MAPS GENERATING THE SAME PRIMAL SPACE

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Abstract. Alexandroff topologies play an enigmatic role in topology. An important family of Alexandroff topologies are the functional Alexandroff spaces introduced by Shirazi and Golestani, and called primal topologies by O. Echi. The primal topology $\mathcal{P}(f)$ on X determined by a function $f : X \to X$ is the topology whose closed sets are the f-invariant subsets of X.

If $(X, \mathcal{P}(f))$ is a primal space, we investigate the collection $\mathcal{F} = \{g : X \to X : \mathcal{P}(g) = \mathcal{P}(f)\}$ of functions on X which determine the primal topology. We give a necessary and sufficient condition for $\mathcal{F}$ to be finite, and when it is finite, we give an enumeration of $\mathcal{F}$.

1. Introduction

Principal spaces (or *Alexandroff spaces*) are topological spaces in which any intersection of open sets is open. These spaces were first introduced by Alexandroff in 1937 and play an important role in several areas including digital topology and computer sciences (see [2, 6, 9, 10, 12, 11, 14, 15, 16]). The survey paper [13] discusses the lattice of topologies on a finite set and the connection to quasi-orders. In [18], interesting results about principal spaces were investigated from a different perspective by viewing them as closed sets of the Cantor cube 2^X.

We will study a particular class of principal spaces, introduced as *functional Alexandroff spaces* by Ayatollah Zadeh Shirazi and Golestani [3], and independently introduced later by Echi [4], who called them *primal spaces*. For any morphism $f : X \to X$ in the category **Set** of sets, Echi associated a principal topology $\mathcal{P}(f)$ whose closed sets are those subsets A which are f-invariant, (i.e, $f(A) \subseteq A$). A topological space (X, τ) is called a *primal space* if there is some mapping $f : X \to X$ such that

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$\tau = \mathcal{P}(f)$.
The following result from [4] allows us to recognize primal spaces by their associated quasi-orders.

Theorem 1.1. [4, Theorem 2.3] *Let X be a principal topological space. Then, X is a primal space if and only if the associated quasi-ordered set $(X, \leq)$ is a causal quasi-forest in which each non-minimal point x has singleton interval $[x, x]$.*

Let X be a topological space. Define an equivalence relation on X by

$$x \sim y \text{ if and only if } \overline{\{x\}} = \overline{\{y\}}.$$

The resulting quotient space $X/_\sim$ is a T_0-space $\mathbf{T}_0(X)$ called the $\mathbf{T}_0$-reflection of X. It is clearly seen that X is a principal space if and only if $\mathbf{T}_0(X)$ is too. In [4], Echi proved that if X is a primal space, then $\mathbf{T}_0(X)$ is also. Another proof of this result, based on the notion of quasihomeomorphism, is given by Lazaar and Haouati in [5].

In this paper, we consider a primal space (X, τ), with the goal of enumerating the maps $f : X \to X$ for which $\tau = \mathcal{P}(f)$. Some particular spaces are studied and discussed.

2. Enumerating maps giving the same primal space

Given a primal space (X, τ), a natural question is to determine all maps f on X such that $\tau = \mathcal{P}(f)$. In particular, we may wish to enumerate the set $\{f : X \to X : \tau = \mathcal{P}(f)\}$.

Before that, let us state some preliminary results. The set of natural numbers will be represented by $\mathbb{N}$, and $\mathbb{N} \cup \{0\}$ will be denoted by $\mathbb{N}_0$. For a map $f : X \to X$, a point $x \in X$ is a *periodic point of* f if $f^n(x) = x$ for some $n \in \mathbb{N}$; the smallest such n is the *period* of that point. A point with period 1 is a *fixed point* of f. Recall that a *flow* in a category $\mathbf{C}$ is a couple (X, f), where X is an object of $\mathbf{C}$ and $f : X \to X$ is a morphism, called the iterator (see [7] and [8]).

The following proposition is found in Theorem 2.1 and Section 2 of [3] and as Proposition 1.2(7) of [4].

Proposition 2.1 ([3, 4])**.** *Let (X, f) be a flow in **Set** and $x \in X$. We equip X with the topology $\mathcal{P}(f)$. Then $f : (X, \mathcal{P}(f)) \to (X, \mathcal{P}(f))$ is a closed continuous map.*

Proposition 2.2. [4, Proposition 2.2] *Let X be a principal space and $x \in X$. Then $\overline{\{x\}} \setminus [x, x]$ is a closed set.*

The next result appears as [3, Lemma 2.6] and [4, Proposition 2.5(2)].

