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The Lattice of Ordered Compactifications of a Direct Sum of Totally Ordered Spaces

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Abstract. The lattice of ordered compactifications of a topological sum of a finite number of totally ordered spaces is investigated. This investigation proceeds by decomposing the lattice into equivalence classes determined by the identification of essential pairs of singularities. This lattice of equivalence classes is isomorphic to a power set lattice. Each of these equivalence classes is further decomposed into equivalence classes determined by admissible partially ordered partitions of the ordered Stone–Čech remainder. The lattice structure within each equivalence class is determined using an algorithm based on the incidence matrix of the partially ordered partition. As examples, the ordered compactification lattices for the spaces $[0, 1) \oplus [0, 1)$, $[0, 1) \oplus [0, 1) \oplus [0, 1)$, $\mathbf{R} \oplus \mathbf{R}$, and $\mathbf{R}\backslash\{0\} \oplus \mathbf{R}\backslash\{0\}$ are determined.

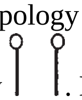
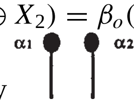
1. Introduction

An *ordered (topological) space* is a triple $(X, \tau, \leq)$ where X is a set, $\leq$ is a partial order on X, and τ is a topology on X having a base of $\leq$-convex sets. A natural compatibility condition between the topology τ and order $\leq$ of an ordered space is the T_2*-ordered property*, which is satisfied if the graph of $\leq$ is closed in the product $(X, \tau) \times (X, \tau)$. A compact T_2-ordered space $(Y, \tau_Y, \leq_Y)$ is an *ordered compactification* of the ordered space $(X, \tau, \leq)$ if (X, τ) is (homeomorphic to) a dense subspace of the compact space (Y, τ_Y) and $\leq_Y$ restricted to X is $\leq$. The set of all ordered compactifications of $(X, \tau, \leq)$ is denoted $K_o(X)$, and can be ordered by $(Y, \tau_Y, \leq_Y) \leq (Y', \tau_{Y'}, \leq_{Y'})$ if there exists a continuous increasing function $f_{Y',Y}\colon Y' \to Y$ that leaves each point of X fixed. This is technically only a preorder on $K_o(X)$, but we will consider it to be a partial order since Y and Y' are homeomorphic and order isomorphic whenever $Y \leq Y'$ and $Y' \leq Y$. With this partial order, $K_o(X)$ is a complete $\vee$-semilattice with largest element $\beta_o X$, the *Stone–Čech ordered-* or *Nachbin- compactification* (see [4]). For each $Y \in K_o(X)$, since $\beta_o X \geq Y$, it follows that Y induces a partition $\{f^{-1}_{\beta_o X,Y}(x) : x \in Y\}$ of $\beta_o X\backslash X$. The blocks of this partition can be given the partial order of $Y\backslash X$, which results in a "partially ordered partition" or "popartition" of $\beta_o X\backslash X$.

Ordered compactifications of totally ordered spaces are studied in Blatter [1] and Kent and Richmond [3]. If X is a totally ordered space, the latter paper characterizes the points of $\beta_o X \backslash X$ as the convex hulls of nonconvergent ultrafilters on X, which are called *singularities*. For any $Y \in K_o(X)$, the blocks of the popartition of $\beta_o X \backslash X$ induced by Y are either singletons or pairs of elements of $\beta_o X \backslash X$. Two singularities form an *essential pair* if the corresponding points $b, c \in \beta_o X \backslash X$ appear together as a block $\{b, c\}$ in the popartition of $\beta_o X \backslash X$ induced by some $Y \in K_o(X)$. Any singularity that is not part of an essential pair is a *simple singularity*. A singularity is *unbounded* if the corresponding point $b \in \beta_o X \backslash X$ is the greatest or least element of $\beta_o X$, and is *bounded* otherwise. For example, given the usual topology and order inherited from the real line, $[0, 1) \cup [2, 3)$ has a simple bounded singularity at 1 and a simple unbounded singularity at 3, while $[0, 1) \cup (1, 2]$ has an essential pair of singularities at 1. The definitions of this paragraph are given in terms of filters on X in [3]. There, it is shown that if E is the set of essential pairs of singularities of a totally ordered space X, then $K_o(X)$ is isomorphic to $\mathcal{P}(E)$, the power set lattice of E.

In this paper, the analysis of $K_o(X)$ is extended to spaces which are direct sums of finitely many totally ordered spaces. Specifically, we will consider $K_o(X)$ for spaces of the form $X = \bigoplus_{i=1}^n X_i$, where each X_i is a totally ordered space and X has the topological sum topology and the direct sum order. The *direct sum order* is defined by $a \leq b$ in X if and only if there exists an $i \in \{1, \ldots, n\}$ such that $a, b \in X_i$ and $a \leq b$ in X_i. To handle the increased complexity of this setting, $K_o(X)$ will be partitioned according to how the essential singularities of the summands are identified (E-equivalence), and then partitioned further according to the induced popartitions of $\beta_o X \backslash X$ (popar equivalence). In Section 2, a matrix algorithm is developed to analyze the structure of each popar equivalence class. In Section 3, $K_o(X)$ is described for the direct sum of three copies of $[0, 1)$. In Section 4, $K_o(R \oplus R)$ is described. Section 5 deals with E-equivalence classes that arise from summands with bounded singularities.

If $X = \bigoplus_{i=1}^n X_i$ is a direct sum of totally ordered spaces, the Stone–Čech ordered compactification $\beta_o X = \beta_o(\bigoplus_{i=1}^n X_i)$ is given by $\bigoplus_{i=1}^n \beta_o X_i$. The elements of $K_o(X)$ are obtained by a combination of identifying compactification points of $\beta_o X$, imposing additional order between compactification points, and imposing additional order between compactification points and the points of the base space.

As an illustration of these ideas, consider $X_1 = X_2 = [0, 1)$, the half open interval with the usual topology and order. We will denote the direct sum $X = X_1 \oplus X_2$ schematically by . Now $\beta_o X = \beta_o(X_1 \oplus X_2) = \beta_o(X_1) \oplus \beta_o(X_2) \approx [0, 1] \oplus [0, 1]$ which will be schematically denoted by 

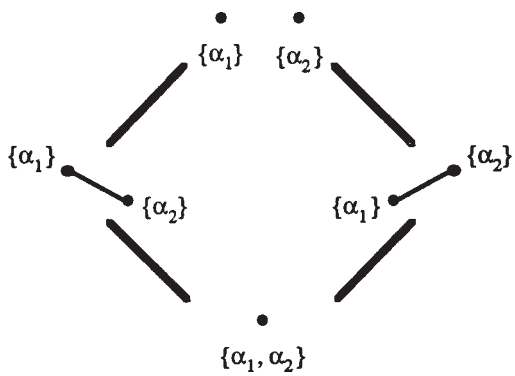
. Throughout the paper α_i will denote the "top" compactification point of the ith summand.

