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Ordered Compactifications of Products of Two Totally Ordered Spaces

DOUGLAS D. MOONEY
The Center for Healthcare Industry Performance Studies, 1550 Old Henderson Road, Columbus, OH 43220, U.S.A. E-mail: Dmooney@chipsonline.com

THOMAS A. RICHMOND
Department of Mathematics, Western Kentucky University, Bowling Green, KY 42101, U.S.A. E-mail: Tom.Richmond@wku.edu

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Abstract. We describe the semilattice of ordered compactifications of $X \times Y$ smaller than $\beta_0 X \times \beta_0 Y$ where X and Y are certain totally ordered topological spaces, and where $\beta_0 Z$ denotes the Stone–Čech ordered- or Nachbin-compactification of Z. These basic cases are used to illustrate techniques for describing the semilattice of ordered compactifications of $X \times Y$ smaller than $\beta_0 X \times \beta_0 Y$ for arbitrary totally ordered topological spaces X and Y. Such products $X \times Y$ provide many counterexamples in the theory of ordered compactifications.

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Introduction

A product of two totally ordered spaces has a certain “flatness”. If both factors are compact totally ordered spaces, we may picture the space as a flattened handkerchief. If the factors are totally ordered but not both compact, we may picture the space as a “threadbare handkerchief”, that is, a handkerchief with some vertical or horizontal threads removed. To compactify such a space topologically, without regard to the order structure, one could restore a missing thread with a single thread or (in some cases) with two threads and one could “drawstring” restored threads, using a short thread to serve as limits of a long open edge. In Section 1 we will present some conditions determining whether or not $X \times Y$ has a smallest ordered compactification. We will see that for threads removed from the center of a handkerchief, the order on points of the base space on either side of the missing thread effectively staple the handkerchief flat and allow only a limited amount of identification of points of the restored threads. For missing outside edge threads, points of the base space are on only one side of the restored thread, so the order does not pin down these edge points as much, and some “drawstringing” is possible along these outer edges. If the northern and western outer edges of the handkerchief are missing, restoring the threads gives a northwest corner point nw that is neither above nor below any point of the base space. Thus, the order on the base space does not pin down the point nw at all, and this corner may be folded over to be identified with any point of any restored thread. In Section 2, we will focus on the basic case of a handkerchief with a single thread removed from the middle. This basic case will be used in Section 3 to develop techniques for describing more general cases.

The *decreasing hull* of a subset A of a poset $(X, \leq)$ is $d_X(A) = \{x \in X : \exists a \in A \text{ with } x \leq a\}$. If the poset X is clear from the context, we may write $d(A)$ for $d_X(A)$. A set A is *decreasing* if $A = d(A)$. Increasing hulls $i(A)$ and increasing sets are defined dually. A set is *monotone* if it is either increasing or decreasing. A set A is $\leq$*-convex* if $A = i(A) \cap d(A)$. Points $x, y \in X$ are *non-comparable*, denoted $x \parallel y$, if $x \not\leq y$ and $y \not\leq x$. If $(X, \leq)$ is a totally ordered set and $a \in X$, we may denote the rays $i(a)$ and $i(a) \setminus \{a\}$ by $[a, \rightarrow)$ and $(a, \rightarrow)$, respectively.

An *ordered* (*topological*) *space* is a triple $(X, \tau, \leq)$ where X is a set, $\leq$ is a partial order on X, and τ is a topology on X having a base of $\leq$-convex sets. A natural compatibility condition between the topology τ and order $\leq$ of an ordered space is the T_2*-ordered property*, which is satisfied if the graph of $\leq$ is closed in the product $(X, \tau) \times (X, \tau)$. A compact T_2-ordered space $(\gamma X, \tau_\gamma, \leq_\gamma)$ is an *ordered compactification* of the ordered space $(X, \tau, \leq)$ if (X, τ) is (homeomorphic to) a dense subspace of the compact space $(\gamma X, \tau_\gamma)$ and $\leq_\gamma$ restricted to X is $\leq$. The set of all ordered compactifications of $(X, \tau, \leq)$ is denoted $K_o(X)$ and can be ordered by $(\gamma X, \tau_\gamma, \leq_\gamma) \leq (\delta X, \tau_\delta, \leq_\delta)$ if there exists a continuous increasing function $f\colon \delta X \to \gamma X$ that leaves each point of X fixed. With this partial order, $K_o(X)$ is a complete $\vee$-semilattice with largest element $\beta_o X$, the *Stone–Čech ordered-* or *Nachbin-compactification* (see [8]). If γX is any compactification of X, then the subspace $\gamma X \setminus X$ of γX is called the *remainder* associated with γX.

If $(X, \tau, \leq)$ is an ordered topological space whose order $\leq$ is a total order, i.e., if $(X, \tau, \leq)$ is a *totally ordered space*, then the problem of characterizing the ordered compactifications of X is greatly simplified. Such characterizations appear in Blatter [1] and Kent and Richmond [5]. If X is a totally ordered space, the latter paper characterizes the points of $\beta_o X \setminus X$ as the convex hulls of nonconvergent ultrafilters on X, which are called *singularities*. We will follow the terminology given in [5] but rather than repeat the technical definitions of singularities, we will provide characterizations of them. Often we will not distinguish between a singularity, which is a nonconvergent ultrafilter on X, and the limit of the singularity in $\beta_o X$. If $\alpha \in \beta_o X \setminus X$ and α is the maximum (minimum) element of $\beta_o X$, then we say α is an *unbounded increasing* (*decreasing*) *simple singularity* of X. The adjective “simple” indicates that the point $\alpha \in \beta_o X \setminus X$ may not be identified with any other remainder points of $\beta_o X \setminus X$ to form smaller ordered compactifications. The adjective “increasing” indicates that α is the limit of a net (or ultrafilter) from below. There may also be *simple bounded singularities* on X which introduce a remainder point, neither maximun nor minimum in $\beta_o X$, which can not be identified with other remainder points to form any smaller ordered compactification. *Essential singularities* always occur in pairs $\{\alpha^-, \alpha^+\}$, with α^- increasing and α^+ decreasing, and have the property that the *essential pair* $\{\alpha^-, \alpha^+\} \subseteq \beta_o X \setminus X$ may be identified to a single point to form a smaller ordered compactification of X. A simple example will serve to illustrate these ideas. Consider the subset $X = (0, 1) \cup (2, 3] \cup (4, 5)$ of the real line with the usual topology and order. Now $\beta_o X = X \cup \{0, 1, 2, 4, 5\}$ with the subspace topology and order from the real line. Now $0 \in \beta_o X \setminus X$ is (the limit of) an unbounded decreasing simple singularity, $5 \in \beta_o X \setminus X$ is (the limit of) an unbounded increasing simple singularity, $4 \in \beta_o X \setminus X$ is (the limit of) a bounded decreasing simple singularity, and the points $\{1, 2\} \subset \beta_o X \setminus X$ form an essential pair of singularities. (Observe that identifying 1 and 2 in $\beta_o X$ gives a smaller ordered compactification of X.)

For any ordered space X, the ordered compactifications of X are obtained by identifying the remainder points of $\beta_o X$ and/or adding some additional order. The additional order can only be between remainder points and the base space or between remainder points. If the underlying topological compactifications are the same, the ordered compactification with the larger order is the smaller ordered compactification.

1. Identification of Points in $\alpha X \times \eta Y$

Throughout, we assume that X and Y are totally ordered spaces. We start with an ordered compactification of $X \times Y$ of the form $\alpha X \times \eta Y$ where αX and ηY are ordered compactifications of X and Y, respectively, and consider ordered compactifications of $X \times Y$ smaller than $\alpha X \times \eta Y$. If we wanted to describe as much of the semilattice $K_o(X \times Y)$ as possible, we would take $\alpha X \times \eta Y = \beta_o X \times \beta_o Y$, and in case $\beta_o X \times \beta_o Y = \beta_o(X \times Y)$, we would have a description of the entire semilattice $K_o(X \times Y)$. Necessary and sufficient conditions for $\beta_o X \times \beta_o Y$ to equal $\beta_o(X \times Y)$ (where X and Y are totally ordered spaces) are given in [4].