Proposition 2.3 ([3, 4])**.** *Let (X, f) be a flow. Then $(X, \mathcal{P}(f))$ is a T_0-space if and only if each periodic point of (X, f) is fixed.*

Remark 2.4. Let (X,τ) be a primal space, $x \in X$ and $f,\ g : X \to X$ two maps such $\tau = \mathcal{P}(f) = \mathcal{P}(g)$. Then x is an f-periodic point with period n if and only if x is a g-periodic point with period n. Indeed, suppose that x is an f-periodic point. Then $f^n(x) = x$ for some positive integer n which implies that $\overline{\{x\}}^{\mathcal{P}(f)} = \{x, f(x), \dots, f^{n-1}(x)\} = \overline{\{x\}}^{\mathcal{P}(g)} = \{g^k(x) : k \in \mathbb{N}_0\}$. Hence there exists $m \leq n-1$ such that $g^n(x) = g^m(x)$, so we get $\{g^m(x), g^{m+1}(x), \dots, g^{n-1}(x)\} = \overline{\{g^m(x)\}}^{\mathcal{P}(g)} = \overline{\{g^n(x)\}}^{\mathcal{P}(f)}$. Thus $|\{g^m(x), g^{m+1}(x), \dots, g^{n-1}(x)\}| = |\overline{\{g^n(x)\}}^{\mathcal{P}(f)}|$ which implies that $n - m = n$. Therefore $m = 0$ and so $g^n(x) = g^0(x) = x$ which means that x is a g-periodic point with period n.

Let (X,τ) be a primal space and $x \in X$. Then, by Remark 2.4, x is said to be a *periodic point* (resp., a *fixed point*) if x is a periodic point (resp., fixed point) of any function f satisfying $\tau = \mathcal{P}(f)$.

Proposition 2.5. *Let (X,τ) be a primal space, $x \in X$ and $f,\ g : X \to X$ two maps such that $\tau = \mathcal{P}(f) = \mathcal{P}(g)$. Then the following properties hold:*

(1) $f(\overline{\{x\}}) = g(\overline{\{x\}})$.
(2) $f^{-1}(\overline{\{x\}}) = g^{-1}(\overline{\{x\}})$.

Proof. (1) If x is a periodic point, then $\overline{\{x\}} = \overline{\{f(x)\}} = \overline{\{g(x)\}}$. It follows from Proposition 2.1 that $f(\overline{\{x\}}) = g(\overline{\{x\}})$.

If x is not a periodic point, by Proposition 2.2 we have $\overline{\{x\}} \setminus \{x\}$ is a closed set. But, we already know that $f(x) \in \overline{\{x\}} \setminus \{x\} \subseteq \overline{\{f(x)\}}$ and so $\overline{\{x\}} \setminus \{x\} = \overline{\{f(x)\}}$. Likewise $\overline{\{x\}} \setminus \{x\} = \overline{\{g(x)\}}$. Thus, $\overline{\{f(x)\}} = \overline{\{g(x)\}}$. Therefore, $f(\overline{\{x\}}) = g(\overline{\{x\}})$.

(2) If $y \in f^{-1}(\overline{\{x\}})$, then $f(y) \in \overline{\{x\}}$. On the other hand, we have $\overline{\{y\}} = \{f^n(y) : n \in \mathbb{N}_0\} = \{g^n(y) : n \in \mathbb{N}_0\}$ so $g(y) = f^n(y)$ for some $n \in \mathbb{N}_0$.

If $n = 0$, then we get $g(y) = y = f(y) \in \overline{\{x\}}$.

If $n \geq 1$, then we get $g(y) = f^n(y) \in f^{n-1}(\overline{\{x\}}) \subseteq \overline{\{x\}}$.

Therefore, $f^{-1}(\overline{\{x\}}) \subseteq g^{-1}(\overline{\{x\}})$.

A dual argument gives the converse containment. □

Notation 2.6. Let (X,τ) be a primal space, $A \subseteq X$, $x \in X$ and $n \in \mathbb{N}$.

(1) $\mathcal{F} = \{f : X \to X : \tau = \mathcal{P}(f)\}$.
(2) A_p denotes the set of all periodic points in A.
(3) $A_{p=n}$ denotes the set of all periodic points in A of period $p = n$.
(4) $A_{p\geq n}$ denotes the set of all periodic points in A of period $p \geq n$.
(5) $\mathcal{V}(x)$ denotes the smallest open set containing x.
(6) $X^{-1}_{p\geq n}$ denotes the inverse image of $X_{p\geq n}$ by any map $f \in \mathcal{F}$.
(7) $\mathcal{X} = X^{-1}_{p\geq 2} \setminus X_{p=2}$.

Remark 2.7. To see that the notation of 2.6(6) is well-defined, let (X,τ) be a primal space and $f,\ g : X \to X$ two maps such that $\tau = \mathcal{P}(f) = \mathcal{P}(g)$. We can easily see that $f(X_{p\geq n}) \subseteq X_{p\geq n}$, so $X_{p\geq n}$ is a closed set of X. Thus

$$\begin{aligned} f^{-1}(X_{p\geq n}) = f^{-1}(\bigcup_{x\in X_{p\geq n}} \overline{\{x\}}) &= \bigcup_{x\in X_{p\geq n}} f^{-1}(\overline{\{x\}}) \\ &= \bigcup_{x\in X_{p\geq n}} g^{-1}(\overline{\{x\}}) \\ &= g^{-1}(X_{p\geq n}). \end{aligned}$$

Proposition 2.8. *Let (X,τ) be a primal space, $x \in X$, and $f,\ g : X \to X$ two maps such that $\tau = \mathcal{P}(f) = \mathcal{P}(g)$. Then $f(x) \neq g(x)$ implies that $x \in \mathcal{X}$.*

Proof. We will prove the contrapositve: $x \notin \mathcal{X}$ implies $f(x) = g(x)$. If $x \notin \mathcal{X}$, then two cases arise.