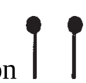
The ordered compactifications of X are obtained by identifying the compactification points of $\beta_o X$ and/or adding some additional order. The identification of the compactification points corresponds to a partition of the set $\{\alpha_1, \alpha_2\}$ of compactification points. The additional order can only be between compactification points and the base space or between compactification points. The instances of the latter case give an order relation on the blocks of the partition of $\beta_o X \backslash X$, that is, a popartition of $\beta_o X \backslash X$.

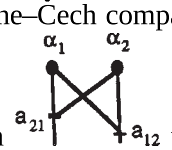
In general, there are not ordered compactifications of a space X corresponding to every partially ordered partition of $\beta_o X \backslash X$; those that do arise from an ordered compactification will be called *admissible popartitions* of $\beta_o X \backslash X$. For example, the 1-block partition $\{\{-\infty, +\infty\}\}$ of $\beta_o \mathbf{R} \backslash \mathbf{R}$ is not admissible since $+\infty$ and $-\infty$ may not be identified in any ordered compactification of the real line **R**.

We will divide $K_o(X)$ into popar equivalence classes by saying that two ordered compactifications of X are popar equivalent if they correspond to the same partially ordered partition of $\beta_o X \backslash X$. The partially ordered partitions of a set form a lattice when ordered by $P \leq Q$ if and only if Q is a refinement of P and the map $[x]_Q \mapsto [x]_P$ is increasing. This lattice is equivalent to the lattice of quasiorders on the set and to the lattice of principal topologies (topologies closed under arbitrary intersections) on the set (see [2]).

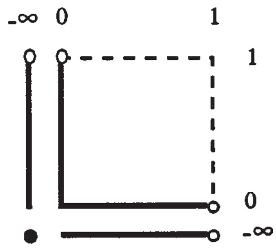
For a space $X = [0, 1) \oplus [0, 1)$, the lattice of popartitions of $\beta_o X \backslash X = \{\alpha_1, \alpha_2\}$ is shown below.

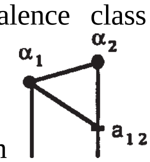
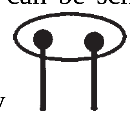


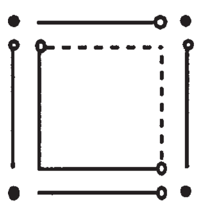
The popar equivalence class of the ordered Stone–Čech compactification 

consists of ordered compactifications of the form 

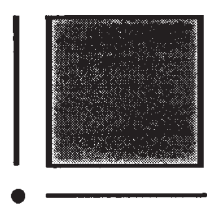
where a_{ij} is the largest element of the jth summand such that $\alpha_i \geq [0, a_{ij}]$. The point a_{ij} must come from the set $\{-\infty\} \cup [0, 1)$, where $a_{ij} = -\infty$ indicates the ith compactification point α_i is not related to any points of the jth summand. For two ordered compactifications with the same underlying topological compactification, as is the case within any popar equivalence class, we recall that the larger ordered compactification is the one with the smaller order. Thus, the ordered compactifications of the popar equivalence class of this current example are determined by the points $(a_{12}, a_{21}) \in \{-\infty\} \cup [0, 1) \times \{-\infty\} \times [0, 1)$ and are ordered by the (dual) product order on this product. This popar equivalence class can be depicted by the diagram below.



Similarly one can perform an analysis on the popar equivalence class  which contains ordered compactifications of the form  where $a_{12} \in \{-\infty\} \cup [0, 1)$; hence this equivalence class can be schematically represented as . The case  is analogous. Finally  is the only element in its popar equivalence class. Putting the appropriate equivalence classes together gives us $K_o([0, 1) \oplus [0, 1))$, which, with the dual order, is represented schematically by



or



A similar analysis of the space $X = [0, 1) \oplus [0, 1) \oplus [0, 1)$ quickly becomes complicated. In this case there are 29 partially ordered partition equivalence classes to consider (see [2]). For each of these classes, a diagram needs to be considered and care must be taken not to introduce prohibited order. Spaces with bottom or middle holes further complicate the situation. Clearly a systematic analysis of the equivalence class structure is needed. This is the theme of Section 2. Section 3 will use the results of Section 2 to analyze the space $X = [0, 1) \oplus [0, 1) \oplus [0, 1)$.

2. A Matrix Algorithm

In this section, we develop a matrix algorithm for investigating a popar equivalence class of ordered compactifications determined by a given partially ordered partition of the ordered Stone–Čech remainder. In this section, we will only consider direct sums of totally ordered spaces *without* essential singularities or simple bounded singularities. In other words, the summands only have top or bottom holes, but not holes in the middle. Spaces with holes in the middle (i.e. spaces which have closed bounded noncompact subsets) will be examined in Section 5.

We will begin with some notation. If $X = \bigoplus_{i=1}^{n} X_i$, then $T \subseteq \{1, \ldots, n\}$ will be the set of indices of summands with "top" singularities (i.e. increasing unbounded singularities) and $B \subseteq \{1, \ldots, n\}$ will be the set of indices of summands with "bottom" singularities (i.e. decreasing unbounded singularities). We will consider partitions $\mathcal{P}$ of the disjoint union $T \overset{\bullet}{\cup} B$ of form $\mathcal{P} = \{T_1, \ldots, T_k, B_1, \ldots, B_l\}$ where $T_i \subseteq T$ for all $i \in \{1, \ldots, k\}$ and $B_j \subseteq B$ for all $j \in \{1, \ldots, l\}$. An *admissible partial order of* $\mathcal{P}$ will be one such that 1) $\forall i \in \{1, \ldots, k\}$ and $\forall j \in \{1, \ldots, l\}$, $T_i \not\leq B_j$ and 2) If $T_i \cap B_j \neq \emptyset$ when considered as subsets of $\{1, \ldots, n\}$, then $T_i > B_j$.

