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NEIGHBORHOOD SPACES AND CONVERGENCE

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Abstract. We study neighborhood spaces (X, ν) in which the system $\nu(x)$ of neighborhoods at a point $x \in X$ is a system of subsets of X containing x which need not be a filter, but must only be a stack, i.e., closed under the formation of supersets. We investigate continuity, separation, compactness, and convergence of centered stacks in this setting.

1. Neighborhood spaces

Before the definition of a topological space was standardized to its current form, Felix Hausdorff [6] and others defined topological spaces in terms of a system of neighborhoods at each point. This approach has continued to be fruitfully studied (cf. [3]) under the name of neighborhood spaces, with various conditions on the systems of neighborhoods at each point. Replacing filters of neighborhoods by p-stacks of neighborhoods, Kent and Min [7] and Min [8] have investigated neighborhood spaces in which the intersection of two neighborhoods of a point x need not be a neighborhood of x, but must only be nonempty (i.e., must have the pairwise intersection property). Thus, the neighborhoods of a given point need not form a filter. We continue the study of neighborhood spaces from [7].

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In particular, we generalize some usual topological concepts including separation and compactness to these spaces and study their behavior. We also introduce a concept of convergence in a neighborhood space and study its properties. To express our convergence, we use centered stacks (called rasters) while in [7] p-stacks, i.e., stacks with the pairwise-intersection property, are used.

Suppose $\mathcal{R}$ is a collection of subsets of a nonempty set X. Consider the following conditions on $\mathcal{R}$.
(a) $A \in \mathcal{R}, A \subseteq B \Rightarrow B \in \mathcal{R}$. ($\mathcal{R}$ is *expansive* or is a *stack*.)
(b) $A_1, A_2 \in \mathcal{R} \Rightarrow A_1 \cap A_2 \neq \emptyset$. ($\mathcal{R}$ has the *pairwise intersection property.*)
(b$'$) $A_1, A_2, \ldots, A_n \in \mathcal{R} \Rightarrow A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$. ($\mathcal{R}$ is *centered*, or has the *finite intersection property.*)

If $\mathcal{R}$ satisfies (a) and (b), it is called a *p*-stack (see [7], [8]). A *raster* on X is a collection $\mathcal{R}$ of subsets of X satisfying conditions (a) and (b$'$) above. Clearly every raster is a p-stack and every filter is a raster, but not conversely. A centered system $\mathcal{B}$ of subsets of X is called a *base* of a raster $\mathcal{R}$ on X if $\mathcal{R}$ is the smallest raster containing $\mathcal{B}$, i.e., if $\mathcal{R} = \langle\mathcal{B}\rangle$ where $\langle\mathcal{B}\rangle = \{B : \exists A \in \mathcal{B},\ A \subseteq B \subseteq X\}$. We then say that $\mathcal{B}$ *generates* $\mathcal{R}$. For every set X, $\mathbf{R}_X$ denotes the set of all rasters on X and, for every point $x \in X$, $\dot{x}$ denotes the filter of all supersets of $\{x\}$.

A *neighborhood space* is a pair (X, ν) where X is a set and $\nu : X \to \mathbf{R}_X$ is a map such that, for every $x \in X$, $\nu(x) \subseteq \dot{x}$. Then $\nu(x)$ is a raster for each $x \in X$ which will be called the *neighborhood raster* at x and the elements of $\nu(x)$ will be called *neighborhoods* of x. Since every neighborhood p-stack is a raster, the neighborhood spaces introduced here coincide with those studied in [7].

In the natural way, we may define interior and closure operators in terms of neighborhoods:
$I_\nu(A) = \{x \in A : A \in \nu(x)\}$,
$cl_\nu(A) = \{x \in X : A \cap V \neq \emptyset \text{ for all } V \in \nu(x)\}$.
It is evident that $I_\nu(X) = X$, cl_ν is grounded ($cl_\nu\emptyset = \emptyset$), I_ν is intensive ($I_\nu(A) \subseteq A$ for all $A \subseteq X$), cl_ν is extensive ($A \subseteq cl_\nu(A)$ for all $A \subseteq X$) and both I_ν and cl_ν are monotonic. A subset $A \subseteq X$ is said to be *open* (respectively, *closed*), if $I_\nu(A) = A$ (respectively, $cl_\nu(A) = A$).