Identifying any pair of compactification points from a topological compactification results in a topological compactification. Thus, only the order properties restrict identification of points. Specifically, any identification of remainder points (and the resulting transitivity) must maintain a partial order that extends that of $X \times Y$ and is T_2-ordered. We observe that to maintain the T_2-ordered property, only convex sets in $\alpha X \times \eta Y$ consisting entirely of remainder points may be identified to a point.

We first consider identification of remainder points arising from unbounded singularities. If such singularities exist, let us denote the (limit of the) increasing unbounded singularity of X by ∞_X and denote the decreasing unbounded singularity of X by $-\infty_X$, with ∞_Y and $-\infty_Y$ defined similarly. If the relevant singularities exist, we will call $T = \alpha X \times \{\infty_Y\}$ the *top edge* of $\alpha X \times \eta Y$, $B = \alpha X \times \{-\infty_Y\}$ the *bottom edge* of $\alpha X \times \eta Y$, $L = \{-\infty_X\} \times \eta Y$ the *left edge* of $\alpha X \times \eta Y$, and $R = \{\infty_X\} \times \eta Y$ the *right edge* of $\alpha X \times \eta Y$. Since there are no points of $X \times Y$ greater than the points of $T \cup R$ or less than the points of $B \cup L$, the points of these edges are not as fixed by the order on $X \times Y$ as points of $\alpha X \times \eta Y$ which are between points of $X \times Y$. Thus, additional identifications among these points will be possible. In the proposition below, we will interpret T, B, L, or R to be $\emptyset$ if the corresponding unbounded singularity does not exist. In the following results, we will allow the possibility that some identification has already taken place among remainder points. Thus, we will consider ordered compactifications $\gamma(X \times Y) \leq \alpha X \times \eta Y$. If $f\colon \alpha X \times \eta Y \to \gamma(X \times Y)$ is the natural quotient map, we will denote $f^{-1}(\{p\})$ by $[p]$. For $p \in \gamma(X \times Y)$ we will write $p \in T$ to mean $[p] \subseteq T$, and similar conventions apply for B, L, and R.

The following proposition says that any convex closed subset of $T \cup R$ or of $B \cup L$ can be identified to a point.

PROPOSITION 1.1. *Suppose $\gamma(X \times Y) \leq \alpha X \times \eta Y$ is an ordered compactification of $X \times Y$. Let C be a closed convex subset of $\gamma(X \times Y)$ with $[C] \subseteq T \cup R$ (or $[C] \subseteq B \cup L$). Collapse C to a point $\overline{C}$ and order the quotient space $\delta(X \times Y) = \gamma(X \times Y)/\{C\}$ by taking $x \leq_\delta y$ iff $x \leq y$ or there exist $c, c' \in C$ with $x \leq c$ and $c' \leq y$. Then $\delta(X \times Y)$ is a T_2-ordered compactification of $X \times Y$.*

Proof. First note that the relation described is antisymmetric, for $x \leq_\delta \overline{C}$ and $\overline{C} \leq_\delta x$ imply by the convexity of C that $\overline{C} =_\delta x$. Thus, $\leq_\delta$ is a partial order. Observe that if $c' \in C \subseteq T \cup R$, then $c' \leq y$ implies $y \in T \cup R$. Thus, the new order introduced by $\leq_\delta$ does not involve two points of $X \times Y$, so $\leq_\delta$ extends the order on $X \times Y$. It remains to verify that $\delta(X \times Y)$ is T_2-ordered. If $x \not\leq_\delta y$ and $x, y \neq_\delta \overline{C}$, then $x \not\leq y$ and either $i(x)$ or $d(y)$ is disjoint from C. If $i(x) \cap C = \emptyset$, then $i(x) \cap [d(C) \cup d(y)] = \emptyset$ in $\gamma(X \times Y)$. But since $\{x\}, \{y\}$ and C are compact, $i(x)$ and $d(C) \cup d(y)$ are closed (p. 44, [8]). Now we may separate these disjoint oppositely directed monotone closed sets in the T_4-ordered space $\gamma(X \times Y)$ by disjoint open monotone sets. Since these sets are saturated (one completely contains C), these open sets separate x and y in $\delta(X \times Y)$ after C is collapsed to the point $\overline{C}$. If $x \not\leq_\delta \overline{C}$, then $i(x) \cap C = \emptyset$, and the argument for this case and its dual is similar to the previous case. □

One immediate result of the proposition above is that any single edge T, B, L, or R of $\alpha X \times \eta Y$ may be identified to a point, resulting in a T_2-ordered compactification of $X \times Y$.

COROLLARY 1.2. *If X has a decreasing unbounded singularity and Y has an increasing unbounded singularity, then then $K_o(X \times Y)$ is not a lattice. In particular, $X \times Y$ has no smallest ordered compactification.*

Proof. Identifying the top edge T of $\beta_o X \times \beta_o Y$ to a point gives an ordered compactification of $X \times Y$, as does identifying the left edge L. These two ordered compactifications have no lower bounds in $K_o(X \times Y)$, since identifying $T \cup L$ to a point destroys the order on $X \times Y$. □

A natural related question that remains open is whether any quotient of $\alpha X \times \eta Y$ that gives a topological compactification of $X \times Y$ (i.e., any T_2-quotient, or any upper semicontinuous quotient) whose non-singleton blocks are convex subsets of $\alpha X \times \eta Y$ completely contained in either $T \cup R$ or $B \cup L$ gives an ordered compactification of $X \times Y$.

If X has an unbounded decreasing singularity and Y has an unbounded increasing singularity, the northwest corner point $(-\infty_X, \infty_Y)$ of $\gamma(X \times Y)$ has a remainder point for each coordinate. If this occurs, we say the point $nw = (-\infty_X, \infty_Y)$ *exists.* Likewise, saying that the southeast corner point $se = (\infty_X, -\infty_Y)$ exists implies that X has an unbounded increasing singularity and Y has an unbounded decreasing singularity.

PROPOSITION 1.3. *Suppose $\gamma(X \times Y) \leq \alpha X \times \eta Y$ is an ordered compactification of $X \times Y$ and the northwest corner point $nw = (-\infty_X, \infty_Y)$ of $\gamma(X \times Y)$ exists. Then for any remainder point p of $\gamma(X \times Y) \setminus (X \times Y)$, there is an ordered compactification of $X \times Y$ in which p and nw are identified. The dual statement holds if the southeast corner se exists.*

Proof. Notice that $\{nw, p\}$ is convex if $p \notin T \cup L$. If $p \in T \cup L$ (or $p \in B \cup R$, for that matter), then nw and p may be identified along with a closed convex segment by Proposition 1.1. If $C = \{nw, p\}$ is convex, then the details of the proof of Proposition 1.1 can be repeated to verify that $\delta(X \times Y) = \gamma(X \times Y)/\{C\}$ is an ordered compactification of $X \times Y$. □

Identification of remainder points of the top edge T generally are restricted, as the following proposition shows.

PROPOSITION 1.4. *Let $\gamma(X \times Y) \leq \alpha X \times \eta Y$ be an ordered compactification of $X \times Y$. Suppose $t \in T \setminus L$ is a top remainder point ($\neq nw$). Then t may be identified with another remainder point p of $\gamma(X \times Y)$ (possibly along with other points) if and only if $p \in T \cup R$.*

Proof. From Proposition 1.1, t and p may be identified along with a convex subset of $T \cup R$ if $p \in T \cup R$. If $p \in L \setminus T$, then there exist noncomparable points of $X \times Y$ in the convex hull (in $\gamma(X \times Y)$) of $\{p, t\}$, so identifying p and t would introduce new order on $X \times Y$.