We will denote the top compactification points by the symbol α, with α_{T_i} being the compactification point corresponding to the partition element T_i. Similarly, the symbol ω will be used for bottom compactification points. The symbol a_{ij} will determine the added order between the ith top compactification point and the jth summand, i.e. α_{T_i} is greater than or equal to a_{ij} and all points beneath a_{ij} in the jth summand and α_{T_i} is greater than no other points of the jth summand. Similarly z_{ij} will denote the least point of the jth summand that is above ω_{B_i}. If there is no order between α_{T_i} and the jth summand, then we write $a_{ij} = -\infty$; if there is no order between ω_{B_i} and the jth summand then we will write $z_{ij} = \infty$. The symbol $-\infty$ is less than every element in its summand. The symbol ∞ is greater than every element in its summand.

Identifying the equivalence classes of a partially ordered partition of $\beta_o X \backslash X$ results in one topological compactification of X. The order between the compactification points is determined by the partially ordered partition and thus the compactification points are ordered with respect to each other in exactly one way. The many different ordered compactifications are obtained by defining order between the compactification points and the points of the base space, which is the same as defining the a_{ij}'s and the z_{ij}'s. In principle the a_{ij}'s and the z_{ij}'s could take any value in the jth summand, but many of their values are either forced by the partially ordered partition, or restricted by it. Notice that no choice of a_{ij} and z_{ij} will add order to the base space, but some choices will add order to the partially ordered partition which is illegal since the partially ordered partition was given *a priori*.

Consider two blocks of a partially ordered partition (T_i and T_j, T_i and B_j, or B_i and B_j) indexed by i and j, and a summand indexed by s. There are six possible order arrangements for the blocks, namely: $T_i > B_j$, $T_i \not> B_j$, $T_i \geq T_j$, $T_i \not\geq T_j$, $B_i \geq B_j$, and $B_i \not\geq B_j$ (equality is impossible between top and bottom blocks). For each of these cases, either the index s is in one of the two blocks or it is not.

This presents twelve cases and each implies a set of restrictions, which we list below.

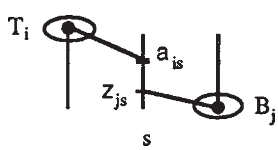
I. Suppose that s is in one of the two blocks (T_i and T_j, T_i and B_j, or B_i and B_j) then

a. $T_i > B_j$ implies $a_{is} \neq -\infty$ and $z_{js} \neq \infty$;
b. $T_i \not> B_j$ implies $a_{is} = -\infty$ and $z_{js} = \infty$;
c. $T_i \geq T_j$ implies $a_{is} = \begin{cases} \alpha_{T_j} & \text{if } s \in T_j, \\ \alpha_{T_i} & \text{if } s \in T_i; \end{cases}$
d. $T_i \not\geq T_j$ implies $a_{is} \neq \alpha_{T_j}$;
e. $B_i \leq B_j$ implies $z_{is} = \begin{cases} \omega_{B_j} & \text{if } s \in B_j, \\ \omega_{B_i} & \text{if } s \in B_i; \end{cases}$
f. $B_i \not\leq B_j$ implies $z_{is} \neq \omega_{B_j}$.

II. Suppose s is not in any of either of the two blocks (T_i and T_j, T_i and B_j, or B_i and B_j) then

a. $T_i > B_j$ implies no restrictions on a_{is} or z_{js} except that they cannot be the top or bottom compactification points if that would change the partially ordered partition;
b. $T_i \not> B_j$ implies $a_{is} \not\geq z_{js}$;
c. $T_i \geq T_j$ implies $a_{is} \geq a_{js}$;
d. $T_i \not\geq T_j$ implies no restrictions on a_{is} and a_{js} except that they cannot be the top compactification point of the s summand (if any) if that adds order to the partially ordered partition;
e. $B_i \geq B_j$ implies $z_{is} \geq z_{js}$;
f. $B_i \not\geq B_j$ implies no restrictions on z_{is} and z_{js} except that they cannot be the bottom compactification point of the s summand (if any) if that adds order to the partially ordered partition.

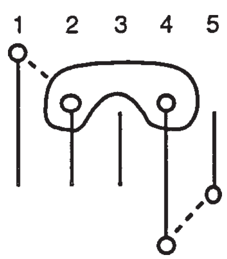
These restrictions are evident. We detail IIb, but others can be reasoned similarly. Consider the following diagram.



This diagram illustrates that if $a_{is} \geq z_{js}$ then order would be introduced between α_{T_i} and ω_{B_j} that was not already present. Requiring $a_{is} < z_{js}$ prevents this problem.

A matrix is a natural bookkeeping tool to keep track of all the a_{ij}'s and z_{ij}'s. In this section, an algorithm is presented which constructs a matrix whose rows correspond to the compactification points (or partition blocks), whose columns correspond to the summands, and whose entries are the a_{ij}'s and z_{ij}'s. The algorithm systematically takes care of all of the possible restrictions of the a_{ij}'s and z_{ij}'s. To justify the algorithm, note that items under the heading I are implied by the incidence matrix of the partially ordered partition. In Steps 1 through 4 of the algorithm below, we start with the incidence matrix (and its dual) and encode Ib, Ic, and Ie which force certain choices of the a_{ij}'s and z_{ij}'s. Specifically, Step 1 combines the incidence matrix and its dual to create a matrix which has one row and column for each partition block. Step 2 encodes all restrictions that follow from the top-to-top, top-to-bottom, bottom-to-top, and bottom-to-bottom partition relations. Step 3 merely switches the columns of the matrix from representing compactification points or partition elements to summands. Since both the top and bottom blocks get converted to summands separately, each summand occurs as two separate columns. Step 4 combines summand information from the two columns into a single column. This gives a matrix of the desired size. Step 5 incorporates of the rest of the restrictions. Several of the restrictions are restatements of the restrictions Ia, IIb, IIc, and IIe; but note that by restricting the a_{ij}'s to $\{-\infty\} \cup \beta_o X_j \backslash \{\alpha_j\}$ and the z_{ij}'s to $(\beta_o X_j \cup \{\infty\}) \backslash \{\omega_j\}$ we incorporate Id and If and the "... no restrictions except ..." requirements of IIa, IId, and IIf.

Throughout this section we will use a running example to illustrate notation and operations. This example is denoted schematically by



For this example, $T = \{1, 2, 4\}$ and $B = \{4, 5\}$. The disjoint union $T \dot{\cup} B$ is partitioned by $T_1 = \{1\}$, $T_2 = \{2, 4\}$, $B_1 = \{4\}$ and $B_2 = \{5\}$. The partition $\mathcal{P} = \{T_1, T_2, B_1, B_2\}$ is partially ordered by $B_1 < T_2 < T_1$ and $B_1 < B_2$. This popartition is admissible.

