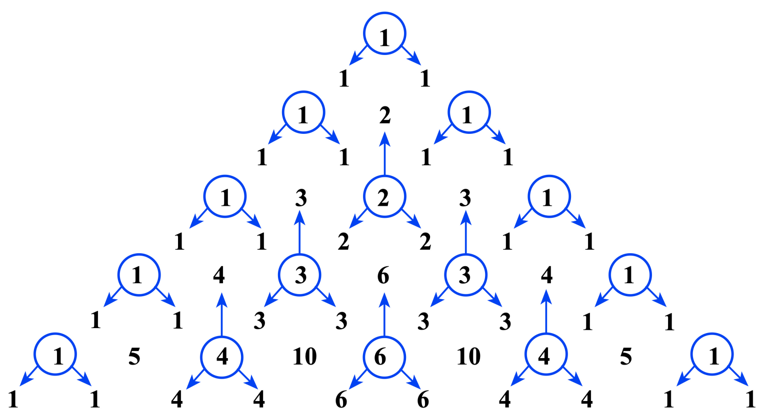

Figure 6. Every entry $R(n, k)$ of the fundamental triangle comes from a nearby circled entry of Pascal's triangle.

This gives rise to formulas for $R(m, j)$, based on the parity of m and j:

$$\begin{aligned} R(2n+1, 2k+1) &= R(2n+1, 2k) \\ &= R(2n-2, 2k-1) \\ &= R(2n, 2k) \\ &= \binom{n}{k}. \end{aligned} \tag{1}$$

We may track the provenance of an entry of Pascal's triangle: it arises as the sum of two entries in the row above it, which arose from entries above them, and so on. In

Pascal's triangle, you may superimpose an inverted copy of the triangle with the top 1 positioned on an entry, and the entry is realized as the dot product of the overlapping entries–from the original and from the inverted copy–on any higher row. For any entry in the fundamental triangle corresponding to a point on the map with two southerly paths arriving–that is, for any $R(2n, 2k)$—a similar pattern arises, as suggested in Figure 7. Here, however, we use every other entry of the even rows (which gives the embedded copy of Pascal's triangle) and every entry of the odd rows.

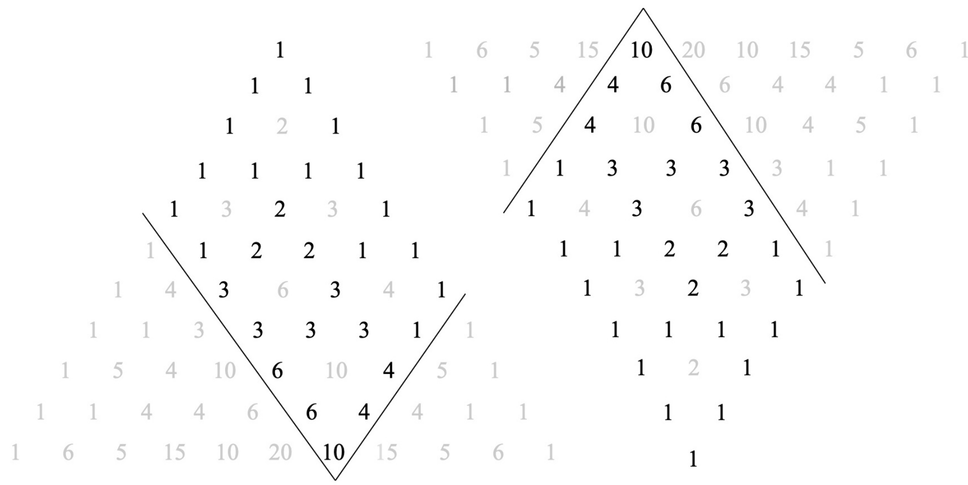

Figure 7. Provenance of $R(2n, 2k)$. For any row, using the highlighted entries from the inverted copy of the fundamental triangle as coefficients for the highlighted entries of the original will give $R(10, 6) = 10$.

Sums of rows and slant diagonals

It is natural to ask which properties of Pascal's triangle will hold, in some form, for the fundamental triangle. The properties of Pascal's triangle we consider may be found in [**3**]. We start by looking at the sums of the rows. Recall that the sum of the n-th row of Pascal's triangle is 2^n and the alternating sum $\sum_{k=0}^{n}(-1)^k\binom{n}{k}$ of the entries of the n-th row of Pascal's triangle is zero. By observing that row $2n+1$ of the fundamental triangle is simply row n of Pascal's triangle with every entry repeated, and row $2n$ of the fundamental triangle is obtained by intersplicing row n of Pascal's triangle with row $n+1$ of Pascal's triangle with the initial and final 1s removed (see Figure 8), the following results are easily obtained.