If the remainder point $p = [(p_x, p_y)] \notin L \cup T \cup R$, and $t = [(t_x, \infty_Y)] \in T \setminus L$, consider first the case $t_x \leq p_x$. Then there exist points $y_1, y_2 \in Y$ with $p_y < y_1 < y_2$ and there exist points $x_1, x_2 \in X$ with $x_1 \leq t_x \leq p_x \leq x_2$. Assuming X has more than one point, we may choose x_1 and x_2 to be distinct. Now $(x_1, y_2) \| (x_2, y_1)$ in $X \times Y$, but $(x_1, y_2) \leq t$ and $p \leq (x_2, y_1)$, so identifying p and t would introduce new order on $X \times Y$. In the case $t_x > p_x$, there exist $y_1, y_2 \in Y$ with $p_y < y_1 < y_2$ and there exist $x_1, x_2 \in X$ with $x_1 \leq p_x < t_x \leq x_2$. Now $(x_1, y_2) \| (x_2, y_1)$ in $X \times Y$ but $(x_2, y_1) \geq p$ and $t \geq (x_1, y_2)$, which prevents the identification of p and t. □

Now we consider identification involving remainder points (a, b), $(c, d) \in (\alpha X \times \eta Y) \setminus (X \times Y)$ that are not on the edge. We will see that if there are distinct points $x, w \in X$ with $a \leq x < w \leq c$ (or dually ordered) and distinct points $y, z \in Y$ with $b \leq y < z \leq d$ (or dually ordered), then the remainder points (a, b) and (c, d) cannot be identified, for this would introduce new order on unrelated points of the base space (namely, on the unrelated corners of the rectangle $\{x, w\} \times \{y, z\}$). Let us consider the ways in which the prohibiting inequalities above cannot arise. Suppose $a < c$. Then there are only three possible ways that there are not distinct points $x, w \in X$ with $a \leq x < w \leq c$:

Case 1. $\{a, c\}$ forms an essential pair of singularities (i.e., a and c are consecutive points of αX, and are both remainder points).

Case 2. One point of $\{a, c\}$ is a point of X, the other is a remainder point, and there are no points of X strictly between a and c. It follows that a covers c (or dually) in αX and one of them is a simple singularity. We will refer to this case by saying $\{a, c\}$ is a *Case* 2 *gap*.

Case 3. $a, c \in \alpha X \setminus X$ and there are not two points of X between them. Unless $\{a, c\}$ is an essential singularity (covered in Case 1), this could only occur if a is an increasing simple singularity, c is a decreasing simple singularity, and there exists a single point $x \in X$ between a and c.

DEFINITION 1.5. A pair of oppositely directed simple singularities with a single isolated point of X between them (as in Case 3 above) will be called an *almost essential pair of* (*simple*) *singularities.* An almost essential pair of singularities *encompasses* three points: the two singularities and the enclosed point of X.

Using the analogy of the threadbare handkerchief, an almost essential pair of singularities in one of the factors of $X \times Y$ corresponds to a thread isolated between two open edges (i.e., between two missing threads).

PROPOSITION 1.6. *Suppose X and Y are totally ordered spaces, and αX and ηY are ordered compactifications of X and Y respectively. Distinct points (a, b) and (c, d) from $(\alpha X \times \eta Y) \setminus (X \times Y \cup T \cup B \cup R \cup L)$ with $a \leq c$ may be identified (possibly along with other points) to produce an ordered compactification of $X \times Y$ smaller than $\alpha X \times \eta Y$ only if one of the following conditions holds.*

(1) *$a = c \in X$ and $\{b, d\}$ is an essential pair of singularities of Y.*

(2) *$a = c \in \alpha X \setminus X$ and*

  (a) *$\{b, d\}$ is an essential pair of singularities of Y, or*

  (b) *b and d are distinct points encompassed by an almost essential pair of singularities of Y, or*

  (c) *$\{b, d\}$ is a Case 2 gap.*

(3) *a and c are distinct points encompassed by an almost essential pair of singularities of X and*

(a) $b = d \in \eta Y \setminus Y$, *or*

(b) $\{b, d\}$ *is an essential pair of singularities of* Y, *or*

(c) $d < b$ *and* b *and* d *are encompassed by an almost essential pair of singularities of* Y, *or*

(d) $d < b$ *and* $\{b, d\}$ *is a Case* 2 *gap.*

(4) $\{a, c\}$ *is a Case* 2 *gap and*

(a) $b = d \in \eta Y \setminus Y$, *or*

(b) $d < b$ *and* $\{b, d\}$ *is an essential pair of singularities of* Y, *or*

(c) $d < b$ *and* $\{b, d\}$ *are encompassed by an almost essential pair of singularities of* Y, *or*

(d) $d < b$ *and* $\{b, d\}$ *is a Case* 2 *gap.*

(5) $\{a, c\}$ *is an essential pair of singularities in* X *and*

(a) $b = d$, *or*

(b) $\{b, d\}$ *is an essential pair of singularities of* Y, *or*

(c) b *and* d *are distinct points encompassed by an almost essential pair of singularities of* Y, *or*

(d) $\{b, d\}$ *is a Case* 2 *gap.*

Proof. Suppose $a = c \in X$. Note that $b, d \in \eta Y \setminus Y$, and without loss of generality, $b < d$. If $\{b, d\}$ is not an essential pair, then there exists $y \in Y$ such that $b < y < d$, and therefore (a, y) is a point of $X \times Y$ in the convex hull of $\{(a, b), (c, d)\}$. The point (a, y) could not be separated from the identified point in a T_2-ordered manner. Thus, if $a = c \in X$, the only way identification of (a, b) and (c, d) is possible is the case described in (1).

Suppose $a = c \in \alpha X \setminus X$, and without loss of generality, $b < d$. There exist points $a^-, c^+ \in X$ with $a^- < a = c < c^+$. Unless $\{b, d\}$ is as described in (2), there exist points $y, z \in Y$ with $b \le y < z \le d$. Now $(a^-, z) \| (c^+, y)$ in $X \times Y$ and therefore in any ordered compactification of $X \times Y$, yet $(a^-, z) < (c, d)$ and $(a, b) < (c^+, y)$, so identifying (a, b) and (c, d) would introduce new order on $X \times Y$. Thus, if $a = c \in \alpha X \setminus X$, identification is possible only in the cases described in (2).

Suppose a and c are distinct points encompassed by an almost essential pair, and let $x \in X$ be the point such that $a \le x \le c$. If $b = d$, the dual arguments of the first two paragraphs of this proof show that identification is only possible if $b = d \in \eta Y \setminus Y$. If $b < d$ and $\{b, d\}$ is not an essential pair, then there exists $y \in Y$ with $b \le y \le d$, and $(a, b) \le (x, y) \le (c, d)$ prevents the identification of (a, b) and (c, d). If $d < b$ and $\{b, d\}$ is not as described in (3), then there exist $y, z \in Y$ such that $d \le y < z \le b$. Since $\{a, c\}$ is encompassed by an almost essential pair, there exist $a^-, c^+ \in X$ with $a^- \le a < c \le c^+$. Now $(a^-, z) \| (c^+, y)$ in $X \times Y$ but $(a^-, z) \le (a, b)$ and $(c, d) \le (c^+, y)$ prevent the identification of (a, b) and (c, d).

An argument analagous to the preceding paragraph disposes of the cases listed in (4).