2.1. STEP 1: THE COMBINED INCIDENCE MATRIX

Form the $(k + l) \times (k + l)$ incidence matrices $M_{\geq}$ and $M_{\leq}$ for the partial orders $\geq$ and $\leq$ on $\mathcal{P}$. Let M be the $(k + l) \times (k + l)$ matrix whose top k rows agree with $M_{\geq}$ and whose bottom l rows agree with $M_{\leq}$. We call M the combined incidence matrix for the partially ordered partition.

In our example

$$M = \begin{array}{c} \\ T_1 \\ T_2 \\ B_1 \\ B_2 \end{array} \begin{array}{c} \begin{array}{cccc} T_1 & T_2 & B_1 & B_2 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

It is convenient to think of $M = (m_{ij})$ in terms of four submatrices $T' \times T'$, $T' \times B'$, $B' \times T'$, and $B' \times B'$ corresponding to the order from top-to-top, top-to-bottom, bottom-to-top, and bottom-to-bottom respectively.

2.2. STEP 2: REFINEMENT

As we have seen, the order between top compactification points and that between bottom compactification points will force some a's to be α's and some z's to be ω's. Similarly, the order relation (or lack thereof) between top and bottom compactification points will introduce further restriction of the choices of a_{ij}'s and z_{ij}'s. Codifying these restrictions requires several steps starting from the combined incidence matrix M.

We alter $M = (m_{ij})$ as follows:
If $(i, j) \in T' \times T'$ and $m_{ij} = 1$, replace m_{ij} by α_{T_j}.
If $(i, j) \in T' \times T'$ and $m_{ij} = 0$, replace m_{ij} by a generic letter a.
If $(i, j) \in T' \times B'$ and $m_{ij} = 1$, replace m_{ij} by a generic letter a.
If $(i, j) \in T' \times B'$ and $m_{ij} = 0$, replace m_{ij} by $-\infty$.
If $(i, j) \in B' \times T'$ and $m_{ij} = 1$, replace m_{ij} by a generic letter z.
If $(i, j) \in B' \times T'$ and $m_{ij} = 0$, replace m_{ij} by ∞.
If $(i, j) \in B' \times B'$ and $m_{ij} = 1$, replace m_{ij} by a ω_{B_j}.
If $(i, j) \in B' \times B'$ and $m_{ij} = 0$, replace m_{ij} by a generic letter z.
In our example, we have

$$M = \left(\begin{array}{cc|cc} \alpha_{T_1} & \alpha_{T_2} & a & -\infty \\ a & \alpha_{T_2} & a & -\infty \\ \hline z & z & \omega_{B_1} & \omega_{B_2} \\ \infty & \infty & z & \omega_{B_2} \end{array}\right).$$

2.3. STEP 3: EXPANSION

Next the $(k+l) \times (k+l)$ matrix M is expanded to form a $(k+l) \times 2n$ matrix D.

Take the ith column of D (for $1 \leq i \leq n$) to be the jth column of M if $i \in T_j$. If $i \notin T_j$ for any j then fill the ith column with generic a's in the top k rows and generic z's in the bottom l rows. For $1 \leq i \leq n$, take the $(n+i)$th column of D to be the $(k+j)$th column of M if $i \in B_j$. If $i \notin B_j$ for any j then fill the $(n+i)$th column with generic a's in the top k rows and generic z's in the bottom l rows.

For our example, we have

$$D = \left(\begin{array}{ccccc|ccccc} \alpha_{T_1} & \alpha_{T_2} & a & \alpha_{T_2} & a & a & a & a & a & -\infty \\ a & \alpha_{T_2} & a & \alpha_{T_2} & a & a & a & a & a & -\infty \\ z & z & z & z & z & z & z & z & \omega_{B_1} & \omega_{B_2} \\ \infty & \infty & z & \infty & z & z & z & z & z & \omega_{B_2} \end{array}\right).$$

2.4. STEP 4: COLLAPSE

The matrix D has one row for each popartition element (i.e. each compactification point) and two columns for each summand. Despite this redundancy, we observe that some information comes only from the top down and other information comes only from the bottom up.

To include all this information in an efficient $(k+l) \times n$ matrix C, we "collapse" D by taking the left half of D and replacing all generic a's and z's by the corresponding entries in the right half. Equivalently,

$$c_{ij} = \begin{cases} d_{ij} & \text{if } d_{ij+n} \text{ is not an } \alpha, \omega, \text{ or } \pm\infty, \\ d_{ij+n} & \text{if } d_{ij+n} \text{ is an } \alpha, \omega, \text{ or } \pm\infty. \end{cases}$$

This step is called "collapsing" or "superimposing". Observe that there are no conflicts in this collapsing. The matrix D has two columns for each summand. One of these contains the a_{ij} and z_{ij} restrictions imposed by the popartition block that the top compactification point of the summand (if any) is in. The other column has restrictions imposed by the popartition block that the bottom compactification point (if any) is in. Collapsing the matrix combines the restrictions from the top with the restrictions from the bottom into a single column. Since we had an admissable popartition to begin with, there can be no conflict. Addressing this issue more concretely, observe that in Step 1 we put only a's, α's, and $-\infty$'s in the top half of M and z's, ω's, and ∞'s in the bottom. Since a's and z's do not conflict with anything, the only conflicts can be α's with $-\infty$'s and ω's with ∞'s. An α in a column of D implies there is order between the compactification point and everything in the summand, while a $-\infty$ implies there is no order between the compactification point and the summand. Since these assignments came from an incidence matrix of an order (and its dual), it is impossible for full order and no order to be assigned at the same time. Therefore an α will never be superimposed on a $-\infty$. Similarly an ω will never be superimposed on an ∞.

For our example, we have

$$C = \begin{pmatrix} \alpha_{T_1} & \alpha_{T_2} & a & \alpha_{T_2} & -\infty \\ a & \alpha_{T_2} & a & \alpha_{T_2} & -\infty \\ z & z & z & \omega_{B_1} & \omega_{B_2} \\ \infty & \infty & z & \infty & \omega_{B_2} \end{pmatrix}.$$

2.5. STEP 5: FINISHING

We consider two submatrices associated with C. The top k rows will be denoted by A, the bottom l rows will be denoted by Z. We will index the elements according to their appropriate submatrix. Thus a_{ij} will be the (i, j)th element of the submatrix A of C and z_{ij} will be the (i, j)th element of the submatrix Z of C. We see that many of the a_{ij} and z_{ij} have already been specified. To create an ordered compactification, we can choose the remaining (currently unspecified) a_{ij}'s from $\{-\infty\}\cup\beta_o X_j\backslash\{\alpha_j\}$ (where α_j is the top compactification point of the jth summand, if there is one) subject to the restrictions:

1. $a_{sj} \geq a_{tj}$ if $T_s > T_t$,
2. $a_{ji} \neq -\infty$ if $T_j > B_m$ where $i \in B_m$.