Case 1: $x \in X_{p=2}$. Then $\overline{\{x\}} = \{x, f(x)\} = \{x, g(x)\}$. Since $x \in X_{p=2}$ then $g(x) \neq x$ and thus $f(x) = g(x)$.

Case 2: $x \in (X^{-1}_{p\geq 2})^c$. We already know that $\overline{\{x\}} = \{f^n(x) : n \in \mathbb{N}_0\} = \{g^n(x) : n \in \mathbb{N}_0\}$. If $g(x) = x$ or $f(x) = x$ then we get $f(x) = x = g(x)$. Otherwise, we have $f(x) \in \overline{\{g(x)\}}^{\mathcal{P}(g)}$ and $g(x) \in \overline{\{f(x)\}}^{\mathcal{P}(f)} = \overline{\{f(x)\}}^{\mathcal{P}(g)}$. Then $f(x) = g^n(g(x))$ for some $n \geq 0$ and $g(x) = g^m(f(x))$ for some $m \geq 0$. Hence, $f(x) = g^{n+m}(f(x))$. Since $x \in (X^{-1}_{p\geq 2})^c$ then $n + m < 2$. Thus, $n = 0$ or $m = 0$ and so $f(x) = g(x)$. $\square$

Now, we are in a position to give one of the main results of this section.

Theorem 2.9. *Let (X,τ) be a topological space. Then the following statements are equivalent.*

(1) *X is a primal T_0-space.*
(2) *The following properties hold:*
   (i) *There exists a unique map $f : X \to X$ such that $\tau = \mathcal{P}(f)$.*
   (ii) *If x and y are distinct points in X and $\{x,y\}$ is open, then $\{x\}$ or $\{y\}$ is open.*

Proof. (1) $\Longrightarrow$ (2).

(i) The existence of such a map follows immediately from the definition of a primal space.

Now, for the uniqueness, let $f,\ g : X \to X$ be two maps such that $\tau = \mathcal{P}(f) = \mathcal{P}(g)$. If $f \neq g$, then there exists $x \in X$ such that $f(x) \neq g(x)$. Hence, by Proposition 2.8, we get $x \in \mathcal{X}$. But since (X,τ) is a T_0-space, then by Proposition 2.3 we have $\mathcal{X} = \emptyset$ and this gives a contradiction. Thus, $f = g$.

(ii) Let x and y be two distinct points of X such that $\{x, y\}$ is open. Suppose that neither $\{x\}$ nor $\{y\}$ is open. Then we have $\mathcal{V}(x) = \mathcal{V}(y) = \{x, y\}$. Hence, $x \in \mathcal{V}(y)$, so $y = f^n(x)$ for some nonzero integer n, and likewise $y \in \mathcal{V}(x)$ implies $x = f^m(y)$ for some nonzero integer m. Thus, we get $x = f^m(y) = f^m(f^n(x)) = f^{n+m}(x)$, so x is a periodic point of period $n+m$. By Proposition 2.3, $n + m = 1$ which implies that $n = 0$ or $m = 0$, giving the contradiction that $x = y$.

(2) $\Longrightarrow$ (1). Since there exists a map $f : X \to X$ such that $\tau = \mathcal{P}(f)$ then X is a primal space. Thus, it remains to prove that X is a T_0-space.

Let $a \in X$ be a periodic point of period $p \geq 1$. Then $\overline{\{a\}} = \{a_1, a_2, \dots, a_p\}$ where $a_1 = a$, $f(a_p) = a_1$ and $a_{i+1} = f(a_i)$ for each $1 \leq i \leq p-1$.

• If $p \geq 3$, take the map $g : X \to X$ defined by:

$$\begin{aligned} &g(x) = f(x) \text{ if } x \neq a_i \text{ for each } 1 \leq i \leq p.\\ &g(a_i) = a_{i-1} \text{ for each } 2 \leq i \leq p.\\ &g(a_1) = a_p. \end{aligned}$$

Since $p \geq 3$ then we can suppose without loss of generality that $a_1 \neq a_2 \neq a_p$.

Now, we will prove that $\mathcal{P}(f) = \mathcal{P}(g)$. Since $\mathcal{P}(f)$ and $\mathcal{P}(g)$ are principal topologies, it is sufficient to show that $\overline{\{x\}}^{\mathcal{P}(f)} = \overline{\{x\}}^{\mathcal{P}(g)}$ for all $x \in X$, that is $\{f^n(x) : n \in \mathbb{N}_0\} = \{g^n(x) : n \in \mathbb{N}_0\}$ for all x in X. Let $x \in X$.

If $f^n(x) \neq a_i$ for each $1 \leq i \leq p$ and each $n \in \mathbb{N}_0$ then $f^n(x) = g^n(x)$ for each $n \in \mathbb{N}_0$, hence $\{f^n(x) : n \in \mathbb{N}_0\} = \{g^n(x) : n \in \mathbb{N}_0\}$.