Let (X, ν) be a neighborhood space and, for every subset $A \subseteq X$, let $\overline{A}$ denote the complement of A in X, i.e., $\overline{A} = X - A$.

We clearly have $cl_\nu(A) = \overline{I_\nu(\overline{A})}$ and, consequently, $A \subseteq X$ is closed if and only if its complement $\overline{A}$ is open.

Because of the above facts, neighborhood spaces may be studied as forms of closure spaces, where the emphasis is on the closure operator rather than the system of neighborhoods at each point.

The following statement immediately follows from Theorem 2.12 of [7].

Proposition 1.1. *Let (X, ν) be a neighborhood space. Then*

(a) $I_\nu(A \cap B) = I_\nu(A) \cap I_\nu(B)$ *for all* $A, B \subseteq X$ *if and only if* $\nu(x)$ *is a filter for every* $x \in X$,

(b) I_ν *is idempotent if and only if* $\nu(x) = \langle\{A \subseteq X \colon I_\nu(A) \in \nu(x)\}\rangle$ *for all* $x \in X$.

Recall that a closure operator cl on X is *additive* if $cl(A \cup B) = cl(A) \cup cl(B)$ for all $A, B \subseteq X$.

Corollary 1.2. *Let (X, ν) be a neighborhood space. Then*

(a) cl_ν *is additive if and only if* $\nu(x)$ *is a filter for every* $x \in X$,

(b) cl_ν *is idempotent if and only if* $\nu(x) = \langle\{A \subseteq X \colon I_\nu(A) \in \nu(x)\}\rangle$ *for all* $x \in X$.

Remark 1.3. (1) If a neighborhood space (X, ν) has the property that $\nu(x)$ is a filter for every $x \in X$ and $\nu(x) = \langle\{A \subseteq X \colon I_\nu(A) \in \nu(x)\}\rangle$ for all $x \in X$, then cl_ν is a Kuratowski closure operator on X (and vice versa) by Corollary 1.2. Therefore, we will call (X, ν) a *topological space* in this case (of course, every usual topological space is a neighborhood space with neighborhoods given in the usual way).

(2) A grounded, extensive and additive (hence also monotonic) closure operator is called a *pretopology* (or a Čech closure operator - cf. [1]). By Corollary 1.2, cl_ν is a pretopology if and only if $\nu(x)$ is a filter for every $x \in X$. Therefore, we will call (X, ν) a *pretopological space* in this case. (Of course, every usual pretopological space is a neighborhood space - see [1].)

(3) A grounded, extensive, monotonic and idempotent closure operator is called a *supratopology* - cf. [7], [8]. By Corollary 1.2, cl_ν is a supratopology if and only if (X, ν) has the property that $\nu(x) = \langle\{A \subseteq X : I_\nu(A) \in \nu(x)\}\rangle$ for all $x \in X$. Therefore, we will call (X, ν) a *supratopological space* in this case.

(4) Let (X,ν) be a neighborhood space with an *open base*; that is, for every $x \in X$, $\nu(x)$ has a base consisting of open sets. Then I_ν (respectively, cl_ν) is idempotent by Proposition 1.1(b).

Let (X,ν) and (Y,μ) be neighborhood spaces. We say that (X,ν) is a subspace of (Y,μ) if $X \subseteq Y$ and $\nu(x) = \{X \cap A : A \in \mu(x)\}$.

We note that if $\mathcal{R}$ is a raster on X and $f : X \to Y$ a function, then $f(\mathcal{R})$ need not be a raster, but only a centered system. We will interpret the image raster $f(\mathcal{R})$ to mean the raster generated by $\{f(R) : R \in \mathcal{R}\}$. Similar remarks apply to inverse images of rasters.