Suppose $a \neq c$ and $\{a, c\}$ is an essential pair of singularities. Unless we have the situation described in (5), there are two distinct points of Y between (or equal to) b and d. If $b < d$, there exist points $y, z \in Y$ with $b \leq y < z \leq d$. Because $\{a, c\}$ is an essential pair, there exist $a^-, c^+ \in X$ with $a^- < a < c < c^+$. Now $(c^+, y) \| (a^-, z)$ in $X \times Y$, yet $(a^-, z) \leq (c, d)$ and $(a, b) \leq (c^+, y)$. This prevents the identification of (a, b) and (c, d). Similarly, if $d < b$, there exist $y, z \in Y$ with $d \leq y < z \leq b$, so $(c^+, y) \| (a^-, z)$ in $X \times Y$. But $(a^-, z) \leq (a, b)$ and $(c, d) \leq (c^+, y)$ and this prevents the identification. □

This proposition provides necessary conditions for the identification of non-edge points of $\alpha X \times \eta Y$ to get a smaller ordered compactification. That these conditions are also sufficient will follow from the (f, g) characterization given in the next section.

Proposition 1.6 could be stated in the more general setting of T_2-ordered extensions of $X \times Y$ that are quotients of $\alpha X \times \eta Y$. Compactness was not used in the proof.

LEMMA 1.7. *Suppose X and Y are totally ordered spaces having no simple singularities. For a totally ordered space Z, let σZ denote the smallest ordered compactification of Z. The product order $\leq_p$ on $\sigma X \times \sigma Y$ is the unique order on $\sigma X \times \sigma Y$ that makes this space* (*with the product topology*) *an ordered compactification.*

Proof. Note that σZ exists since Z is totally ordered (see [1] or [5]). Since σX has only essential pairs of singularities filled by single remainder points, every point $a \in \sigma X$ is the limit of monotone nets (a_λ^-) and (a_γ^+) in X where $a_\lambda^- \leq a \ \forall \lambda$ and $a \leq a_\gamma^+ \ \forall \gamma$. Likewise, each point b of σY is the limit of nets in Y converging to b from above and below. We first show that $\leq_p$ is the largest compactification order on $\sigma X \times \sigma Y$. Suppose $\sqsubseteq$ is an order on $\sigma X \times \sigma Y$ that makes $\sigma X \times \sigma Y$ an ordered compactification of $X \times Y$. We wish to show that $(a, b) \sqsubseteq (c, d)$ implies $(a, b) \leq_p (c, d)$, or equivalently, $(a, b) \not\leq_p (c, d)$ implies $(a, b) \not\sqsubseteq (c, d)$. Suppose $(a, b) \not\leq_p (c, d)$. The three cases in which this could happen are considered below.

Case 1: $c < a$ and $d < b$. From the nets (c_λ^-) converging to c from below in X and the net (b_γ^-) converging to b from below in Y, construct the net $(c_\lambda^-, b_\gamma^-)$ in $X \times Y$ indexed by the product of the indexing sets for the coordinate nets. Similarly, construct the net (a_ν^+, b_η^+) in $X \times Y$. Now $c_\lambda^- \leq c < a \leq a_\nu^+ \ \ \forall \lambda, \ \ \forall \nu$ and $b_\gamma^- \leq b \leq b_\eta^+ \ \ \forall \gamma, \ \ \forall \eta$ implies $(c_\lambda^-, b_\gamma^-) \leq (a_\nu^+, b_\eta^+)$ in $X \times Y$ for any $\lambda, \gamma, \nu, \eta$. In any T_2-order $\sqsubseteq$ on $\sigma X \times \sigma Y$ that extends the order $\leq$ on $X \times Y$, we must have $\lim(c_\lambda^-, b_\gamma^-) \sqsubseteq \lim(a_\nu^+, b_\eta^+)$, that is, $(c, b) \sqsubseteq (a, b)$. A similar analysis of nets $(c_\lambda^-, d_\gamma^-) \leq (a_\nu^+, d_\nu^+)$ in $X \times Y$ shows $(c, d) \sqsubseteq (a, d)$ not only in the product order $\leq_p$, but in any compactification order $\sqsubseteq$ on $\sigma X \times \sigma Y$. To show that $(a, b) \not\sqsubseteq (c, d)$, observe that if $(a, b) \sqsubseteq (c, d)$, then transitivity of $\sqsubseteq$ would imply $(c, b) \sqsubseteq (a, d)$. However, since there exist $c^-, a^+ \in X$ with $c^- \leq c < a \leq a^+$ and there exist $b^-, d^+ \in Y$ with $d \leq d^+ < b^- \leq b$, we have $(c^-, b^-) \leq (c, b) \sqsubseteq (a, d) \leq$

(a^+, d^+), yet $(c^-, b^-) \not\leq (a^+, d^+)$ in $X \times Y$. Thus, if $\sqsubseteq$ is to extend the order $\leq$ on $X \times Y$, we must have $(a, b) \not\sqsubseteq (c, d)$.

Case 2: $c < a$ and $b \leq d$. Then there exist $a^-, c^+ \in X$ such that $c \leq c^+ < a^- \leq a$ and there exist $b^-, d^+ \in Y$ such that $b^- \leq b \leq d \leq d^+$. If Y has only one point, the result is trivial. Assuming Y has more than one point, we may further choose such b^- and $d^+ \in Y$ with $b^- < d^+$. Now $(c^+, d^+) \| (a^-, b^-)$ in $X \times Y$, and using a net argument, we must have $(a^-, b^-) \sqsubseteq (a, b)$ and $(c, d) \sqsubseteq (c^+, d^+)$. Thus, $(a, b) \sqsubseteq (c, d)$ would imply $(a^-, b^-) \sqsubseteq (c^+, d^+)$, contrary to $(a^-, b^-) \not\leq (c^+, d^+)$ in $X \times Y$.

Case 3: $a \leq c$ and $d < b$. Then there exist $a^-, c^+ \in X$ such that $a^- \leq a \leq c \leq c^+$ and (if X has more than one point), $a^- < c^+$. Also, there exist $d^+, b^- \in Y$ such that $d \leq d^+ < b^- \leq b$. Now $(a^-, b^-) \| (c^+, d^+)$ in $X \times Y$, $(a^-, b^-) \sqsubseteq (a, b)$, and $(c, d) \sqsubseteq (c^+, d^+)$. Thus, $(a, b) \sqsubseteq (c, d)$ would give the contradiction that $(a^-, b^-) \sqsubseteq (c^+, d^+)$ though $(a^-, b^-) \not\leq (c^+, d^+)$ in $X \times Y$.

Thus, in each of the three cases, $(a, b) \not\leq_p (c, d)$ implies $(a, b) \not\sqsubseteq (c, d)$. It follows that $\leq_p$ is the largest compactification order on $\sigma X \times \sigma Y$.