If $f^k(x) = a_i$ for some positive integers k and i with $1 \leq i \leq p$. Without loss of generality, we will take k to be the smallest such positive integer, so that $f^n(x) \notin \{a_1, a_2, \dots, a_p\}$ for each $n < k$. Then

$$\begin{aligned} \overline{\{x\}}^{\mathcal{P}(f)} &= \{x, f(x), \dots, f^{k-1}(x), f^k(x), f^{k+1}(x), \dots, f^{k+p-1}(x), \dots\}\\ &= \{x, f(x), \dots, f^{k-1}(x), a_i, a_{i+1}, \dots, a_{p-1}, a_p, a_1, \dots\}\\ &= \{x, f(x), \dots, f^{k-1}(x), a_1, a_2, \dots, a_p\}, \end{aligned}$$

and

$$\begin{aligned} \overline{\{x\}}^{\mathcal{P}(g)} &= \{x, g(x), \dots, g^{k-1}(x), g^k(x), g^{k+1}(x), \dots, g^{k+p}(x), \dots\}\\ &= \{x, f(x), \dots, f^{k-1}(x), a_i, g(a_i), \dots, g^{p-1}(a_i), \dots\}\\ &= \{x, f(x), \dots, f^{k-1}(x), a_1, a_2, \dots, a_p\}. \end{aligned}$$

Thus $\overline{\{x\}}^{\mathcal{P}(f)} = \overline{\{x\}}^{\mathcal{P}(g)}$ and consequently $\mathcal{P}(f) = \mathcal{P}(g)$. However, we have $g(a_1) = a_p \neq a_2 = f(a_1)$. Thus $f \neq g$ which leads to a contradiction. Thus, $p \ngeq 3$.

• If $p = 2$, then necessarily we have $f^{-1}(\overline{\{a\}}) \setminus \overline{\{a\}} \neq \emptyset$. Indeed, $f^{-1}(\overline{\{a\}}) \setminus \overline{\{a\}} = \emptyset$ means that $f^{-1}(\overline{\{a\}}) = \overline{\{a\}}$ which implies that $\overline{\{a\}} = \{a, f(a)\}$ is an open subset of X. Then, by hypothesis, we get $\{a\}$ or $\{f(a)\}$ is open. If $\{a\}$ is open, we have $f(f(a)) = a$ so $f(a) \in \mathcal{V}_f(a) = \{a\}$. Hence, $f(a) = a$ which is a contradiction ($p = 2$). If $\{f(a)\}$ is open, we know that $a \in \mathcal{V}_f(f(a)) = \{f(a)\}$. Hence, $f(a) = a$ which is also a contradiction ($p = 2$).

Let $y \in f^{-1}(\overline{\{a\}}) \setminus \overline{\{a\}}$ and take the map $g : X \to X$ defined by:

$$g(x) = f(x) \text{ if } x \neq y.$$
$$g(y) = f^2(y).$$

Let $x \in X$.

If $f^n(x) \neq y$ for each $n \in \mathbb{N}_0$ then $f^n(x) = g^n(x)$ for each $n \in \mathbb{N}_0$. Therefore, $\{f^n(x) : n \in \mathbb{N}_0\} = \{g^n(x) : n \in \mathbb{N}_0\}$.

If $f^k(x) = y$ for some positive integer k, take k to be the smallest such positive integer. Then

$$\begin{aligned}\overline{\{x\}}^{\mathcal{P}(f)} &= \{x, f(x), \ldots, f^k(x), f^{k+1}(x), \ldots, f^{k+p-1}(x), \ldots\}\\ &= \{x, f(x), \ldots, y, f(y), f^2(y)\},\end{aligned}$$

and

$$\begin{aligned}\overline{\{x\}}^{\mathcal{P}(g)} &= \{x, g(x), \ldots, g^k(x), g^{k+1}(x), \ldots, g^{k+p}(x), \ldots\}\\ &= \{x, f(x), \ldots, f^k(x), g^{k+1}(x), \ldots, g^{k+p}(x), \ldots\}\\ &= \{x, f(x), \ldots, y, g(y), \ldots, g^p(y), \ldots\}\\ &= \{x, f(x), \ldots, y, f^2(y), f(y)\}.\end{aligned}$$

Thus $\overline{\{x\}}^{\mathcal{P}(f)} = \overline{\{x\}}^{\mathcal{P}(g)}$ and consequently $\mathcal{P}(f) = \mathcal{P}(g)$. However $f \neq g$ since $g(y) = f^2(y) \neq f(y)$, which is a contradiction, so $p \neq 2$.

Therefore $p = 1$ and thus X is a primal T_0-space by Proposition 2.3. □

The necessity of the T_0 condition in Theorem 2.9 is shown in the next example.