A map $f : (X,\nu) \to (Y,\mu)$ is called *continuous* if and only if, for every $x \in X$, $\mu(f(x)) \subseteq f(\nu(x))$ or, equivalently, $f^{-1}(\mu(f(x))) \subseteq \nu(x)$. By [7], Proposition 2.9, we have:

Proposition 1.4. *Let $f : (X,\nu) \to (Y,\mu)$ be a map. Then the following conditions are equivalent:*
(a) f is continuous,
(b) $f(cl_\nu(A)) \subseteq cl_\mu(f(A))$ for all $A \subseteq X$,
(c) $f^{-1}(I_\mu(B)) \subseteq I_\nu(f^{-1}(B))$ for all $B \subseteq Y$.

Corollary 1.5. *Let $f : (X,\nu) \to (Y,\mu)$ be a map. If f is continuous, then $f^{-1}(B)$ is open whenever $B \subseteq Y$ is open. The converse is true provided that I_ν is idempotent.*

The previous Corollary and Remark 1.3(4) immediately imply:

Corollary 1.6. *Let $f : (X,\nu) \to (Y,\mu)$ be a map and let $\nu(x)$ have an open base for every $x \in X$. Then f is continuous if and only if the inverse image under f of every open subset of Y is open.*

2. Separation and compactness

Definition 2.1. A neighborhood space (X,ν) is said to be
- (a) *separated* provided that, whenever $x, y \in X$ are different points, there are $M \in \nu(x)$ and $N \in \nu(y)$ with $M \cap N = \emptyset$,
- (b) *compact* if $\bigcap \mathcal{T} \neq \emptyset$ for every centered system $\mathcal{T}$ of closed subsets of (X,ν).

Separated neighborhood spaces are called T_2 in [7]. Clearly, (X,ν) is compact if and only if every open cover of (X,ν) has a finite subcover. The example below shows that the concept of compactness introduced here is strictly weaker than that given in [7], where a neighborhood space (X, ν) is said to be compact if every cover $\mathcal{A} = \{A_x;\ x \in X\}$ of X where $A_x \in \nu(x)$ for every $x \in X$ has a finite subcover. Both concepts of compactness coincide for neighborhood spaces with an open base.

Example 2.2. Let $X = [0, \infty) \times (0, 1] \cup \{(0, 0)\}$. For $(a, b) \in X \setminus \{(0, 0)\}$, let $N(a, b)$ be the union of the line segment from $(0, 0)$ through (a, b) extended to $(c, 1)$ with the horizontal ray having left endpoint $(c, 1)$. Let $N(0, 0) = \{(0, 0)\}$. For each $(a, b) \in X$, take $\nu(a, b) = \langle\{N(a, b)\}\rangle$. Now $\{N(a, b) : (a, b) \in X\}$ is a cover of X by neighborhoods which has no finite subcover. If U is an open set containing $(0, 1)$, then $N(0, 1) \subseteq U$ and U is a neighborhood of each $(x, y) \in N(0, 1)$, and thus $U = X$. It follows that any cover of X by open sets has a finite subcover.

Theorem 2.3. *A neighborhood space (X, ν) is separated if and only if $\{x\} = \bigcap\{cl_\nu(N) : N \in \nu(x)\}$ for every point $x \in X$.*

Proof. Let $\{x\} = \bigcap\{cl_\nu(N) : N \in \nu(x)\}$ for every $x \in X$. Let $x, y \in X$ be different points. Then there is $N \in \nu(x)$ with $y \notin cl_\nu(N)$. Thus, $y \in \overline{cl_\nu(N)} = I_\nu(\overline{N})$ and we have $\overline{N} \in \nu(y)$. As $N \cap \overline{N} = \emptyset$, (X, ν) is separated.

