Instant Insanity II

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Instant Insanity II is a sliding mechanical puzzle developed by Phil Orbanes (see Figure 1). It involves aligning four rows of colors by sliding the tiles as in the popular 15 puzzle (Figure 2), and by mechanical rotations reminiscent of Rubik's Cube. It resembles the 1967 Parker Brothers puzzle Instant Insanity. Versions of the latter puzzle appeared over a century ago, and the graph theoretic solution is widely known [1, 2, 6, 9].

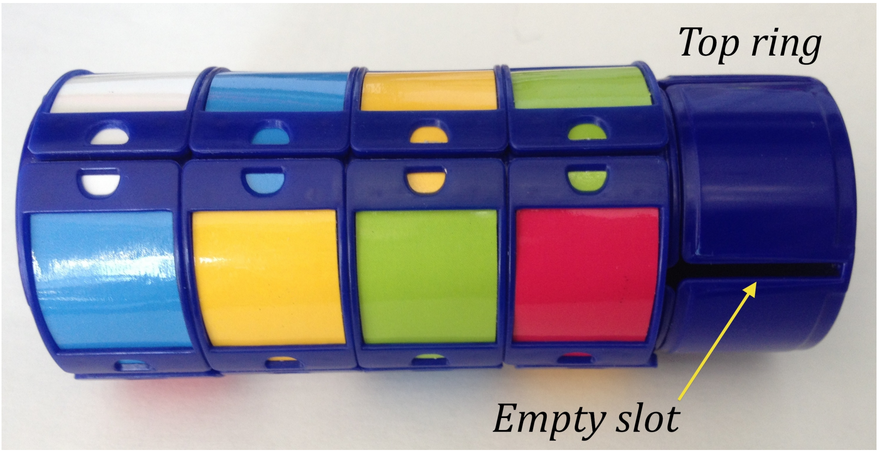


Figure 1: Instant Insanity II.

To appear in *The College Mathematics Journal*, Vol. 44, No. 4, Sept. 2013.
http://dx.doi.org/10.4169/college.math.j.44.4.265 MSC: 05C57, 00A08

Figure 2: The 15 puzzle.

Instant Insanity II consists of 4 *linear* columns and 4 *circular* rows of colored tiles on a cylinder. Above the 4 rows of tiles, at the top of the cylinder, there is an extra row, which we call the *top ring*, with two adjacent empty slots. The bottom row of the cylinder and the top ring, with its two slots for tiles, rotate. Each tile may slide up or down its column, alone or with other tiles of the column, if the column contains an empty slot. The 16 tiles are of five colors: four blue tiles, and three each of red, green, yellow, and white tiles. Instant Insanity II's *classic challenge* is to configure the tiles so that each row and column contains four different colors. The *ultimate challenge* involves a further aspect of the puzzle. Following a circular path around each row, each tile has, at each juncture, a small semicircular "halfmoon" of color. Positioned with the top ring on top, the halfmoon on the right of each tile has the same color as the tile, while the left halfmoon has a different color. Among the tiles of a given color, the nonmatching halfmoons have distinct colors. The ultimate challenge is to find a classical solution in which the pair of semicircles at the junctures match.

The Instant Insanity II box proclaims that there are 24 solutions to the ultimate challenge. We show that the $4! = 24$ permutations of the rows of any solution are also attainable solutions, so the box suggests that all solutions are permutations of a single solution. We show that there are, in fact, 48 solutions, arising as row permutations of two fundamentally different ultimate solutions. We also count the number of classical solutions and give an algorithm for producing them.

Every permutation of the tiles can be obtained

Even a configuration of the tiles obtained by breaking the tiles off and gluing them back on is actually possible by legal moves of the puzzle. To emphasize the significance of this, we mention that for another sliding puzzle, the famous 15 puzzle, the obtainable configurations correspond in a natural way to the even permutations in the symmetric group S_{15} [5, 8, 10]. Thus, only half the potential configurations of the 15 puzzle are obtainable.

We first show (i) that any two tiles on the top row may be transposed, and (ii) any vertically adjacent tiles in a single column may be transposed.

To show (i), let A and B be any tiles in the top row. Slide A and B to the empty slots on the top ring, rotating the top ring as necessary so the empty slots align with A and B. Then rotate the top ring and drop tiles B and A into transposed positions.

To show (ii), let A and B be vertically adjacent tiles. We first assume A is in the first row. We raise A to the top ring and rotate it, then raise B to the top ring. Now we rotate the top ring and drop A and then B into their positions. If A is not in the top row, more work is required. Figures 3 and 4 show sequences of moves to transpose A and B if A is in the second or third row. In these figures the rows of the matrices are the circular rings on the puzzle, and the arrows indicate the appropriate moves.