Example 2.10. Let $X = \{a_1, a_2, \ldots, a_n\}$ be a finite set with $n > 2$, and let $\tau = \{\emptyset, X\}$ be the indiscrete topology on X. It is clear that (X, τ) is a primal space. Furthermore, we can find two maps f and g from X to X such that $\tau = \mathcal{P}(f) = \mathcal{P}(g)$. For example, let f be defined by $f(a_i) = a_{i+1}$, for $i \in \{1, 2, \ldots, n-1\}$ and $f(a_n) = a_1$, and g by $g(a_{i+1}) = a_i$, for $i \in \{1, 2, \ldots, n-1\}$ and $g(a_1) = a_n$. Since $|X| > 2$ then $f \neq g$.

The following example shows that the property (i) in Theorem 2.9 is essential.

Example 2.11. Let $X = \mathbb{N}_0$ and let τ be the topology on X generated by $\{\{3n\}, \{3n, 3n{+}1, 3n{+}2\} : n \in \mathbb{N}_0\}$. Then (X, τ) satisfies the property (ii) in Theorem 2.9. Indeed, let x and y be two distinct points in X such that $\{x, y\}$ is open. Then there exist $n, m \in \mathbb{N}_0$ such that $\{x, y\} = \{3n, 3m\}$ which implies that $\{x\}$ and $\{y\}$ are open. Clearly, $\overline{\{3n+1\}} = \overline{\{3n+2\}} = \{3n+1, 3n+2\}$ for each $n \in \mathbb{N}_0$ and consequently (X, τ) is not T_0. However (X, τ) is a primal space generated by the map

$$\begin{aligned} f\colon\ & \mathbb{N}_0 \to \mathbb{N}_0 \\ & n \mapsto \begin{cases} f(n) = n-1 \text{ if } n \equiv 2 \pmod 3 \\ f(n) = n+1 \text{ otherwise.} \end{cases} \end{aligned}$$

Now, we give an example showing that the property (ii) in Theorem 2.9 is essential.

Example 2.12. Let $X = \mathbb{N}_0$ and let τ be the topology on X generated by $\{\{2n, 2n+1\} : n \in \mathbb{N}_0\}$. Then (X, τ) satisfies the property (i) in Theorem 2.9. Indeed, take the map

$$\begin{aligned} f\colon\ & \mathbb{N}_0 \to \mathbb{N}_0 \\ & n \mapsto \begin{cases} f(n) = n+1 \text{ if n is even.} \\ f(n) = n-1 \text{ if n is odd.} \end{cases} \end{aligned}$$

Then, one can easily see that f is the unique map satisfying $\tau = \mathcal{P}(f)$. However, (X, τ) is a primal space which is not T_0 since $\overline{\{2n\}} = \overline{\{2n+1\}} = \{2n, 2n+1\}$ for each $n \in \mathbb{N}_0$.

Theorem 2.13. *Let (X, τ) be a primal space. Then we have*

$$|\mathcal{F}| < \infty \textit{ if and only if } |\mathcal{X}| < \infty.$$

Proof. Let (X, τ) be a primal space and $f : X \to X$ a map such that $\tau = \mathcal{P}(f)$.

First, we remark that

$$\mathcal{X} = (X^{-1}_{p\geq 2} \setminus X_{p\geq 2}) \cup X_{p\geq 3}.$$

Suppose that $\mathcal{X}$ is infinite. Then either $X^{-1}_{p\geq 2} \setminus X_{p\geq 2}$ is infinite or $X_{p\geq 3}$ is infinite.

Case 1. $X_{p\geq 3}$ is infinite. Let $\{x_i : i \in I\} \subsetneq X_{p\geq 3}$ be a complete set of equivalence class representatives (with respect to the equivalence relation $\sim$ defined by: $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$) so that $\{[x_i] = [x_i, x_i] : i \in I\}$ is a partition of $X_{p\geq 3}$. Since $X_{p\geq 3}$ is infinite then $\{x_i : i \in I\}$ is infinite, too.

Let $i \in I$ and set $[x_i] = \{a_1, a_2, \dots, a_p\}$, where $a_1 = x_i$ and $p = |[x_i]|$. Then we let the restriction of f on $[x_i]$ be defined by: $f(x_i) = a_2, \dots, f(a_{p-1}) = a_p$ and $f(a_p) = x_i$.

Now, for each $i \in I$ we define the map $f_i : X \to X$ as following:

$$f_i(x) = f(x) \text{ if } x \neq a_k \text{ for each } 1 \leq k \leq p.$$
$$f_i(a_k) = a_{k-1} \text{ for each } 2 \leq k \leq p.$$
$$f_i(a_1) = a_p.$$

We have already seen, in the proof of Theorem 2.9, that $\mathcal{P}(f_i) = \mathcal{P}(f)$. Furthermore, for each $i \neq j \in I$ we have $f_i(x_i) = f_i(a_1) = a_p \neq a_2 = f(a_1) = f(x_i) = f_j(x_i)$ (since $|[x_i]| \geq 3$). Therefore, $f_i \neq f_j$ for each $i \neq j \in I$ and consequently $\mathcal{F}$ is infinite.

Case 2. $X^{-1}_{p\geq 2} \setminus X_{p\geq 2}$ is infinite.

Then, if $y \in X^{-1}_{p\geq 2} \setminus X_{p\geq 2}$ we have $f(y) \in X_{p\geq 2}$.