Conversely, given a point $x \in X$, we clearly have $x \in \bigcap\{cl_\nu(N) :\ N \in \nu(x)\}$. Suppose that $\{x\} \neq \bigcap\{cl_\nu(N) : N \in \nu(x)\}$. Then there is a point $y \in X$, y different from x, such that $y \in \bigcap\{cl_\nu(N) : N \in \nu(x)\}$. Since $y \neq x$, there are $M \in \nu(y)$ and $N \in \nu(x)$ with $M \cap N = \emptyset$. Consequently, $M \subseteq \overline{N}$, which yields $\overline{N} \in \nu(y)$. We have $y \in I_\nu(\overline{N}) = \overline{cl_\nu(N)}$. Therefore, $y \notin cl_\nu(N)$, which is a contradiction. □

The following two statements are obvious:

Theorem 2.4. *Let (Y, μ) be a neighborhood space and (X, ν) a subspace of (Y, μ) such that X is closed in (Y, μ). If (Y, μ) is compact, then (X, ν) is compact too.*

Theorem 2.5. *Let (X, ν) and (Y, μ) be neighborhood spaces and $f : (X, \nu) \to (Y, \mu)$ a continuous surjection. If (X, ν) is compact, then (Y, μ) is compact too.*

Remark 2.6. Further statements may be obtained by reformulating some well-known facts about topological spaces to our neighborhood setting. For example, we have: Let (Y, μ) be a separated neighborhood space such that cl_μ is additive and idempotent (so that cl_μ is a Kuratowski closure operator). If (X,ν) is a compact subspace of (Y,μ), then X is closed in (Y,μ).

Definition 2.7. Let (X_i,ν_i), $i\in I$ be a system of neighborhood spaces. The *product* of this system is a neighborhood space (X,ν) where $X=\prod_{i\in I}X_i$ and $\nu(x)=\langle\{\bigcap_{i\in I'}\mathrm{pr_i}^{-1}N_i : I'\subseteq I$ finite and $N_i\in\nu(x_i)$ for all $i\in I'\}\rangle$ for every $x=(x_i : i\in I)\in X$.

In [7], the Cartesian product of neighborhood spaces is used (i.e., the product that is the initial neighborhood space with respect to projections). This product differs from the one introduced in Definition 2.7. Unlike the Cartesian product of neighborhood spaces, the product from Definition 2.7 has the property that, for topological spaces, it coincides with their usual (Cartesian) product. Consequently, the following, Tychonoff theorem for neighborhood spaces (unlike the Tychonoff theorem proved in [7]) generalizes the well-known Tychonoff theorem for topological spaces.

Theorem 2.8. *Let (X_i,ν_i), $i\in I$ be a system of supratopological spaces. Then the product of this system is compact if and only if (X_i,ν_i) is compact for every $i\in I$.*

Proof. Let (X,ν) be the product of the system (X_i,ν_i), $i\in I$. If (X,ν) is compact, then all (X_i,ν_i), $i\in I$, are compact by the previous theorem because all projections are clearly continuous surjections. Conversely, let (X_i,ν_i) be compact for every $i\in I$ and let $\mathcal{S}$ be a centered class of closed subsets of X. Then, by the Axiom of Choice (Zorn's Lemma), there is a maximal centered class $\mathcal{T}$ of subsets of X with $\mathcal{S}\subseteq\mathcal{T}$ (of course, $\mathcal{T}$ is an ultrafilter). The maximality of $\mathcal{T}$ implies that (1) $\mathcal{T}$ is closed under finite intersections and (2) $\mathcal{T}$ contains every subset $M\subseteq X$ having the property $M\cap T\neq\emptyset$ for all $T\in\mathcal{T}$. Since $\mathcal{T}$ is centered and cl_{ν_i} is idempotent for every $i\in I$, $\{cl_{\nu_i}(\mathrm{pr}_i(T))\} : T\in\mathcal{T}\}$ is a centered system of closed subsets of X_i for all $i\in I$. Thus, as (X_i,ν_i) is compact for every $i\in I$, we have $\bigcap_{T\in\mathcal{T}}cl_{\nu_i}(\mathrm{pr}_i(T))\neq\emptyset$ for every $i\in I$. Let $x_i\in\bigcap_{T\in\mathcal{T}}cl_{\nu_i}(\mathrm{pr}_i(T))$ be an arbitrary point and $N_i\in\nu(x_i)$ an arbitrary neighborhood. Then $N_i\cap\mathrm{pr}_i(T)\neq\emptyset$ for each $T\in\mathcal{T}$. Consequently, we have $\mathrm{pr}_i^{-1}(N_i)\cap T\neq\emptyset$ for every $T\in\mathcal{T}$. It follows from (2) that $\mathrm{pr}_i^{-1}(N_i)\in\mathcal{T}$. Put $x=(x_i : i\in I)$. Then, by the assumptions of the statement, there exists a base $\mathcal{B}$ of $\nu(x)$ given by $\mathcal{B}=\{\bigcap_{i\in I'}\mathrm{pr_i}^{-1}N_i : I'\subseteq I$ finite and $N_i\in\nu(x_i)$ for all $i\in I'\}$.