After the a_{ij}'s are chosen, all the remaining unspecified z_{ij}'s can be chosen from $(\beta_o X_j \cup \{\infty\})\backslash\{\omega_j\}$ subject to:

1. $z_{sj} \geq z_{tj}$ if $B_s > B_t$,
2. $z_{ji} \neq \infty$ if $B_j < T_m$ where $i \in T_m$,
3. $z_{sj} > a_{tj}$ if $T_t \not> B_s$.

Referring to the running example, we have

$$C = \begin{pmatrix} A \\ Z \end{pmatrix} = \left(\begin{array}{ccccc} \alpha_{T_1} & \alpha_{T_2} & a_{13} & \alpha_{T_2} & -\infty \\ a_{21} & \alpha_{T_2} & a_{23} & \alpha_{T_2} & -\infty \\ \hline z_{11} & z_{12} & z_{13} & \omega_{B_1} & \omega_{B_2} \\ \infty & \infty & z_{23} & \infty & \omega_{B_2} \end{array}\right),$$

where $a_{13} \geq a_{23}$ since $T_1 > T_2$, and $a_{i3} \in \{-\infty\}\cup\beta_o X_3$. Similarly $z_{13} \geq z_{23}$ where $z_{i3} \in \beta_o X_3 \cup \{\infty\}$. Also $z_{23} > a_{13}$ since $T_1 \not\geq B_2$. Thus $z_{13} \geq z_{23} > a_{13} \geq a_{23}$. Since $B_1 < T_2$ and $2 \in T_2$, we have $z_{12} \neq \infty$, and thus $z_{12} \in \beta_o X_2$. Similarly, $z_{11} \in \beta_o X_1$. There are no restrictions on $a_{21} \in \{-\infty\} \cup \beta o X_1\backslash\{\alpha_1\}$.

The last part of this section will be concerned with the order structure within a popar equivalence class and between elements of different popar equivalence classes.

THEOREM 2.1. *If Y and Y' are two ordered compactifications within a popar equivalence class with associated matrices C and C', then $Y \leq Y'$ if and only if every element of the first k rows of C is greater than or equal to the corresponding element of C' and every element of the last l rows of C is less than or equal to the corresponding element of C'.*

Proof. The theorem follows immediately from the fact that for topologically equivalent ordered compactifications, the more order an ordered compactification has the lower it is in the lattice of ordered compactifications. □

Thus for our example the lattice structure of the popar equivalence class is the following set with the product order:

$$\begin{aligned}&[\{-\infty\} \cup \beta_o X_1\backslash\{\alpha_1\}]^d \times \beta_o X_1 \times \beta_o X_2 \\ &\quad \times\{(u, v, x, y) \mid u, v \in \beta_o X_3 \cup \{\infty\}, x, y \in [\{-\infty\} \cup \beta_o X_3]^d \\ &\qquad \text{and } u \geq v > x \geq y\},\end{aligned}$$

where X^d represents the poset X with the dual order.

In general, Theorem 2.1 will not work when comparing ordered compactifications in different popar equivalence classes because the matrices C and C' may be of different sizes. This can be corrected if the matrix C obtained by the algorithm of this section is expanded to a $2n \times n$ matrix, which will be called E in Theorem 2.2 below. The ith row of E will be the row of the matrix C which corresponds to the top partition element containing the ith summand. If the ith summand is not in any partition element (i.e. the ith summand has no top hole) then fill the row with x's. Similarly the $(i+n)$th row of E will be the row of the matrix C which corresponds to the bottom partition element containing the ith summand. Again the row will be filled with x's if the ith summand is not in any bottom partition element (i.e. the ith summand has no bottom hole).

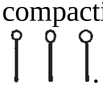
The following theorem generalizes Theorem 2.1 to include comparing compactifications in different partially ordered partition equivalence classes. In addition to making use of the expanded matrix E, the order structure of the partially ordered partitions is included.

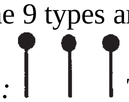
THEOREM 2.2. *Let Y and Y' be ordered compactifications of X. Let E and E' be the associated E matrices. Let a and a' denote the top elements (first n rows) and z and z' denote the bottom elements (2nd n rows) of E and E' respectively.*

$Y \leq Y'$ if and only if

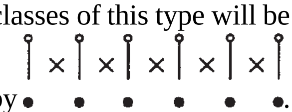
1. *The popar class of Y is smaller than the popar class of Y' in the popar lattice, and*
2. *$a'_{ij} \leq a_{ij}$ and $z'_{ij} \geq z_{ij}$ for all $1 \leq i \leq 2n$ and $1 \leq j \leq n$.*

3. An Example

In this section, we apply the results of the previous section to characterize the lattice of compactifications of the space $[0, 1) \oplus [0, 1) \oplus [0, 1)$, represented schematically by . There are 29 partially ordered partitions of the ordered Stone–Čech remainder points (see [2]). For any $X = \bigoplus_{i=1}^{n} X_i$ where each X_i is $[0, 1)$, each partially ordered partition is admissable. Popar equivalence classes with isomorphic lattice structures generated by the methods of the previous section will be said to be of the same *isomorphic type*. There are nine isomorphic types. The types, the number of partially ordered partitions in each type, and the lattice structure for each of the 9 types are given below.

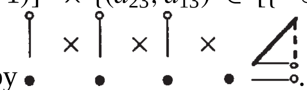
Type 1:  There is only one popartition of this type.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} \alpha_1 & a_{12} & a_{13} \\ a_{21} & \alpha_2 & a_{23} \\ a_{31} & a_{32} & \alpha_3 \end{pmatrix}$$

The six a_{ij} entries of the matrix C can each be chosen without restriction from $\{-\infty\} \cup [0, 1)$, so the dual of the popartition equivalence classes of this type will be $[\{-\infty\} \cup [0, 1]^6$, which can be represented schematically by .

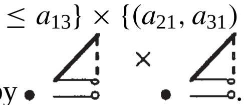
Type 2:  There are six popartitions of this type.