The first part of Case 1 shows that if $(c, d) \leq_p (a, b)$ then $(c, d) \sqsubseteq (a, b)$, that is, $\leq_p$ is the smallest compactification order on $\sigma X \times \sigma Y$. □

COROLLARY 1.8. *Suppose X and Y are totally ordered spaces having no simple singularities. For a totally ordered space Z, let σZ denote the smallest ordered compactification of Z. Then there is a smallest ordered compactification less than $\beta_o X \times \beta_o Y$, and it is $\sigma X \times \sigma Y$.*

Proof. If X and Y have no simple singularities, then they have no unbounded singularities, no almost essential pairs, and no Case 2 gaps. Thus, by Proposition 1.6, identification of points $(a, b), (c, d) \in \beta_o X \times \beta_o Y$ is possible only if $a = c$ and $\{b, d\}$ is an essential pair, or $\{a, c\}$ is an essential pair and $b = d$, or $\{a, c\}$ and $\{b, d\}$ are both essential pairs. All of this “possible” identification has already occurred in $\sigma X \times \sigma Y$, and (again applying Proposition 1.6) no further identification (of points in $\sigma X \times \sigma Y$) is possible. Thus, $\sigma X \times \sigma Y$ is topologically the smallest ordered compactification below $\beta_o X \times \beta_o Y$. Lemma 1.7 shows that there can be no larger compactification order on $\sigma X \times \sigma Y$, so $\sigma X \times \sigma Y$ is a minimal ordered compactification below $\beta_o X \times \beta_o Y$. Using the fact that all ordered compactifications below $\beta_o X \times \beta_o Y$ are constructed from the identifications described in Proposition 1.6 and have orders that extend the order on $X \times Y$, it is easy to see that $\sigma X \times \sigma Y$ is the smallest ordered compactification below $\beta_o X \times \beta_o Y$. □

COROLLARY 1.9. *Suppose X and Y are totally ordered spaces having only essential singularities. For a totally ordered space Z, let σZ denote the smallest ordered compactification of Z. If $\beta_o(X \times Y) = \beta_o X \times \beta_o Y$ then $\sigma(X \times Y)$ exists and $\sigma(X \times Y) = \sigma X \times \sigma Y$.*

THEOREM 1.10. *If X is a totally ordered space having a simple singularity, Y is a totally ordered space having a simple singularity oppositely directed from that* *of X, and neither X nor Y has unbounded singularities, then $K_o(X \times Y)$ is not a lattice. In particular, $X \times Y$ has no smallest ordered compactification.*

Proof. We will exhibit two ordered compactifications below $\sigma X \times \sigma Y$ which have no infimum. Suppose X has an increasing simple singularity and Y has a decreasing simple singularity. Then there exist points $x \in X$, $a \in \sigma X \setminus X$, $y \in Y$, $b \in \sigma Y \setminus Y$ such that x covers a in σX and b covers y in σY. It is easily verified that an ordered compactification of $X \times Y$ is obtained by identifying (a, b) and (x, b), or by identifying (a, b) and (a, y) in $\sigma X \times \sigma Y$. In any ordered compactification below these two ordered compactifications, the points (a, b), (x, b), and (a, y) would have to be identified, and since only convex sets can be identified, (x, y) would have to be identified along with these points. However, $(x, y) \in X \times Y$ cannot be identified with remainder points in any ordered compactification of $X \times Y$, so there is no ordered compactification below these two. □

2. The Basic Case via (f, g) Pairs

To construct the semilattice $d_{K_o(X \times Y)}(\beta_o X \times \beta_o Y)$ of ordered compactifications below $\beta_o X \times \beta_o Y$ where X and Y are totally ordered spaces, we focus on a simple case in this section. In Section 3 we will see that the general cases can be built up from variations of this simple case, with a few restrictions on how the simple cases interact.

The basic building block for the semilattice $d_{K_o(X \times Y)}(\beta_o X \times \beta_o Y)$ is obtained by considering $X \times Y$ where X is a compact totally ordered space and Y is a totally ordered space whose only singularities are a single essential pair. Let $\beta_o Y \setminus Y = \{\alpha^+, \alpha^-\}$, with $\alpha^+ > \alpha^-$.

From Proposition 1.6, the only identification of compactification points of $\beta_o X \times \beta_o Y = X \times (Y \cup \{\alpha^+, \alpha^-\})$ is between points of form $x_o^+ = (x_o, \alpha^+)$ and $x_o^- = (x_o, \alpha^-)$, that is, between corresponding points on the open edges $P = X \times \{\alpha^+\}$ and $M = X \times \{\alpha^-\}$. See Figure 1.

By considering which points of $\beta_o X \times \beta_o Y$ are already related to a given point $a \in M$, it is easy to see that the only additional order involving a in an ordered compactification of $X \times Y$ could come from putting a larger than some closed decreasing segment of P to the left of a. If we are given an ordered compactification $\gamma(X \times Y)$ of $X \times Y$ which is topologically equivalent to $\beta_o X \times \beta_o Y$ (i.e., one which differs from $\beta_o X \times \beta_o Y$ only in the addition of order), then $\forall a \in M$ there exists a largest element $f(a) \in P \cup \{-\infty\}$ that is less than or equal to a. Here we interpret $f(a) = -\infty$ to mean that a is not larger than any element of P. This defines a function f: $M \to P \cup \{-\infty\}$:

$$f(a) = \begin{cases} \max\{P \cap d_{\gamma(X \times Y)}(a)\} & \text{if } P \cap d_{\gamma(X \times Y)}(a) \neq \emptyset, \\ -\infty & \text{otherwise.} \end{cases}$$

That f is well-defined follows from the fact that P is compact and totally ordered and $d_{\gamma(X\times Y)}(a)$ is compact. Similarly, the ordered compactification $\gamma(X \times Y)$ defines a function $g\colon P \to M \cup \{\infty\}$:

$$g(x) = \begin{cases} \min\{M \cap i_{\gamma(X\times Y)}(x)\} & \text{if } M \cap i_{\gamma(X\times Y)}(x) \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

Thus, $g(x)$ is the smallest element of M that is greater than or equal to x, with $g(x) = \infty$ if no element of M is greater than x.

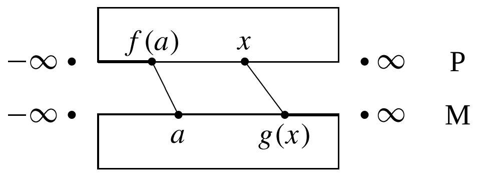

Figure 1.

Since $M = X \times \{\alpha^-\} \approx X \approx X \times \{\alpha^+\} = P$, we may consider f and g as functions from X to $X \cup \{\pm\infty\}$. Let us extend the topology and order on X to $X \cup \{\pm\infty\}$ by taking ∞ and $-\infty$ to be topologically isolated, with $-\infty < x < +\infty\ \ \forall x \in X$. Now f and g may be extended to functions on $X \cup \{\pm\infty\}$ by taking ∞ and $-\infty$ to be fixed points of f and g.

If $\gamma(X \times Y)$ is an ordered compactification of $X \times Y$ which involves identification of points of $\beta_o X \times \beta_o Y$, $\gamma(X \times Y)$ determines functions f and g as above. It is easy to see that $x_o^- \in M$ is identified in $\gamma(X \times Y)$ with its corresponding point $x_o^+ \in P$ if and only if $f(x_o) = x_o = g(x_o)$. The set fix$\{f\}$ of fixed points of f equals the set fix$\{g\}$ of fixed points of g, and the points of these sets, excluding $\pm\infty$, are the points of $\beta_o X \times \beta_o Y$ identified in $\gamma(X \times Y)$. Furthermore, fix$\{f\}$ must be closed: otherwise there would be a net in fix$\{f\}$ converging to a point $x \notin$ fix$\{f\}$. As points of $\gamma(X \times Y)$, this net of identified points of $\beta_o X \times \beta_o Y$ would converge to the copy x^+ of x in P and to the copy x^- of x in M, contrary to uniqueness of limits in a Hausdorff space.