Using (i) and (ii) we now show that any tile A in the top row may be transposed with any other tile B: If B is not in the top row, we repeatedly apply (ii) until B rises to the top. If A and B are not in the same column, we then apply (i) to transpose A and B. In either case, A may now be transposed with the tiles below it by applying (ii) until A descends to the original position of B. Now, if any top row tile A may be transposed with any other tile B, then any two arbitrary tiles B and C may be transposed by picking any top row tile A and transposing A and B, B and C, and then A and C.

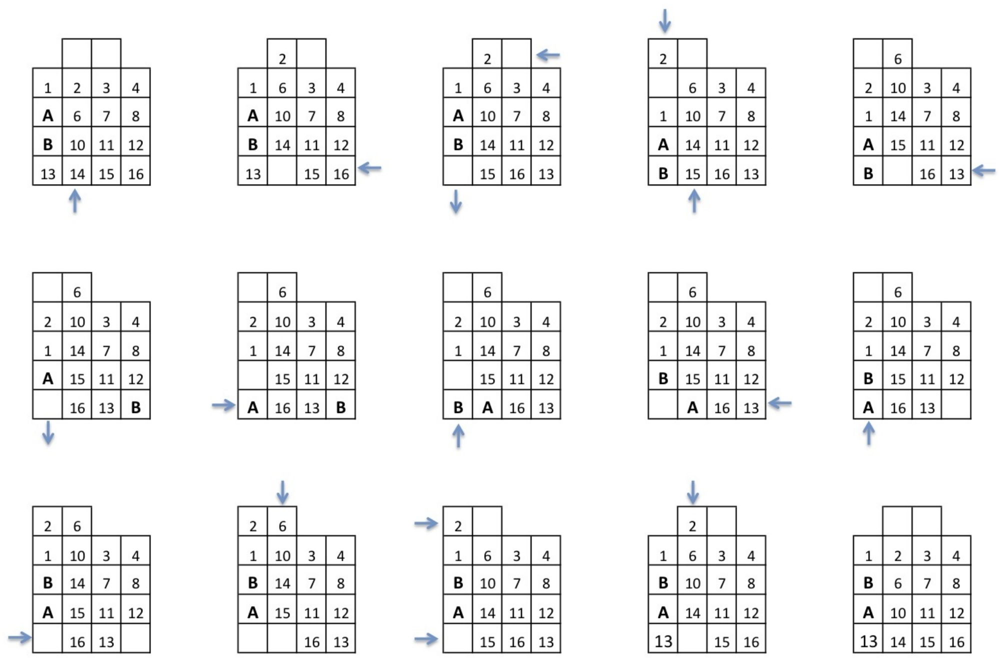


Figure 3: Transposing tiles in the second and third rows.

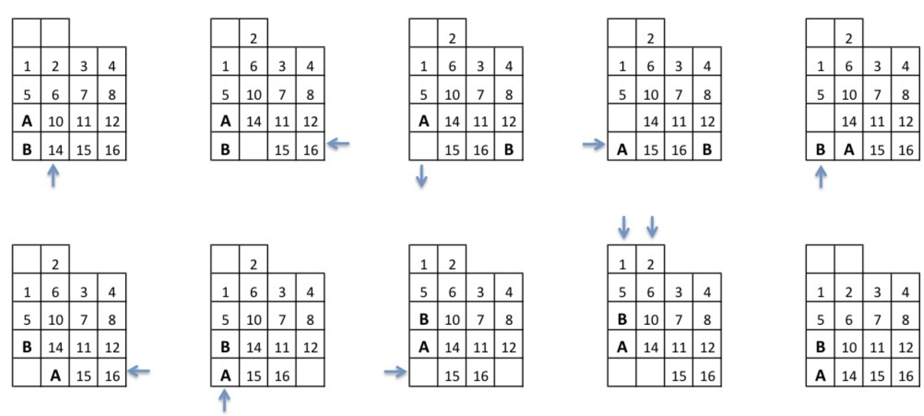


Figure 4: Transposing tiles in the third and fourth rows.

Counting classic solutions

Now that we know every possible permutation of tiles is achievable, let us count those that are solutions. A solution is a tile configuration with the proper color configuration. To count all the solutions, we count the number of proper color configurations, then count the number of ways the tiles of a given color may be permuted within their positions in a color configuration. However, we do not distinguish between permutations which are quarter rotations of the cylinder, since these may be realized without moving tiles.

In a solution, each row and column contains four different colors. For the classic challenge, ignoring the semicircular halfmoons, we need only know that there are 4 blue, 3 red, 3 green, 3 yellow, and 3 white tiles. Because there are 4 blue tiles, each of the 4 rows and columns of a solution must contain a blue tile, plus 3 tiles of other colors. Suppose a row contains, say, blue, red, green, and yellow tiles. No other row can have these four colors, for then these two rows would account for 2 blue, 2 red, 2 green, and 2 yellow tiles, leaving 2 blue, 1 red, 1 green, 1 yellow, and 3 white tiles for the remaining two rows. But it is impossible to put 3 white tiles in two rows which have all distinct colors. By this reasoning, each row and column of a solution includes one blue tile, and no two rows and no two columns have the same set of colors.