Thus, $\mathcal{B} \subseteq \mathcal{T}$ by (1). Therefore, every $T \in \mathcal{T}$ has the property $T \cap P \neq \emptyset$ for all $P \in \mathcal{B}$. Hence, $x \in cl_\nu(T)$ for every $T \in \mathcal{T}$. It follows that $\bigcap_{T\in\mathcal{T}} cl_\nu(T) \neq \emptyset$. Since $\mathcal{S} \subseteq \{cl_\nu(T) : T \in \mathcal{T}\}$, we have $\bigcap \mathcal{S} \neq \emptyset$. □

Supratopological (neighborhood) spaces which are not topological in general are quite common and occur in various branches of mathematics, particularly in algebra, geometry and logic - see the following example:

Example 2.9. (1) Let $\mathcal{G}$ be a universal algebra (of a given type) having no nullary operations and let X be the underlying set of $\mathcal{G}$. For every subset $A \subseteq X$, let $\langle A\rangle$ be the subalgebra of $\mathcal{G}$ generated by A. Put $\nu(x) = \{A \subseteq X;\ x \notin \langle \overline{A}\rangle\}$ for each $x \in X$. Then (X,ν) is a supratopological space. Clearly, $cl_\nu(A) = \langle A\rangle$ whenever $A \subseteq X$.

(2) Let V be a (real) vector space and, for every subset $A \subseteq V$, let $[A]$ denote the convex hull of A. Put $\nu(x) = \{A \subseteq V;\ x \notin [\overline{A}]\}$ for all $x \in V$. Then (V,ν) is a supratopological space. Clearly, $cl_\nu(A) = [A]$ whenever $A \subseteq V$.

(3) In a formal (axiomatic) theory, let $\vdash$ be the derivation defined on the set $\mathcal{F}$ of all formulas (including all axioms). If $A \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$, let $A \nvdash \varphi$ mean that φ is not derivable from A. For every $\varphi \in \mathcal{F}$, put $\nu(\varphi) = \{A \subseteq \mathcal{F}; \overline{A} \nvdash \varphi\}$. Then $(\mathcal{F},\nu)$ is a supratopological space. Clearly, $cl_\nu(A) = \{\varphi \in \mathcal{F}; A \vdash \varphi\}$ for all $A \subseteq X$.