Set $X^{-1}_{p\geq 2} \setminus X_{p\geq 2} = \{y_i : i \in I\}$ and let $[f(y_i), f(y_i)] = \{a_1, a_2, \dots, a_p\}$, where $a_1 = f(y_i)$ and $p = |[f(y_i), f(y_i)]|$. Then we let the restriction of f on $[f(y_i), f(y_i)]$ be defined by: $f(f(y_i)) = a_2, \dots, f(a_{p-1}) = a_p$ and $f(a_p) = f(y_i)$.

Now, for each $y_i \in X^{-1}_{p\geq 2} \setminus X_{p\geq 2}$ we define the map $f_i : X \to X$ as follows:

$$f_i(x) = f(x) \text{ if } x \neq y_i.$$
$$f_i(y_i) = a_r \text{ with } 2 \leq r \leq p.$$

We shall prove that $\mathcal{P}(f_i) = \mathcal{P}(f)$. But, since $\mathcal{P}(f)$ and $\mathcal{P}(f_i)$ are principal topologies, it is sufficient to show that $\overline{\{x\}}^{\mathcal{P}(f)} = \overline{\{x\}}^{\mathcal{P}(f_i)}$ for all $x \in X$, that is $\{f^n(x) : n \in \mathbb{N}_0\} = \{f_i^n(x) : n \in \mathbb{N}_0\}$. Let $x \in X$.

If $f^n(x) \neq y_i$ for each $n \in \mathbb{N}_0$ then $f^n(x) = f_i^n(x)$ for each $n \in \mathbb{N}_0$. Hence, $\{f^n(x) : n \in \mathbb{N}_0\} = \{f_i^n(x) : n \in \mathbb{N}_0\}$.

If $f^k(x) = y_i$ for some positive integer k, then $f^n(x) \neq y_i$ for each $n \neq k$. Thus

$$\begin{aligned}
\overline{\{x\}}^{\mathcal{P}(f)} &= \{x, f(x), \dots, f^k(x), f^{k+1}(x), \dots, f^{k+p}(x), \dots\} \\
&= \{x, f(x), \dots, y_i, f(y_i), f^2(y_i), \dots, f^p(y_i), \dots\} \\
&= \{x, f(x), \dots, y_i, a_1, a_2, \dots, a_p\}.
\end{aligned}$$

and

$$\begin{aligned}
\overline{\{x\}}^{\mathcal{P}(f_i)} &= \{x, f_i(x), \dots, f_i^k(x), f_i^{k+1}(x), \dots, f_i^{k+p}(x), \dots\} \\
&= \{x, f(x), \dots, f^k(x), f_i^{k+1}(x), \dots, f_i^{k+p}(x), \dots\} \\
&= \{x, f(x), \dots, y_i, f_i(y_i), \dots, f_i^p(y_i), \dots\} \\
&= \{x, f(x), \dots, y_i, a_r, a_{r+1}, \dots, a_p, a_1, \dots, a_{r-1}\} \\
&= \{x, f(x), \dots, y_i, a_1, a_2, \dots, a_p\}.
\end{aligned}$$

Thus, $\overline{\{x\}}^{\mathcal{P}(f)} = \overline{\{x\}}^{\mathcal{P}(f_i)}$ and consequently $\mathcal{P}(f) = \mathcal{P}(f_i)$. However, for each $i \neq j \in I$ we have $f_i(y_i) = a_r \neq a_1 = f(y_i) = f_j(y_i)$ (since

$|[f(y_i), f(y_i)]| \geq 2$). Therefore, $f_i \neq f_j$ for each $i \neq j \in I$ and consequently $\mathcal{F}$ is infinite.

For the converse, suppose that $\mathcal{F} = \{f_i : i \in I\}$ and let $i,\ j \in I$ with $i \neq j$. Then there exists $x \in \mathcal{X}$ such that $f_i(x) \neq f_j(x)$. So, there exists $x \in \mathcal{X}$ such that $f_{i|\mathcal{X}}(x) \neq f_{j|\mathcal{X}}(x)$ and consequently $|\mathcal{F}| = |\{f_i : i \in I\}| = |\{f_{i|\mathcal{X}} : i \in I\}|$. On the other hand, we have $\{f_{i|\mathcal{X}} : i \in I\} \subseteq \mathcal{X}^{\mathcal{X}}$. Since $|\mathcal{X}|$ is assumed to be finite, then $|\{f_{i|\mathcal{X}} : i \in I\}| < \infty$ and finally $|\mathcal{F}| < \infty$. □

Notation 2.14. Suppose (X, τ) is a primal space.

(1) If X has periodic points, let $\Lambda = \{a_j : j \in J\} \subseteq X_p$ be a set of representatives (with respect to the equivalence relation $\sim$ defined by: $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$) such that $\{[a_j, a_j] : j \in J\}$ is a partition of X_p.
(2) If X has no periodic points, take $\Lambda = X_p = \emptyset$.
(3) For each $a_j \in \Lambda$, let p_j be the period of a_j.
(4) For each $a_j \in \Lambda$, let $q_j = |f^{-1}(\overline{\{a_j\}}) \setminus \overline{\{a_j\}}|$.
(5) For $n \in \mathbb{N}$, let $\Lambda_{p \geq n} = \{a_j \in \Lambda : p_j \geq n\}$ and $\Lambda_{p=n} = \{a_j \in \Lambda : p_j = n\}$.