Proposition. (a) The sum of the $(2n+1)$-st row of the fundamental triangle is 2^{n+1}. The sum of the $2n$-th row is $3 \cdot 2^n - 2$.

(b) The alternating sum $\sum_{k=0}^{n}(-1)^k R(n, k)$ is zero if n is odd and $2 - 2^n$ if n is even.

The sums of the slant diagonals of Pascal's triangle are the Fibonacci numbers. The slant diagonals of the fundamental triangle are shown in Figure 8. To more easily see the diagonals, the triangle is drawn as a right triangle instead of the usual equilateral format.

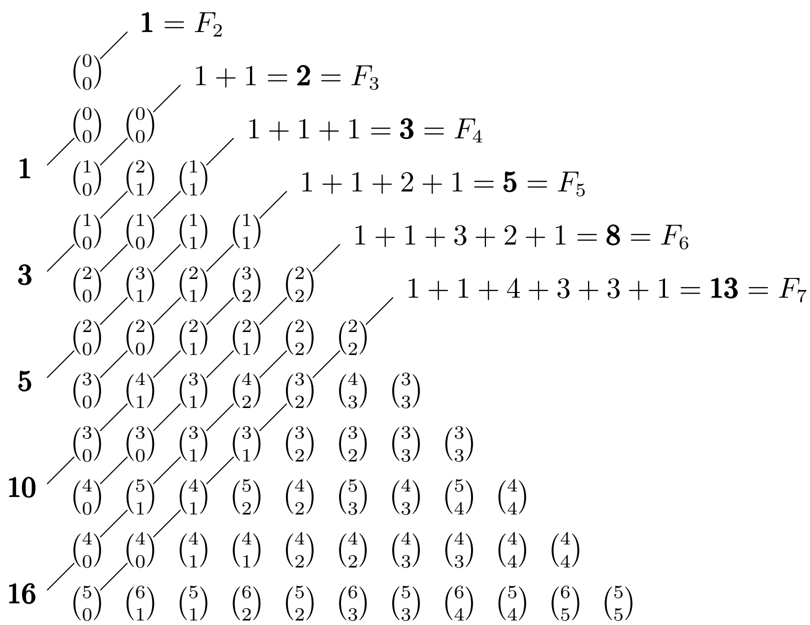

Figure 8. Slant diagonals of the fundamental triangle.

The even slant diagonal sums, shown on the right of the triangle in Figure 8, are easily recognizable. The odd slant diagonal sums, shown on the left in Figure 8, are not as obvious.

Proposition. The sum of the $2n$-th slant diagonal (counting from zero) of the fundamental triangle is the Fibonacci number F_{n+2}. The sum of the $(2n-1)$-st slant diagonal is $F_n + F_{n+2} - k$, where $k = 1$ if n is even and $k = 2$ if n is odd.

In lieu of a formal proof, we present a few illustrative examples to show how these formulas arise. For example, the sum of the tenth slant diagonal is, after rearranging terms,

$$13 = \left[\binom{5}{0} + \binom{4}{1} + \binom{3}{2}\right] + \left[\binom{4}{0} + \binom{3}{1} + \binom{2}{2}\right].$$

The bracketed terms above are slant diagonals of Pascal's triangle, which are Fibonacci numbers. In this case, they are $F_6 = 8$ and $F_5 = 5$. Thus, when $n = 5$, the sum of the $2n$-th slant diagonal of the fundamental triangle is $F_5 + F_6 = F_7 = F_{n+2}$.

For the $(2n-1)$-st slant diagonal, we first consider $n = 5$. Then the sum of the $(2n-1)$-st = 9-th slant diagonal is

$$16 = \left[\binom{4}{0} + \binom{3}{1} + \binom{2}{2}\right] + \left[\binom{5}{1} + \binom{4}{2}\right].$$

The first bracketed term is again the sum of a slant diagonal of Pascal's triangle, and is $5 = F_5$. The second bracketed term is almost the sum of a slant diagonal of Pascal's triangle, but with the initial $1 = \binom{6}{0}$ and final $1 = \binom{3}{3}$ missing. Thus, the second bracketed terms is $F_7 - 2$. Adding both bracketed terms, the sum of the $(2n-1)$-st slant diagonal when $n = 5$ is $F_5 + F_7 - k$, where $k = 2$.