Remark 2.10. The Tychonoff theorem for topological spaces permits the construction of compactifications. Analogously to topological spaces, a *compactification* of a separated neighborhood space (X,ν) is a compact separated neighborhood space (Y,μ) and a continuous injection $\alpha : X \to Y$ with continuous inverse f^{-1} : $\alpha(X) \to X$ such that $cl_\mu(\alpha(X)) = Y$. Let $C^*(X)$ be the collection of continuous, bounded real-valued functions on (X,ν). A subcollection $C \subseteq C^*(X)$ is said to separate points from closed sets if $x \notin cl_\nu(F)$ implies there exists $f \in C$ with $f(x) \notin cl_{\mathbb{R}}(f(cl_\nu(F)))$ (where $cl_{\mathbb{R}}$ denotes the Kuratowski closure operator on $\mathbb{R}$). A separated neighborhood space (X,ν) is said to be *completely regular* if $C^*(X)$ separates points from closed sets. If (X,ν) is a separated neighborhood space and $C \subseteq C^*(X)$ separates points from closed sets, then the evaluation map $e_C : X \to \Pi_{f\in C} cl_{\mathbb{R}}(f(X))$ defined by $\mathrm{pr}_f(e_C(x)) = f(x)$ is a continuous injection and for each $A \subseteq X$, $e_C(I_\nu(A))$ is open, and it follows that e_C embeds X densely into $cl(e_C(X))$, giving a compactification of (X, ν). Taking $C = C^*(X)$ gives the largest compactification (in the usual order) known as the Čech-Stone compactification. See [2].

3. Convergence

Let $X = \prod_{i\in I} X_i$ and $\mathcal{B}_i \in \mathbf{R}_{X_i}$ for each $i \in I$. As usual, we put $\prod_{i\in I} \mathcal{B}_i = \{\prod_{i\in I} M_i : M_i \in \mathcal{B}_i \text{ for each } i \in I\}$. Clearly, $\prod_{i\in I} \mathcal{B}_i$ is centered but in general it is not a raster. We will interpret $\prod_{i\in I} \mathcal{B}_i$ to be a raster, namely the raster generated by $\prod_{i\in I} \mathcal{B}_i$.

Definition 3.1. Let (X, ν) be a neighborhood space, $x \in X$ a point and $\mathcal{R} \in \mathbf{R}_X$.

(a) We say that $\mathcal{R}$ *converges* to x relative to ν, in symbols $\mathcal{R} \xrightarrow{\nu} x$, if $\nu(x) \subseteq \mathcal{R}$.

(b) A point $x \in X$ is called a *cluster point* of $\mathcal{R}$ provided that $x \in cl_\nu(A)$ for each $A \in \mathcal{R}$ (i.e., provided that $x \in \bigcap_{A\in\mathcal{R}} cl_\nu(A)$).

Note that the definition of convergence given above is a restriction of that from [7] where the convergence of p-stacks, not only rasters, is considered.

Example 3.2. (1) If (X, ν) is a topological space and $\mathcal{R}$ a filter, we obtain the usual convergence and cluster points.

(2) If (X, ν) is a pretopological space and $\mathcal{R}$ a filter, we obtain the convergence and cluster points introduced in [1].

Remark 3.3. Clearly, we have

(1) $\nu(x) \xrightarrow{\nu} x$ for all $x \in X$,

(2) $I_\nu(A) = \{x \in A : A \in \mathcal{R} \text{ whenever } \mathcal{R} \xrightarrow{\nu} x\}$ for all $A \subseteq X$ and

(3) a map $f : (X, \nu) \to (Y, \mu)$ is continuous if and only if for all $x \in X$, $\mathcal{R} \xrightarrow{\nu} x$ implies $f(\mathcal{R}) \xrightarrow{\mu} f(x)$.

For the usual closure in topological spaces, as well as for the closure in neighborhood spaces defined in terms of Kent and Min's p-stacks, we have

$$cl_\nu(A) = \{x \in X : \exists \text{ p-stack } \mathcal{H} \text{ such that } \mathcal{H} \xrightarrow{\nu} x \text{ and } A \in \mathcal{H}\}.$$

This property depends upon the fact that the closure was defined in terms of the intersection of two sets $A \cap V$. Thus, this equality need not hold in the raster setting. In general, we have only

$$cl_\nu(A) \supseteq \{x \in X : \exists \mathcal{R} \in \mathbf{R}_X \text{ such that } \mathcal{R} \xrightarrow{\nu} x \text{ and } A \in \mathcal{R}\},$$

as seen in the example below.