The functions f and g determined by $\gamma(X \times Y)$ are increasing and satisfy $f(x) \leq x \leq g(x)\ \ \forall x \in X \cup \{\pm\infty\}$. Since $x \leq g(x)$ and $f(g(x))$ is the largest element less than or equal to $g(x)$, we have $x \leq f(g(x)) \leq g(x)$. With the dual argument, this shows that f and g satisfy the *quadruple inequality*:

$$f(x) \leq g(f(x)) \leq x \leq f(g(x)) \leq g(x) \quad \text{for any } x \in X \cup \{\pm\infty\}.$$

The functions f and g determined by an ordered compactification $\gamma(X \times Y)$ need not be continuous, as we will now see. Suppose the compact totally ordered space X is chosen such that its largest and smallest elements x_∞ and $x_{-\infty}$ are not topologically isolated points. Let $\gamma(X \times Y)$ be the ordered compactification of $X \times Y$ obtained by adding $(x_\infty, \alpha^-) \geq (x_{-\infty}, \alpha^+)$ to the order on $\beta_o X \times \beta_o Y$. Since only one point was added to the graph of the order on $\beta_o X \times \beta_o Y$, this is clearly a closed order. The functions f and g determined by $\gamma(X \times Y)$ are given by $f(x) = -\infty$ if $x \notin \{x_\infty, \infty\}$, $f(x_\infty) = x_{-\infty}$, $f(\infty) = \infty$ and $g(x) = \infty$ if $x \notin \{x_{-\infty}, -\infty\}$, $g(x_{-\infty}) = x_\infty$, $g(-\infty) = -\infty$. Now f is not continuous (from the left) at x_∞ and g is not continuous (from the right) at $x_{-\infty}$. The theorem below summarizes some of our previous results and shows that the one-sided nature of this example was not coincidental.

THEOREM 2.1. *Suppose X is a compact totally ordered space and Y is a totally ordered space whose only singularity is an essential pair. Suppose f and g are the functions determined by an ordered compactification $\gamma(X \times Y) \in d_{K_o(X\times Y)}(\beta_o X \times \beta_o Y)$. Then*

(1) *f and g are increasing.*
(2) *f and g satisfy the quadruple inequality $f(x) \leq g(f(x)) \leq x \leq f(g(x)) \leq g(x), \forall x \in X \cup \{\pm\infty\}$.*
(3) $\text{fix}\{f\} = \text{fix}\{g\}$ *and this set is closed.*
(4) *f is continuous from the right and g is continuous from the left.*

Proof. We have already verified (1), (2), and (3). We will show $f\colon M \to P \cup \{-\infty\}$ is continuous from the right. Suppose x_λ is a net in M decreasing to $a \in M$. Since $P \cup \{\pm\infty\}$ is a compact totally ordered space, the decreasing net $f(x_\lambda)$ must converge, say to b. Now $a \leq x_\lambda \Rightarrow f(a) \leq f(x_\lambda) \Rightarrow f(a) \leq b$. If $f(a) \neq b$, then b is strictly larger than $f(a)$ = the largest element of $P \cup \{\pm\infty\}$ which is $\leq a$. Thus, $b \not\leq a$ in the T_2-ordered compactification $\gamma(X \times Y)$, so there should exist a decreasing neighborhood of a disjoint from an increasing neighborhood of b. But every neighborhood of a contains points $x_\lambda \geq f(x_\lambda) \geq b$, so every decreasing neighborhood of a contains b. This contradiction shows that $f(a) = b$, so f is continuous from the right. □

THEOREM 2.2. *If X and Y are as in Theorem* 2.1 *and f, g are functions on $X \cup \{\pm\infty\}$ satisfying* (1)–(4) *of Theorem* 2.1, *then f and g determine an ordered compactification $\gamma(X \times Y) \in d_{K_o(X\times Y)}(\beta_o X \times \beta_o Y)$.*

Proof. $\gamma(X \times Y)$ is obtained from $\beta_o X \times \beta_o Y$ by interpreting $f\colon M \to P \cup \{\pm\infty\}$ and $g\colon P \to M \cup \{\pm\infty\}$ as

$$f(x) = \text{the largest element of } P \text{ which is } \leq x, \quad \text{and}$$
$$g(x) = \text{the smallest element of } M \text{ which is } \geq x,$$

with (the two copies, in M and P, of) any fixed points of f being identified. Conditions (3) and (4) guarantee that this quotient is compact and T_2-ordered, while conditions (1) and (2) imply that the order on this quotient extends the order on $X \times Y$. □

The previous two theorems imply that, with X and Y as stated, the elements of $K_o(X \times Y)$ below $\beta_o X \times \beta_o Y$ are characterized by the (f, g) pairs of functions satisfying (1)–(4) of Theorem 2.1 Recalling that adding order to an ordered compactification results in a smaller ordered compactification, it is not difficult to see that if $\alpha_i(X \times Y)$ corresponds to (f_i, g_i), then $\alpha_1(X \times Y) \leq \alpha_2(X \times Y)$ if and only if $f_1(x) \geq f_2(x)$ and $g_1(x) \leq g_2(x)$ for all x. It is routine to verify that if the pair (f_λ, g_λ) is determined by an ordered compactification $\alpha_\lambda(X \times Y)$ for every $\lambda \in \Lambda$, then the pair $(\inf_{\lambda\in\Lambda} f_\lambda, \sup_{\lambda\in\Lambda} g_\lambda)$ satisfies conditions (1)–(4) of Theorem 2.1 and thus corresponds to an ordered compactification, namely $\sup_{\lambda\in\Lambda} \alpha_\lambda(X \times Y)$. The dual statement for $\inf_{\lambda\in\Lambda} \alpha_\lambda(X \times Y)$ holds if this infimum exists. Thus, $d_{K_o(X\times Y)}(\beta_o X \times \beta_o Y)$ and the collection of (f, g) pairs satisfying (1)–(4) of Theorem 2.1 are not only equivalent sets, but also have the same poset structure. We will see later in this section that an (f, g) pair is completely determined by f alone.

Condition (1) and the middle two inequalities of condition (2) in Theorem 2.1 imply that (P, g, f, M) is a Galois connection (see Proposition 2.5 below or Proposition 4.1 in [2]). We recall the definition and several basic facts about Galois connections to streamline our discussion.

DEFINITION 2.3. Consider the posets $(P, \leq)$ and $(Q, \leq')$. If $f: P \to Q$ and $g: Q \to P$ are functions such that for all $p \in P$ and all $q \in Q$

$$p \leq g(q) \text{ iff } f(p) \leq' q$$

then the quadruple (P, f, g, Q) is called a *Galois connection*.

PROPOSITION 2.4. *For any Galois connection (P, f, g, Q),*

(1) *f and g are increasing.*
(2) *Each of the functions f and g is uniquely determined by the other: $f(p) = \inf(\{q \in Q | p \leq g(q)\})$ and $g(q) = \sup(\{p \in P | f(p) \leq' q\})$.*

PROPOSITION 2.5. *Let $(P, \leq)$ and $(Q, \leq')$ be posets and let $f: P \to Q$ and $g: Q \to P$ be functions. Then the following are equivalent:*

(1) *(P, f, g, Q) is a Galois connection.*
(2) *f is increasing, and $g(q) = \max\{z \in P | f(z) \leq' q\}$ for each $q \in Q$.*
(3) *f and g are increasing, $x \leq' f(g(x))$ for all $x \in Q$ and $g(f(x)) \leq x$ for all $x \in P$.*

The reader is directed to [2] for details on these facts and for a basic introduction to the theory of Galois connections.

LEMMA 2.6. *Suppose f is an increasing function on a compact totally ordered space X. Then the function g on X defined by $g(c) = \sup\{x \in X : f(x) \leq c\}$ is increasing and continuous from the left.*

Proof. Clearly g is increasing. Compactness implies that $g(c) = \max\{x \in X : f(x) \leq c\}$. Showing that if $c_\lambda \to c$ from below then $g(c_\lambda) \to g(c)$ is straightforward. □

LEMMA 2.7. *If f and g are increasing functions on a totally ordered space X satisfying the quadruple inequality, then* $\mathrm{fix}\{f\} = \mathrm{fix}\{g\}$ *and this set is closed.*

Proof. Since $f(a) \leq g(f(a)) \leq a$, if $a \in \mathrm{fix}\{f\}$ we have $a \leq g(a) \leq a$, so $a \in \mathrm{fix}\{g\}$. Dually, $\mathrm{fix}\{g\} \subseteq \mathrm{fix}\{f\}$. If x_o is a limit point of $\mathrm{fix}\{f\} = \mathrm{fix}\{g\}$, then there exists a net x_λ in $\mathrm{fix}\{f\}$ converging to x_o. Without loss of generality, this net converges to x_o either from above or from below, and the one-sided continuity of f or g implies $x_o \in \mathrm{fix}\{g\} = \mathrm{fix}\{f\}$. □

THEOREM 2.8. *Suppose f is an increasing function on a compact totally ordered space $X \cup \{\pm\infty\}$, where $\pm\infty$ are topologically isolated points and $-\infty < x < \infty$ $\forall x \in X$. Suppose f is continuous from the right, and $f(x) \leq x \forall x \in X \cup \{\pm\infty\}$. Then f and the function g given by $g(c) = \sup\{x \in X \cup \{\pm\infty\} : f(x) \leq c\}$ satisfy conditions* (1)–(4) *of Theorem* 2.1.