For $n = 4$, the sum of the $(2n-1)$-st = 7-th slant diagonal is

$$10 = \left[\binom{3}{0} + \binom{2}{1}\right] + \left[\binom{4}{1} + \binom{3}{2}\right].$$

Again the first bracketed term is a slant diagonal of Pascal's triangle. If the second bracketed term had an initial $1 = \binom{5}{0}$, it too would be a slant diagonal of Pascal's triangle. In this case, we see $10 = F_4 + F_6 - 1$, which is $F_n + F_{n+2} - k$, where $k = 1$.

It is easy to see from the pattern that any slant diagonal will have the form of one of the three examples given.

It is known that every positive rational number $\frac{a}{b}$ $(a, b \in \mathbb{N})$ appears (infinitely often) as the quotient of consecutive entries of a row of Pascal's triangle. Since every consecutive pair of entries of the n-th row of Pascal's triangle appears as a consecutive pair of entries of the $(2n+1)$-st row of the fundamental triangle, every rational number appears as the quotient of consecutive entries of the fundamental triangle. Our next result says that we can even achieve this result using the even rows of the fundamental triangle, except for the rational number 1.

Proposition. Every positive rational number $\frac{a}{b} \neq 1$ $(a, b \in \mathbb{N})$ appears (infinitely often) as the quotient of consecutive entries of some even row of the fundamental triangle.

Proof. By the left-right symmetry of the rows of the fundamental triangle, a and b appear as consecutive entries if and only if b and a appear as consecutive entries, so without loss of generality, we will assume $\frac{a}{b} < 1$, so $a < b$.

For any positive integer m with $a + \frac{1}{m} \leq b$ (and there are infinitely many such integers), we have $\frac{2}{m} \leq a + \frac{1}{m} \leq b$, which leads to $0 \leq ma - 1 \leq mb - 2$. Now with $k = ma - 1$ and $n = mb - 1$, this says $0 \leq k \leq n - 1$, so $R(2n, 2k)$ and $R(2n, 2k+1)$ are entries of the fundamental triangle. Now we may use Equation 1 to calculate the quotient of these consecutive entries of the $2n$-th row of the fundamental triangle.

$$\begin{aligned}
\frac{R(2n, 2k)}{R(2n, 2k+1)} &= \frac{\binom{n}{k}}{\binom{n+1}{k+1}} \\
&= \frac{n!}{k!(n-k)!} \cdot \frac{(k+1)!(n-k)!}{(n+1)!} \\
&= \frac{k+1}{n+1} \\
&= \frac{ma}{mb} \\
&= \frac{a}{b}. \quad \blacksquare
\end{aligned}$$

Another familiar property of Pascal's triangle is the hockey stick relation: $\sum_{j=0}^{k} \binom{n+j}{n} = \binom{n+k+1}{k}$. Now we give the analogous properties for the fundamental triangle.

Proposition. With $R(n,k)$ as defined above,

$$\sum_{j=0}^{k} R(n+j,n) = R(n+1+k,n+2) + R(n+2+k,n+2) - [n \pmod 2]$$

and

$$\sum_{j=0}^{k} R(n+j,j) = R(n+k,k-2) + R(m+k+1,k-1) - [n \pmod 2].$$

Not surprisingly, the formulas above takes a different form depending on whether n is even or odd. Note that $[n \pmod 2]$ is 0 if n is even and 1 if n is odd. Also, the second formula is simply the first one with every entry $R(a,b)$ replaced by its left-right symmetric entry $R(a,a-b)$. Figure 9 illustrates the patterns of this Proposition.