Example 3.4. For $(a,b) \neq (0,0)$ in $\mathbb{R}^2$, let $\nu((a,b))$ be the raster generated by the ϵ-balls centered at (a,b). Let $\nu((0,0))$ be the raster generated by $\{B^+_\epsilon(0,0) : \epsilon > 0\} \cup \{B^-_\epsilon(0,0) : \epsilon > 0\}$ where $B^+_\epsilon(0,0) = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\, x^2 + y^2 < \epsilon\}$ and $B^-_\epsilon(0,0) = \{(x,y) \in \mathbb{R}^2 : x \leq 0,\, x^2 + y^2 < \epsilon\}$. Consider $A = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$. Now A intersects every $N \in \nu((0,0))$, so $(0,0) \in cl_\nu(A)$. However, $A \cap B^+_\epsilon(0,0) \cap B^-_\epsilon(0,0) = \emptyset$, so there can be no raster $\mathcal{R}$ containing A such that $\mathcal{R} \xrightarrow{\nu} (0,0)$.

The following result is immediate.

Proposition 3.5. *Let (X,ν) be a neighborhood space and $A \subseteq X$ a subset. Then* $cl_\nu(A) = \{x \in X : \exists \mathcal{R} \in \mathbf{R}_X \text{ with } \mathcal{R} \xrightarrow{\nu} x \text{ and } N \cap A \neq \emptyset \text{ for every } N \in \mathcal{R}\}$.

Proposition 3.6. *Let (X,ν) be a neighborhood space and $x \in X$ a point. If there exists $\mathcal{S} \in \mathbf{R}_X$ with $\mathcal{R} \subseteq \mathcal{S}$ and $\mathcal{S} \xrightarrow{\nu} x$, then x is a cluster point of $\mathcal{R}$. The converse is true if (X,ν) is a pretopological space and $\mathcal{R}$ is a filter.*

Proof. Suppose there exists $\mathcal{S} \in \mathbf{R}_X$ with $\mathcal{R} \subseteq \mathcal{S}$ and $\mathcal{S} \xrightarrow{\nu} x$. Then $\nu(x) \subseteq S$ implies $A \cap N \neq \emptyset$ whenever $A \in \mathcal{R}$ and $N \in \nu(x)$. Thus, $x \in cl_\nu(A)$ for every $A \in \mathcal{R}$. Hence x is a cluster point of $\mathcal{R}$.
Conversely, let (X,ν) be a pretopological space and $\mathcal{R}$ be a filter. Suppose that x is a cluster point of $\mathcal{R}$. Put $\mathcal{B} = \{A \cap N : A \in \mathcal{R},\ N \in \nu(x)\}$. Since $x \in cl_\nu(A)$ for every $A \in \mathcal{R}$, we have $A \cap N \neq \emptyset$ whenever $A \in \mathcal{R}$ and $N \in \nu(x)$. As both $\mathcal{R}$ and $\nu(x)$ are filters, $\mathcal{B}$ is a centered system. Let $\mathcal{S}$ be the raster generated by $\mathcal{B}$, i.e., $\mathcal{S} = \{B \subseteq X : \exists C \in \mathcal{B}, C \subseteq B\}$. We have $\nu(x) \subseteq \mathcal{S}$, hence $\mathcal{S} \xrightarrow{\nu} x$. But we also have $\mathcal{R} \subseteq \mathcal{S}$, and this completes the proof. □

Corollary 3.7. *Let (X,ν) be a neighborhood space and $x \in X$ a point. If there exists $\mathcal{R} \in \mathbf{R_X}$ such that $\mathcal{R} \xrightarrow{\nu} x$, then x is a cluster point of $\mathcal{R}$.*

Corollary 3.8. *Let (X,ν) be a pretopological space, $x \in X$ a point and $\mathcal{R} \in \mathbf{R}_X$ an ultrafilter. Then $\mathcal{R} \xrightarrow{\nu} x$ if and only if x is a cluster point of $\mathcal{R}$.*