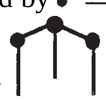
$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} \alpha_1 & \alpha_2 & a_{13} \\ a_{21} & \alpha_2 & a_{23} \\ a_{31} & a_{32} & \alpha_3 \end{pmatrix}$$

All the a_{ij}'s are elements of $\{-\infty\} \cup [0, 1)$. Noting that a_{21}, a_{31}, and a_{32} are unrestricted but $a_{13} \geq a_{23}$, the dual lattice structure for this equivalence class is $[\{-\infty\} \cup [0, 1)]^3 \times \{(a_{23}, a_{13}) \in [\{-\infty\} \cup [0, 1)]^2 : a_{23} \leq a_{13}\}$, which can be represented by .

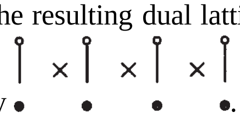
Type 3:  There are three popartitions of this type.

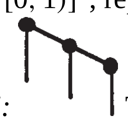
$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} \alpha_1 & \alpha_2 & a_{13} \\ a_{21} & \alpha_2 & a_{23} \\ a_{31} & \alpha_2 & \alpha_3 \end{pmatrix}$$

Throughout this example, the a_{ij}'s are elements of $\{-\infty\} \cup [0, 1)$. Here $a_{13} \geq a_{23}$ and $a_{31} \geq a_{21}$, so the resulting dual lattice structure is $\{(a_{23}, a_{13}) \in [\{-\infty\} \cup [0, 1)]^2 : a_{23} \leq a_{13}\} \times \{(a_{21}, a_{31}) \in [\{-\infty\} \cup [0, 1)]^2 : a_{21} \leq a_{31}\}$, schematically represented by .

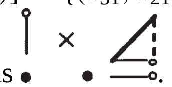
Type 4:  There are three popartitions of this type.

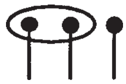
$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} \alpha_1 & a_{12} & a_{13} \\ \alpha_1 & \alpha_2 & \alpha_3 \\ a_{31} & a_{32} & \alpha_3 \end{pmatrix}$$

There are no restrictions on the four a_{ij}'s, so the resulting dual lattice structure is $[\{-\infty\} \cup [0, 1)]^4$, represented schematically by .

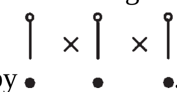
Type 5:  There are six popartitions of this type.

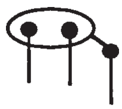
$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ a_{21} & \alpha_2 & \alpha_3 \\ a_{31} & a_{32} & \alpha_3 \end{pmatrix}$$

Here a_{32} is unrestricted, but $a_{21} \geq a_{31}$. The resulting dual lattice structure is $[\{-\infty\} \cup [0,1)] \times \{(a_{31}, a_{21}) \in [\{-\infty\} \cup [0,1)]^2 : a_{31} \leq a_{21}\}$ represented schematically as .

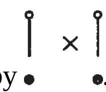
Type 6:  There are three popartitions of this type.

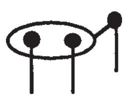
$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} \alpha_{T_1} & \alpha_{T_1} & a_{13} \\ a_{21} & a_{22} & \alpha_{T_2} \end{pmatrix}$$

There are no restrictions on a_{21}, a_{22} or a_{13}, so the resulting dual lattice structure is $[\{-\infty\} \cup [0,1)]^3$, representing schematically by .

Type 7:  There are three popartitions of this type.

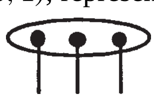
$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} \alpha_{T_1} & \alpha_{T_1} & \alpha_{T_2} \\ a_{21} & a_{22} & \alpha_{T_2} \end{pmatrix}$$

Again, there are no restrictions on the two parameters a_{21} and a_{22}, so the resulting dual lattice structure is $[\{-\infty\} \cup [0,1)]^2$, represented schematically by .

Type 8:  There are three popartitions of this type.

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} \alpha_{T_1} & \alpha_{T_1} & a_{13} \\ \alpha_{T_1} & \alpha_{T_1} & \alpha_{T_2} \end{pmatrix}$$

With no restrictions on the parameter a_{13}, the resulting dual lattice structure is $\{-\infty\} \cup [0,1)$, represented .

Type 9:  There is only popartition of this type.
Since there is only one ordered compactification in this popartition equivalence class, $M = (1)$ and $C = (\alpha_{T_1})$, so the lattice structure consists of a single point.

The Hasse diagram for the lattice of partially ordered partitions on three elements (i.e. the lattice of topologies on 3 elements) shown in Figure 1 is given in [2]. Here, the partially ordered partitions are not individually identified, with only the type of each partially ordered partition given. Filling in the lattice structures for each type as given above into the corresponding positions indicated in the

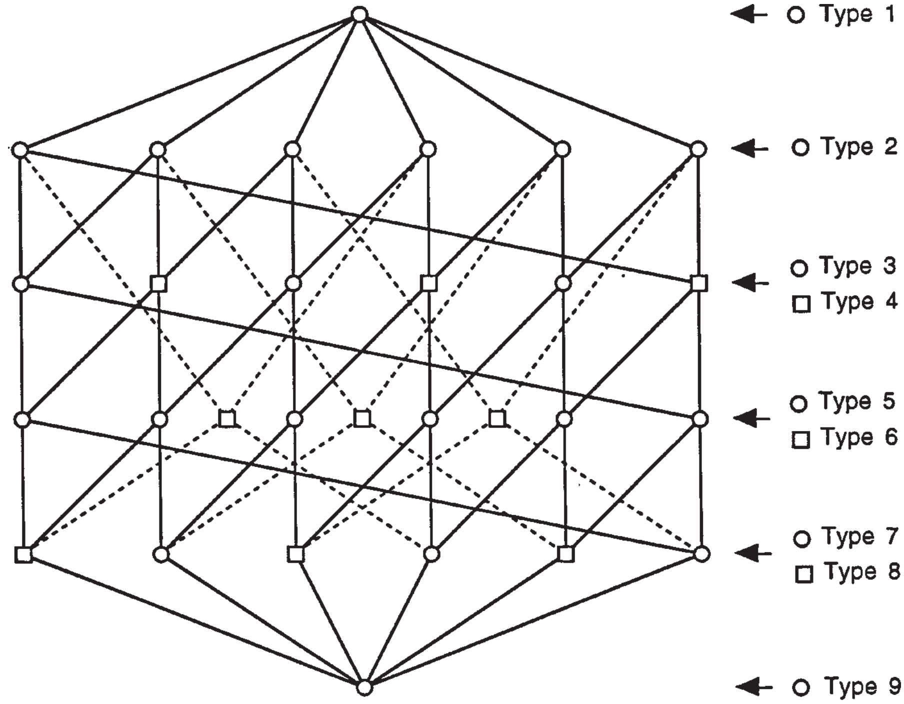


Figure 1.

diagram of Figure 1 would give the lattice for $K_o([0,1) \oplus [0,1) \oplus [0,1))$ where the order between points of different popar equivalence classes would follow from Theorem 2.2.