Suppose that $|\mathcal{X}| < \infty$, $a_j \in \Lambda$, and q_j is infinite. If $p_j \geq 2$, then $a_j \in X_{p\geq 2}$, so $\mathcal{X} = X^1_{p\geq 2} \setminus X_{p=2}$ is infinite, a contradiction. Thus, if q_j is infinite, then $p_j = 1$. In this situation, as a notational convenience for what follows, we interpret $p_j^{q_j} = 1^\infty$ to be 1.

Theorem 2.15. *Let (X, τ) be a primal space such that $|\mathcal{X}| < \infty$. Then*

$$|\mathcal{F}| = \begin{cases} 1 & \text{if } \Lambda = \emptyset \\ \prod_{j \in J} (p_j - 1)! \times p_j^{q_j} & \text{otherwise.} \end{cases}$$

Proof. If $\Lambda = \emptyset$ then (X, τ) is a T_0-space. Hence, by Theorem 2.9, there exists a unique map $f : X \to X$ such that $\tau = \mathcal{P}(f)$ which implies that $|\mathcal{F}| = 1$.

If $\Lambda \neq \emptyset$, Let $\mathcal{A} := \{x \in \Lambda_{p=2} : f^{-1}(\overline{\{x\}}) \setminus \overline{\{x\}} = \emptyset\}$. Then we have

$$\begin{aligned}(\Lambda_{p=1} \cup \mathcal{A}) \cup (\Lambda_{p\geq 2} \setminus \mathcal{A}) &= (\Lambda_{p=1} \cup \mathcal{A} \cup \Lambda_{p\geq 2}) \cap (\Lambda_{p=1} \cup \mathcal{A} \cup \mathcal{A}^c) \\ &= \Lambda \cap X \\ &= \Lambda.\end{aligned}$$

Set $\Lambda_1 = \Lambda_{p=1} \cup \mathcal{A} = \{a_j : j \in J_1\}$ and $\Lambda_2 = \Lambda_{p\geq 2} \setminus \mathcal{A} = \{a_j : j \in J_2\}$ with $J_1, J_2 \subseteq J$. It is clear that $\Lambda_1 \cap \Lambda_2 = \emptyset$. Furthermore, we can easily see that $|\Lambda_2| = J_2 < \infty$ (since $|\mathcal{X}| < \infty$).

Let $f,\ g \in \mathcal{F}$. We have already seen that $f \neq g$ if and only if there exists $x \in \mathcal{X}$ such that $f(x) \neq g(x)$.

Set $\Lambda_2 = \{a_1, a_2, \dots, a_m\}$. It is clear that $\mathcal{X} \subseteq \bigcup\limits_{j=1}^{m} \mathcal{V}(a_j)$ and $\mathcal{V}(a_j) \cap \mathcal{V}(a_i) = \emptyset$ for each $a_i,\ a_j \in \Lambda$ with $i \neq j$.

Then, any map f in $\mathcal{F}$ can be defined by its restrictions on $\mathcal{V}(a_j)$, denoted by $f/\mathcal{V}(a_j)$, for any j between 1 and m.

Therefore, it will be sufficient to determine the number of maps $f/\mathcal{V}(a_j)$ for any fixed j. For simplicity, we will still use the notation f to designate $f/\mathcal{V}(a_j)$.

For this, let $j \in J_2$ and $x \in \mathcal{V}(a_j)$. Then three cases have to be considered.

- If $x \in [a_j, a_j] = \overline{\{a_j\}}$, then $f(x) \in f(\overline{\{a_j\}}) \subseteq \overline{\{a_j\}}$. Hence, $f(x) \in \overline{\{a_j\}} \setminus \{x\}$. Thus we get $p_j - 1$ possibilities for $f(x)$.

Now, $f(f(x)) \in [a_j, a_j] \setminus \{x, f(x)\}$ and $p_j - 2$ possibilities remain for $f^2(x)$. One may do the same thing for $f^3(x), \dots, f^{p_j-2}(x)$ and necessarily $f(f^{p_j-1}(x)) = x$, so we can define $(p_j - 1)!$ maps in this case.

- If $x \in (\overline{\{a_j\}})^{-1} \setminus \overline{\{a_j\}}$ with $(\overline{\{a_j\}})^{-1} = \{x \in X : \overline{\{x\}} \setminus \{x\} \subseteq \overline{\{a_j\}}\}$ then $f(x) \in \overline{\{a_j\}} = [a_j, a_j]$. We have seen in the proof of Theorem 2.13 that there are p_j possibilities for $f(x)$.
- If $x \in \mathcal{V}(a_j) \setminus (\overline{\{a_j\}})^{-1}$, then $x \notin \mathcal{X}$ and consequently we can define only one map in this case.