Proof. Note that compactness implies $g(c) = \max\{x \in X \cup \{\pm\infty\} : f(x) \leq c\}$ and hence from Proposition 2.5 this is a Galois connection. Thus the inner inequalities of the quadruple inequality hold. Furthermore, since $f(x) \leq x$ $\forall x \in X \cup \{\pm\infty\}$, the definition of a Galois connection gives the outer inequalities. The other conditions follow from Lemmas 2.6 and 2.7. □

This allows us to describe the semilattice of ordered compactifications for the product $X \times Y$ where X is compact and the only singularities of Y are an essential pair.

COROLLARY 2.9. *If X and Y are totally ordered spaces, X is compact, and the only singularities of Y are one essential pair, then $K_o(X \times Y)$ is semilattice isomorphic to the set of functions f as in Theorem* 2.8 *ordered by the dual order.*

3. Techniques for the General Case

In this section, we consider the progression from the basic case of Section 2 to the general case. First, we note that if X is compact and Y has only a single simple bounded singularity, then only slight modifications of the case of Section 2 are needed. The space $X \times Y$ considered in Section 2 had two open edges P and M which appeared as the middle horizontal segments in Figure 1. If Y has a simple bounded singularity instead of an essential pair, then one of these edges P or M will be open and the other will contain points of $X \times Y$. (Which one is open depends upon whether Y has an increasing or decreasing simple singularity.) Identification of points of P and M would no longer be possible, so the functions f and g described before can have no fixed points. Indeed, $f(a) < a$ $\forall a \in X$ and $x < g(x) \forall x \in X$, so we have functions f and g satisfying these conditions comparable to conditions (1)–(4) of Theorem 2.1:

(1′) f and g are increasing.

(2′) f and g satisfy the stricter quadruple inequality $f(x) < g(f(x)) \leq x \leq f(g(x)) < g(x)\ \forall x \in X$.
(3′) $\operatorname{fix}\{f\} = \operatorname{fix}\{g\} = \emptyset$.
(4′) f is continuous from the right and g is continuous from the left.

By making the inequalities mentioned here into strict inequalities, the arguments of Section 2 (or direct arguments avoiding references to Galois connections) carry through to give corresponding results to describe this case, in which X is compact and Y has a single simple bounded singularity. For example, observe that if $(f_i, g_i)_{i\in I}$ satisfy the stricter quadruple inequality (2′), then $x < g_i(x)\ \forall i \in I$ implies $x < \sup(g_i(x))$, and substituting $\inf(f_j(x))$ for x shows that $(\inf f_i, \sup g_i)_{i\in I}$ satisfies the first strict inequality in (2′).

These two basic cases, where X is compact and Y has either one simple bounded singularity or one essential pair of singularities, are easily extended to the case of $X \times Y$ where X is compact and Y is any totally ordered space with no unbounded singularities. Let us first introduce some helpful notation for such spaces X and Y.

DEFINITION 3.1. Any ordered compactification $\gamma(X \times Y) \leq \beta_0 X \times \beta_0 Y = X \times \beta_0 Y$ determines a (f, g) pair satisfying (1)–(4) of Theorem 2.1 for each essential pair of singularities e of Y and a (f, g) pair satisfying (1′)–(4′) for each simple singularity of Y. For a given essential pair e of Y, let QI_e denote the set of all (f, g) pairs satisfying (1)–(4) of Theorem 2.1, ordered by $(f_1, g_1) \leq (f_2, g_2)$ iff $f_1 \geq f_2$ and $g_1 \leq g_2$. Similarly, for any simple bounded singularity s of Y, let SQI_s be the set of all (f, g) pairs associated with s satisfying (1′)–(4′) above, with the same order relation.

By Theorem 2.8 and its analog, the *quadruple inequality spaces* QI_e and the *stricter quadruple inequality spaces* SQI_s can be characterized in terms of functions f alone. We can then formulate a general theorem.

THEOREM 3.2. *Suppose X is a compact totally ordered space and Y is a totally ordered space with no unbounded singularities. Let E be the set of essential pairs of singularities of Y and let S be the set of simple bounded singularities of Y. Then the semilattice of ordered compactifications of $X \times Y$ below $\beta_0 X \times \beta_0 Y$ is isomorphic to*

$$\prod_{e\in E} QI_e \times \prod_{s\in S} SQI_s$$

with the product order.

Proof. By Proposition 1.6, no identification is possible between points of $\beta_0 X \times \beta_0 Y = X \times \beta_0 Y$ except between points (x, α^+) and (x, α^-) where $\{\alpha^+, \alpha^-\}$ is an essential pair of singularities of Y. Order may be added as determined by (f, g) pairs associated with the singularities of Y, but no additional order between remainder points associated with different singularities of Y is possible. Thus, the semilattice of ordered compactifications of $X \times Y$ below $\beta_o X \times \beta_o Y$ is the collection of QI_e and SQI_s spaces associated with the essential pairs and the simple bounded singularities of Y. The appropriate order on this collection is the product order. □

Of course, if Y were compact and X had no unbounded singularities, a dual argument would apply. The "slits" in the product $X \times Y$ would be vertical instead of horizontal as depicted in Figure 1. If both X and Y have singularities, then care must be taken at the intersection of the vertical and horizontal slits. The remainder of this paper presents two examples to illustrate the techniques used in and the complexity of such cases. We sketch the details in interest of space and readability.

EXAMPLE 3.3. Consider $X \times Y$ where X and Y each have one essential pair of singularities and no simple singularities, so that $\beta_o X = X \cup \{\alpha^+, \alpha^-\}$ and $\beta_o Y = Y \cup \{\gamma^+, \gamma^-\}$. Each ordered compactification of $X \times Y$ below $\beta_o X \times \beta_o Y$ gives a pair (f, g) associated with the essential pair $\{\gamma^+, \gamma^-\}$ and a pair (s, r) associated with $\{\alpha^+, \alpha^-\}$ satisfying conditions (1)–(4) of Theorem 2.1. Note that $s\colon \{\alpha^+\} \times Y \cup \{\pm\infty\} \to \{\alpha^-\} \times Y \cup \{\pm\infty\}$ and $r\colon \{\alpha^-\} \times Y \cup \{\pm\infty\} \to \{\alpha^+\} \times Y \cup \{\pm\infty\}$. However, the "vertical identification" from the fixed points of s (and therefore of r) and the "horizontal identification" from the fixed points of f (and therefore of g) are not independent. Thus, the functions (f, g) and (s, r) must satisfy additional compatibility conditions. Consider the "corner points" $(\alpha^-, \gamma^-) = x$, $(\alpha^+, \gamma^-) = y$, $(\alpha^+, \gamma^+) = z$ and $(\alpha^-, \gamma^+) = w$ as shown in Figure 2.