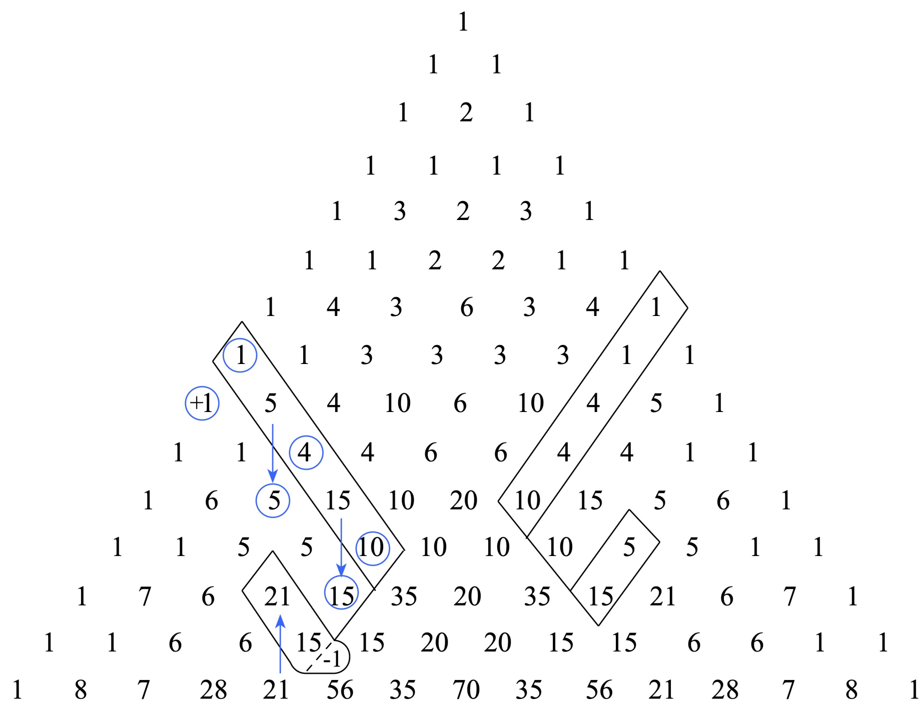

Figure 9. Hockey stick patterns: If the handle starts 1 1 . . ., then the sum of the handle is the sum of the two indicated entries. Otherwise, the sum of the handle is the sum of the two indicated entries, minus 1.

The crux of the proof of the hockey stick patterns is the observation that starting from $R(n,n)$ or $R(n,0)$ in the fundamental triangle, whether n is even or odd, taking every other entry in the handle of a hockey stick gives the entries of a hockey stick handle in Pascal's triangle. For example, for the hockey stick handle 1 1 4 4 10, the terms 1 _ 4 _ 10 in even positions and the terms _ 1 _ 4 _ in odd positions are each hockey stick handles in Pascal's triangle summing to 15 and 5, which are the boxed entries completing the hockey stick. This will always occur for hockey sticks which start 1 1 . . ., that is, starting from $R(n, n)$ or $R(n, 0)$ where n is even. For hockey sticks starting from $R(n, n)$ or $R(n, 0)$ where n is odd, that is, where the handle starts 1 k . . . where $k \neq 1$, a slight modification is required. Referring to the other hockey stick in Figure 9, in the handle 1 5 4 15 10, the terms 1 _ 4 _ 10 in even positions are a hockey stick handle in Pascal's triangle summing to the 15 shown in the lower part of the hockey stick. The terms 5 _ 15 in odd positions appear two rows down, where the circled terms +1, 5, 15 are again a hockey stick handle in Pascal's triangle summing to 21. The hockey stick in Pascal's triangle actually would point us to the 21 on the bottom row of Figure 9, but for consistency, we raise it to fit into the boxed hockey stick. However, the +1 used in this Pascal's hockey stick handle was not present in the fundamental triangle hockey stick handle, so we must subtract 1. A formal proof based on these insights is an exercise in tracking notation and using Equation 1.

We have seen a few of the properties of the fundamental triangle. As with Pascal's triangle, there are many more avenues for exploration. Which rows contain only odd entries? Are there entries which divide every larger entry in its row? How frequently does an integer n appear in the fundamental triangle? (This question is unsolved for Pascal's triangle—see [**1**].) Many of the results about the fundamental triangle depended on whether we start from an odd row (with initial entries 1 1) or an even row (with initial entries 1 k, $k > 1$). If we remove the odd rows of the fundamental triangle, then we may ask how each entry arises from the entries of the row above it, and what are the sums of the slant diagonals. Is there a way to generalize the taxicab quasi-metric on $\mathbb{Z}^2$ to a quasi-metric on $\mathbb{R}^2$? Measuring distances on a one-way street grid provides many opportunities for further investigation.

Summary: Pascal's triangle arises by counting the number of shortest paths from $(0, 0)$ to (n, k) on a square street grid. The length of the shortest path is the Manhattan distance from $(0, 0)$ to (n, k). We consider the case of a square street grid of one-way streets, with successive parallel streets being oppositely directed. We investigate the associated distance function q (which is only a quasi-metric, since $q(A, B)$ may differ from $q(B, A)$) and the arithmetic triangle obtained by counting shortest routes on the one-way grid from $(0, 0)$ to (n, k).

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