Theorem 3.9. *Let (X,ν) and (Y,μ) be neighborhood spaces, $f : (X,\nu) \to (Y,\mu)$ a continuous map, $x \in X$ and $\mathcal{R} \in \mathbf{R}_X$. If $\mathcal{R} \xrightarrow{\nu} x$, then $f(\mathcal{R}) \xrightarrow{\nu} f(x)$.*

Proof. Let $\mathcal{R} \xrightarrow{\nu} x$ and let $N \in \mu(f(x))$. Then $f^{-1}(N) \in \nu(x) \subseteq \mathcal{R}$. Hence, $f(f^{-1}(N)) \in f(\mathcal{R})$. As $N \supseteq f(f^{-1}(N))$, we have $N \in f(\mathcal{R})$. Therefore, $\mu(f(x)) \subseteq f(\mathcal{R})$, which yields $f(\mathcal{R}) \xrightarrow{\nu} f(x)$. □

Let (X,ν) be the product of a system (X_i,ν_i), $i \in I$, of neighborhood spaces and let $\mathcal{R} \in \mathbf{R}_X$. By the previous theorem, given $x = (x_i : i \in I) \in X$, $\mathcal{R} \xrightarrow{\nu} x$ implies $pr_i(\mathcal{R}) \xrightarrow{\nu} x_i$ for each $i \in I$. If the converse implication is also valid, we say that the raster $\mathcal{R}$ is *convergence-compatible* with the product (X,ν). For example, it is well known that, in topological spaces, filters are convergence-compatible with products.

Proposition 3.10. *Let (X,ν) be the product of a system (X_i,ν_i), $i \in I$, of neighborhood spaces and, for every $i \in I$, let $\mathcal{R}_i \in \mathbf{R}_{X_i}$, $x_i \in X_i$ and $\mathcal{R}_i \xrightarrow{\nu} x_i$. If $\prod_{i\in I} \mathcal{R}_i \in \mathbf{R}_X$ is convergence-compatible with (X,ν), then $\prod_{i\in I} \mathcal{R}_i \to (x_i : i \in I)$.*

Proof. The statement follows from the obvious fact that

$$\mathrm{pr}_i\left(\textstyle\prod_{i\in I} \mathcal{R}_i\right) = \mathcal{R}_i.$$ □

The result below is analogous to Theorem 5.3(2) of [7], although our use of rasters requires an additional hypothesis for the converse.

Theorem 3.11. *Let (X,ν) be a neighborhood space and $\mathcal{R} \in \mathbf{R}_X$. If (X,ν) is separated, then from $\mathcal{R} \xrightarrow{\nu} x$ and $\mathcal{R} \xrightarrow{\nu} y$ it follows that $x = y$. The converse is true if (X,ν) is a pretopological space.*

Theorem 3.12. *Let (X,ν) be a neighborhood space. If every $\mathcal{R} \in \mathbf{R}_X$ has a cluster point, then (X,ν) is compact. The converse is true if (X,ν) is supratopological.*

Proof. Suppose that (X,ν) is not compact. Then there exists a centered system $\mathcal{T}$ of closed subsets of X such that $\bigcap \mathcal{T} = \emptyset$. Hence, $\langle \mathcal{T} \rangle \in \mathbf{R}_X$ and $\bigcap\{cl_\nu(A) : A \in \langle \mathcal{T} \rangle\} = \bigcap \langle \mathcal{T} \rangle = \emptyset$. Thus, $\langle \mathcal{T} \rangle$ has no cluster point.

Conversely, let (X, ν) be supratopological and compact. Let $\mathcal{R} \in \mathbf{R}_X$ and put $\mathcal{S} = \{cl_\nu(A);\ A \in \mathcal{R}\}$. The idempotence of the closure operator insures that the elements of $\mathcal{S}$ are closed. Now since $\mathcal{S}$ is a centered family of closed subsets of X, we have $\bigcap \mathcal{S} \neq \emptyset$. Clearly, every point $x \in \bigcap \mathcal{S}$ is a cluster point of $\mathcal{R}$. □

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