Therefore, we can define $(p_j - 1)! \times p_j^{q_j}$ maps for a fixed $j \in J_2$ and so we can define $\prod\limits_{j \in J_2} (p_j - 1)! \times p_j^{q_j}$ maps on $\bigcup\limits_{j=1}^{m} \mathcal{V}(a_j)$.

Since $\mathcal{X} \subseteq \bigcup\limits_{j=1}^{m} \mathcal{V}(a_j)$, then by Proposition 2.8 the number of restrictions of a map g in $\mathcal{F}$ on $\big(\bigcup\limits_{j=1}^{m} \mathcal{V}(a_j)\big)^c$ is equal to 1. Hence, we can define $\prod\limits_{j \in J_2} (p_j - 1)! \times p_j^{q_j}$ maps on X.

Now, one may easily check that $\prod\limits_{j \in J_1} (p_j - 1)! \times p_j^{q_j} = 1$. Indeed, let $a_j \in \Lambda_1$. Then two cases arise.

- If $a_j \in \Lambda_{p=1}$ then we have $(p_j - 1)! \times p_j^{q_j} = (1 - 1)! \times 1^{q_j} = 1$.
- If $a_j \in \mathcal{A}$ then we have $(p_j - 1)! \times p_j^{q_j} = (2 - 1)! \times 2^0 = 1$.

So we get

$$|\mathcal{F}| = \prod_{j \in J} (p_j - 1)! \times p_j^{q_j}.$$

$\square$

Examples 2.16. (1) Let (X, τ) be a topological space where τ is the discrete topology. Then we have $|\Lambda| = |X|$, $p_j = 1$ and $q_j = 0$ for each $j \in J$. Hence, we get

$$|\mathcal{F}| = \prod_{j \in J} (p_j - 1)! \times p_j^{q_j} = (1-1)! \times 1^0 = 1.$$

(2) Let (X, τ) be a topological space such that X is a finite set ($|X| = n$) and τ is the indiscrete topology. Then we have $X_p = X$, $|\Lambda| = 1$, $p_1 = |X| = n$ and $q_1 = 0$. Thus, we get

$$|\mathcal{F}| = \prod_{j \in J} (p_j - 1)! \times p_j^{q_j} = (n-1)! \times n^0 = (n-1)!.$$

(3) Let (X, τ) be a primal T_0-space.
If $\Lambda = \emptyset$, then $|\mathcal{F}| = 1$. If $\Lambda \neq \emptyset$, then $p_j = 1$ for each $a_j \in \Lambda$ (in this case q_j may be finite or infinite). So, we get

$$|\mathcal{F}| = \prod_{j \in J} (p_j - 1)! \times p_j^{q_j} = (1-1)! \times 1^{q_j} = 1.$$

(4) Let (X, τ) be a topological space such that $X = \mathbb{N}_0$ and τ the topology generated by $\{\{2n, 2n+1\} : n \in \mathbb{N}_0\}$. Then we have $X_p = X$, $\Lambda = \{2n : n \in \mathbb{N}_0\}$, $p_j = 2$ and $q_j = 0$ for each $j \in J$. Hence, we get

$$|\mathcal{F}| = \prod_{j \in J} (p_j - 1)! \times p_j^{q_j} = (2-1)! \times 2^0 = 1.$$

(5) Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then we have $X_p = \{b, c\}$, $|\Lambda| = 1$, $p_1 = 2$ and $q_1 = 1$. Thus, we get

$$|\mathcal{F}| = \prod_{j \in J} (p_j - 1)! \times p_j^{q_j} = (2-1)! \times 2^1 = 2.$$

Indeed, the two maps f_1 and f_2 satisfying $\mathcal{P}(f_1) = \mathcal{P}(f_2) = \tau$ are:
$f_1 : X \to X$ defined by $f_1(a) = b$, $f_1(b) = c$ and $f_1(c) = b$.
$f_2 : X \to X$ defined by $f_2(a) = c$, $f_2(b) = c$ and $f_2(c) = b$.

(6) Let (X, τ) be a topological space such that $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then we have $X_p = \{c, d\}$, $|\Lambda| = 1$, $p_1 = 2$ and $q_1 = 2$. So, we get

$$|\mathcal{F}| = \prod_{j \in J} (p_j - 1)! \times p_j^{q_j} = (2-1)! \times 2^2 = 4.$$

Indeed, the four maps f_1, f_2, f_3 and f_4 satisfying $\mathcal{P}(f_1) = \mathcal{P}(f_2) = \mathcal{P}(f_3) = \mathcal{P}(f_4) = \tau$ are:

$f_1 : X \to X$ defined by $f_1(a) = c$, $f_1(b) = c$, $f_1(c) = d$ and $f_1(d) = c$.

$f_2 : X \to X$ defined by $f_2(a) = d$, $f_2(b) = d$, $f_2(c) = d$ and $f_2(d) = c$.

$f_3 : X \to X$ defined by $f_3(a) = c$, $f_3(b) = d$, $f_3(c) = d$ and $f_3(d) = c$.

$f_4 : X \to X$ defined by $f_4(a) = d$, $f_4(b) = c$, $f_4(c) = d$ and $f_4(d) = c$.

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