4. $K_o(\mathbf{R} \oplus \mathbf{R})$

In this section, the lattice of ordered compactifications of the direct sum of two copies of the real line is investigated. By comparing this example with $K_o(\mathbf{R})$, which has only one element, the complexity added to the lattice of ordered compactifications by the topological sum process is evident.

Figure 2 illustrates the 25 admissible partially ordered partitions of $\beta_o(\mathbf{R} \oplus \mathbf{R}) \setminus (\mathbf{R} \oplus \mathbf{R})$ by displaying the largest representative from each popar equivalence class. The space $X = \mathbf{R} \oplus \mathbf{R}$ is denoted by .

Notice that each ordered compactification above has a left-right dual (e.g., g and h). This contributes to the left-right symmetry of $K_o(X)$. The ordered compactifications a, d, s, t, and y are self-(left-right) dual. The lattice structure of these 25 admissible partially ordered partitions is shown below.

Each of the 25 entries in the lattice shown in Figure 3 represents a popar equivalence class. Several of these popar equivalence classes are isomorphic subposets of $K_o(X)$. Left-right duals, for example, will be of the same isomorphic type. There

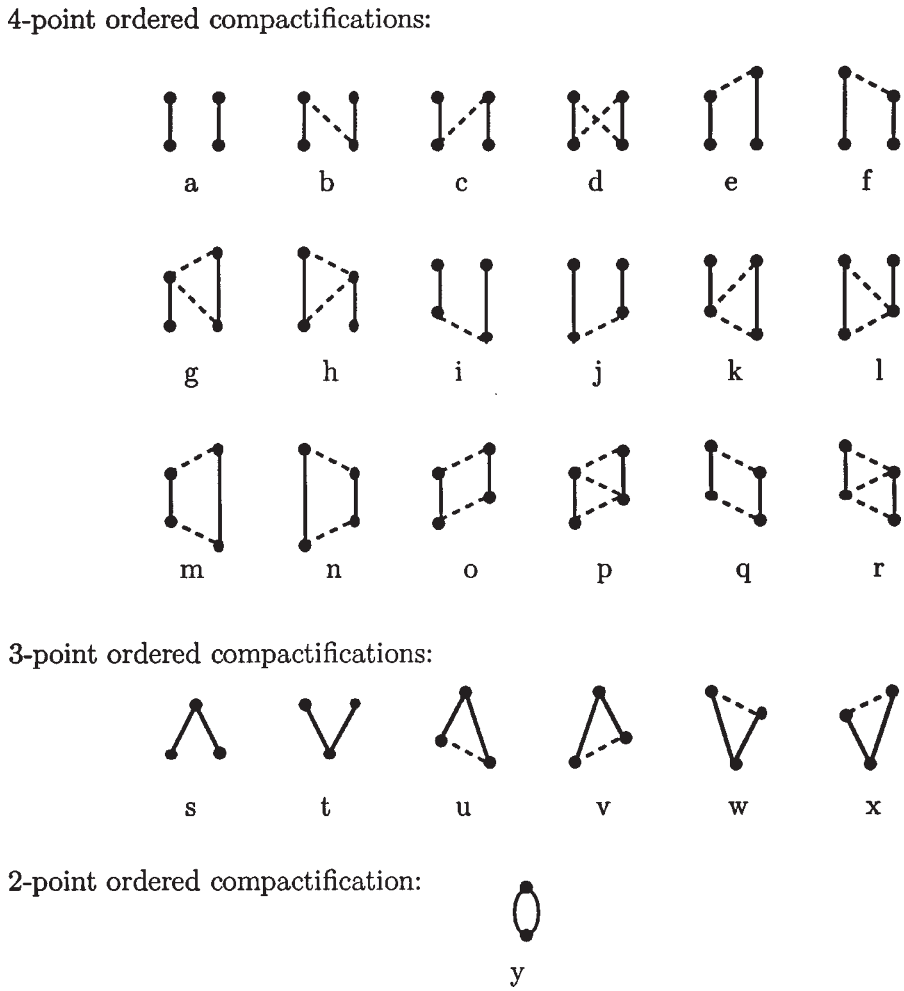


Figure 2.

are only nine non-isomorphic popar equivalence class structures. The popar equivalence class structures corresponding to each of the 25 popar equivalence classes have been listed in Figure 4. A complete lattice diagram for $K_o(\mathbf{R}\oplus\mathbf{R})$ would have these posets at the corresponding vertices of the partially ordered partition lattice in Figure 3. Order between popar equivalence classes is determined according to Theorem 2.2.

5. Summands with Bounded Singularities

In the previous sections, the discussion was limited to topological sums of totally ordered spaces with only unbounded singularities (holes only at the tops or bottoms). Now direct sums of spaces with bounded singularities (holes in the middle)