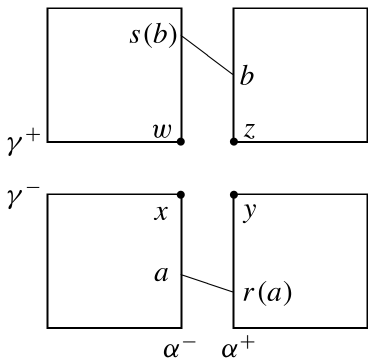

Figure 2.

We consider the possible identification of these corner points and label the cases for reference.

A: No identification between any of the points x, y, z, w.

B: x and w are identified to a point, i.e., α^- is a fixed point of f. This identification implies that the points γ^- and γ^+ in the domain of r are identified to a single point γ. We would like to retain a semicontinuous function $\underline{r}$ with all the properties of r but with this new domain. To do this, we must have $\underline{r}(t) = r(t)$ for $t \notin \{\gamma^-, \gamma^+\}$ and $\underline{r}(\gamma) = r(\gamma^+)$.

C: y and z are identified, i.e., $f(\alpha^+) = \alpha^+$. Then with notation analogous to that in case B, $\underline{s}(\gamma) = s(\gamma^-)$.

D: x and y are identified, i.e., $r(\gamma^-) = \gamma^-$. Then $\underline{f}(\alpha) = f(\alpha^+)$.

E: w and z are identified, i.e., $r(\gamma^+) = \gamma^+$. Then $\underline{g}(\alpha) = g(\alpha^-)$.

F: w and y are identified. Notice that this identification is possible but is not the result of a fixed point of any single function r, s, f or g. The case occurs when $r(\gamma^+) = \gamma^-$ and $f(\alpha^+) = \alpha^-$, and when this occurs, we must have that $s(r(\gamma^+)) = \gamma^+$ and $g(f(\alpha^+)) = \alpha^+$. Thus, this identification is characterized by a fixed point of $s \circ r$ or $g \circ f$ at the upper singularity of an essential pair.

The identifications A–F above are the basic identifications, but certain combinations of these are possible:

G: x, y, and w may be identified. This occurs when identifications A and D above are combined.

H: y, z, and w may be identified. This is the combination of C and E above.

I: $\{x, w\}$ and $\{y, z\}$ may be identified to points. This is the combination of B and C above.

J: $\{x, y\}$ and $\{w, z\}$ may be identified to points. This is the combination of D and E above.

K: $\{x, y, z, w\}$ may be identified to a point. This is the result of A, B, C, and D above.

There is a natural lattice structure $\mathcal{L}$ for the 11 cases A, B, ..., K obtained by considering the order on the 11 topological compactifications of $X \times Y$ obtained from $\beta_0 X \times \beta_0 Y$ by performing the identifications indicated by A, B, ..., K, and no other identifications. The semilattice $d_{K_0(X\times Y)}(\beta_0 X \times \beta_0 Y)$ of ordered compactifications of $X \times Y$ below $\beta_0 X \times \beta_0 Y$ is now obtained by placing a copy of $QI_\alpha \times QI_\gamma$ in each of the 11 positions of $\mathcal{L}$ and performing the indicated identification on all of these according to their position in $\mathcal{L}$. There is a "natural" order on this poset: If $s \leq t$ in $QI_\alpha \times QI_\gamma$ and s appears in a position of $\mathcal{L}$ below the position of t, then the ordered compactification corresponding to s with the additional "crossing" identification determined by its position in $\mathcal{L}$ is below the ordered compactification corresponding to t with its additional "crossing" identification determined by its position in $\mathcal{L}$.

EXAMPLE 3.4. Consider $X \times Y$ where X has an almost essential pair of (simple) singularities and Y has an essential pair of singularities, as depicted in Figure 3.

The 11 identifications A, B, ..., K among the points x, y, z and w as described in the previous example are still possible, as well as the right translations A′, B′, ..., K′ of these to the points y, u, v and z. Formally, let A′, B′, ..., K′ be the identifications obtained by substituting (y, u, v, z) for (x, y, z, w) in the definitions of A, B, ..., K. Note that the identification of y and z is described as C or B′. The identifications G, H, I, J and K are combinations of A, B, C, D, E and F (for example G = B + D). In what follows, we will allow these (acceptable) combinations of A, B, ..., F and will omit references to G, ..., K. Likewise, references to G′, ..., K′ will be omitted.

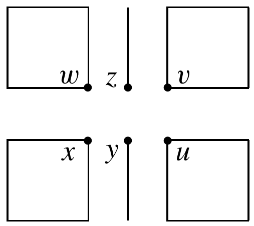

Figure 3.

Combinations of identifications A, ..., F on the left points x, y, z, w and identifications A′, ..., F′ on the right points y, u, v, z are possible. For example, identifying $\{x, w\}$ and $\{u, v\}$ to points corresponds to the combination of B and C′. There are "new" identifications only possible due to the almost essential singularity: $\{x, y, u\}$ may be identified to a point (combine D and D′); $\{w, z, v\}$ may be identified to a point (combine E and E′); $\{w, y, u\}$ may be identified to a point (combine F and D′); and $\{w, z, u\}$ may be identified to a point (combine E and F′). Other identifications of this nature result from (acceptable) combinations larger than those indicated. For example, combining B, D and D′ (or equivalently, G and D′) gives the identification of $\{w, x, y, u\}$. Now $d_{K_o(X\times Y)}(\beta_o X \times \beta_o Y)$ may be described in terms of placing copies of $SQI_1 \times SQI_2 \times QI_1$ into the lattice structure of the admissible identifications involving combinations of A, ..., K, A′, ..., K′.

We remark that the "crossing" of Example 3.4 is the most complex crossing type. The case of $X \times Y$ where X and Y each have an almost essential singularity is not as complex as one may expect due to the fact that there is a point of the base space in the middle of the "crossing" of the singularities which prevents much of the cumbersome identification.

As expected, the techniques of the last two examples may be employed for products of totally ordered factors having more bounded singularities. Then $d_{K_o(X\times Y)}$ $(\beta_o X \times \beta_o Y)$ will be a large product analogous to that of Theorem 3.2, with the (f, g) pairs subject to the "corner restrictions" at every "crossing". If X or Y have unbounded singularities, then further identification ("drawstringing") along the outside edges would be possible, as described in Section 1, and if corner points nw or se exist, they may be folded over and identified with any interior remainder point.

References

1. Blatter, J. (1975) Order compactifications of totally ordered topological spaces, *J. Approx. Theory* **13**, 56–65.

2. Erné, M., Koslowski, J., Melton, A. and Strecker, G. E. (1993) A primer on Galois connections, in S. Andima, R. Kopperman, P. R. Misra, M. E. Rudin and A. R. Todd (eds.), *Papers on General Topology and Applications: Seventh Summer Conference at the University of Wisconsin*, Ann. New York Acad. Sci., Vol. 704, The New York Academy of Sciences, New York, pp. 103–125.
3. Gamon, M. and Kent, D. (1992) A note on ordered planes, *Internat. J. Math. Math. Sci.* **15**(1), 199–202.
4. Kent, D., Richmond, T. and Liu, D. (1995) On the Nachbin compactification of products of totally ordered spaces, *Internat. J. Math. Math. Sci.* **18**(4), 665–676.
5. Kent, D. and Richmond, T. (1988) Ordered compactifications of totally ordered spaces. *Internat. J. Math. Math. Sci.* **11**(4), 683–694.
6. Mooney, D. and Richmond, T. (1997) Zero-dimensional compactifications and ordered compactifications of totally ordered spaces, in M. Rajagopalan and P. Misra (eds.), *Proceedings of the Tennessee Topology Conference*, World Scientific, Singapore, pp. 157–166.
7. Mooney, D. and Richmond, T. (1998) The lattice of ordered compactifications of a direct sum of totally ordered spaces, *Order* **15**, 1–19.
8. Nachbin, L. (1965) *Topology and Order*, Van Nostrand Math. Studies, No. 4, Princeton, NJ.