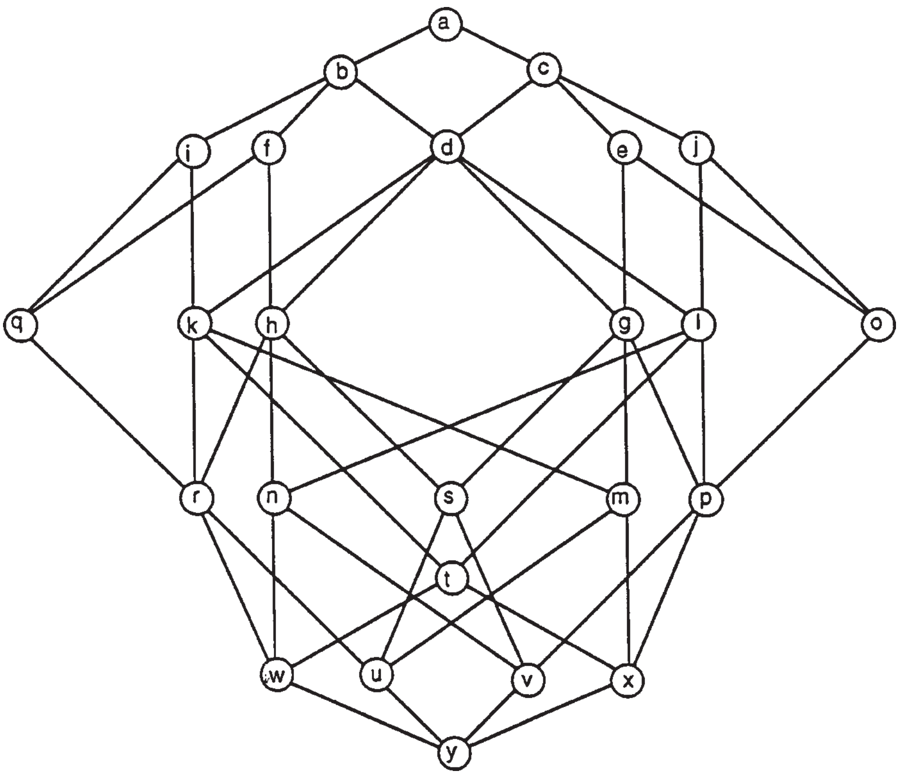


Figure 3.

will be investigated. Throughout this section, X will denote such a space. (For definitions, refer to Section 1 or [3].) Bounded simple singularities are characterized by the fact that in any ordered compactification of the space, they can only introduce a single compactification point that is neither the greatest nor least element of the space. Since they must always have the same effect in every ordered compactification of the space, they have no variability to affect the lattice structure $K_o(X)$ in any way. Thus, the only middle holes of interest in studying the lattice structure are the essential singularities, which always occur in pairs. Recall that if E is the set of essential pairs of singularities of a totally ordered space X, then the lattice $K_o(X)$ is isomorphic of $\mathcal{P}(E)$. More generally, for a direct sum of totally ordered spaces, we will let E denote the union of the sets of essential pairs of singularities from each of the summands. Now $K_o(X)$ can be broken into E-equivalence classes according to which essential singularities from E are compactified with two points. Each of these E-equivalence classes can be further divided into popar equivalence classes which can be analyzed using the methods of Section 2.

DEFINITION 5.1. Two ordered compactifications Y and Y' of a direct sum of totally ordered spaces are E-equivalent if their sets of identified essential singularities are the same.

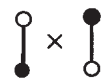
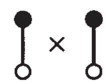
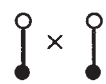
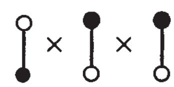
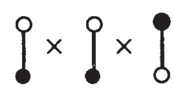
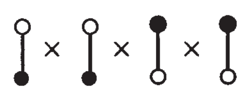
Type	Partially ordered partition	Corresponding poset
1	a, y, o, q	
2	e, f, u, v	
3	i, j, w, x	
4	b, c, m, n, p, r	
5	s	
6	t	
7	g, h	
8	k, l	
9	d	

Figure 4.

A complete set of E-equivalence class representatives will be useful. By the order relation on $K_o(X)$, it is easy to see that each E-equivalence class has a largest element.

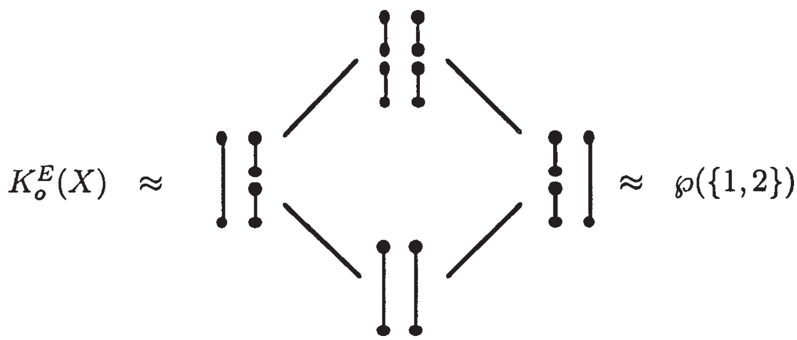
DEFINITION 5.2. The symbol $K_o^E(X)$ will be used to denote the lattice of ordered compactifications of X consisting of the largest elements from each E-equivalence class.

PROPOSITION 5.3. $K_o^E(X) \approx \mathcal{P}(E)$.

The proof is an immediate generalization of the case for a single totally ordered space, as given in Kent and Richmond [3].

In determining $K_o(X)$, note that each element of $K_o^E(X)$ represents the E-equivalence class consisting of all the ordered compactifications of X whose middle holes are filled as indicated by a particular subset of E. Thus, the "middle behavior" of an element of $K_o(X)$ is determined by its representative element from $K_o^E(X)$. With the middle behavior already determined by the position in the lattice $K_o^E(X)$, each of these positions will correspond to a lattice of ordered compactifications of a topological sum of totally ordered spaces *with no essential singularities*, which are the kind of spaces considered in Section 2.

EXAMPLE. Consider $X = \mathbf{R}\backslash\{0\} \oplus \mathbf{R}\backslash\{0\}$ with the usual topology and order. The figure below shows the lattice $K_o^E(X)$ of E-equivalence classes.



The equivalence class associated with the bottom element of this lattice is, of course, $K_o(\mathbf{R} \oplus \mathbf{R})$, which was studied in Section 4. The equivalence class associated with the top element of the lattice will be the same as $K_o(\mathbf{R} \oplus \mathbf{R})$ except that all the intervals appearing in Figure 4 of the previous section will have gaps in them. The other two equivalence classes can be studied similarly with appropriate intervals turned into intervals with gaps and, since symmetry is broken, there will be more isomorphic types.

Using the order provided by the lattice equivalence of the previous proposition, for ordered compactifications Y and Y' of X, we will say $Y' \leq_E Y$ if the associated E-equivalence classes are so related, i.e. if the set of unidentified essential singularities of Y' is a subset of those of Y.

THEOREM 5.4. *If Y and Y' are ordered compactifications of X, then $Y' \leq Y$ if and only if $Y' \leq_E Y$ and the conditions of Theorem* 2.2 *hold, where, in the case of an identification, the two compactification points corresponding to an essential singularity in Y are considered to be equal to the single compactification point for that essential singularity in Y', when comparing entries of the matrix.*

References

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3. Kent, D. and Richmond, T. (1988) Ordered compactifications of totally ordered spaces, *Internat. J. Math. Math. Sci.* **11**(4), 683–694.
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