Figure 5(a) shows one classic solution color configuration. To count the number of such configurations, it helps to think outside the 4×4 box. The 4 rows of a classic solution uniquely determine a fifth column which records the excluded color of each row. This fifth column lists each of the four non-blue colors exactly once. A similar analysis applies to the columns. The result is shown in Figure 5(b). In Figure 5(c), we extend the information in Figure 5(b) to a 5×5 array by filling the vacant corner with blue, resulting in a 5×5 array in which each row and each column contains each of the 5 colors exactly once. The result is a 5×5 Latin square.

B	R	G	W
W	G	Y	B
G	B	R	Y
Y	W	B	R

(a)

B	R	G	W	~Y
W	G	Y	B	~R
G	B	R	Y	~W
Y	W	B	R	~G
~R	~Y	~W	~G	

(b)

B	R	G	W	Y
W	G	Y	B	R
G	B	R	Y	W
Y	W	B	R	G
R	Y	W	G	B

(c)

Figure 5: The correspondence between classic solutions and Latin squares.

These squares have been widely studied (see [3, 4, 7]), and it is known that there are 161,280 of them. Of this number, one fifth have B in the lower right corner. Thus, the number of 4×4 rectangular color configurations for classic solutions is $161,280/5 = 32,256$.

For each of the $32,256$ color configurations, we must also count the number of ways the four blue, three red, three green, three yellow, and three white tiles may be permuted within the positions determined for these colors by the Latin square. This gives $4!3!3!3!3!(32,256)$ rectangular solutions. However, because we do not distinguish between solutions which differ by a quarter rotation of the cylinder, we divide this by 4 to get the number of distinct solutions to the classic challenge, namely $3!3!3!3!3!(32,256) = 250,822,656$. For comparison, the 16 tiles may be arranged in a 4×4 grid in 16! ways, but discounting the quarter rotations of the cylinder, there are $16!/4 = 5,230,697,472,000$ possible configurations of tiles. The percentage of these configurations which are classic solutions is

$$\frac{250,822,656}{5,230,697,472,000} = \frac{6}{125,125} = 0.0000\overline{479520}.$$

We note that for any given classic solution, each of the 4! permutations of the rows or the 4! permutations of the columns, and thus each of the $4!^2 = 576$ permutations of rows and columns also is a solution. Thus, the 250,822,656 classic solutions may be grouped into equivalence classes of 576 solutions derived as row- and column-permutations of a single classic solution.

The ultimate challenge

Figure 6 shows the halfmoon and tile colors for the 16 distinct tiles, arranged in a solution to the ultimate challenge. Observe that column permutations (other than the identity) of an ultimate solution are not ultimate solutions, since the matching halfmoons would be separated.

For every tile, the halfmoon on the right side has the same color as the tile, and the halfmoon on the left tells the color of the tile which must precede that tile in an ultimate solution. For example, the red tile with a green halfmoon on the left must be preceded by a green tile. A directed graph $\mathcal{G}$ represents the possible adjacencies. The vertices of $\mathcal{G}$ are the five colors and we include a directed edge from color X to color Z if there is a halfmoon of color X on a tile of color Z. The 16 tiles provide 16 edges, and the graph is in Figure 7.

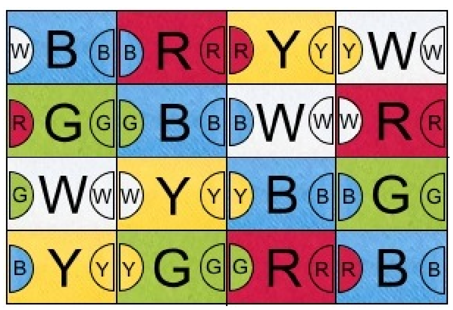


Figure 6: The sixteen Instant Insanity II tiles.

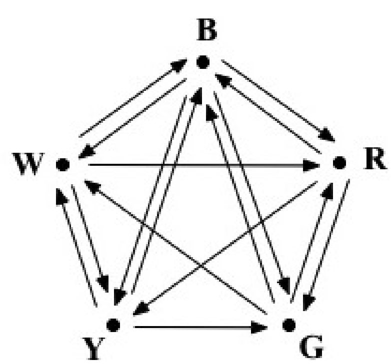


Figure 7: The halfmoon color → tile color graph $\mathcal{G}$.

Each row in a solution to the ultimate challenge corresponds to a circuit of $\mathcal{G}$ beginning at blue and containing four edges and four distinct vertices. The second row of the solution shown in Figure 6 corresponds to the circuit BWRG. From the discussion of color considerations for classic solutions, each of the four rows has a different missing color. By inspection of $\mathcal{G}$, we find 12 acceptable circuits. They are in Table 1, grouped by the colors used and numbered for convenience.

Table 1: Circuits that may be rows of an ultimate solution.

No Red	No Green	No Yellow	No White
(1) BWYG	(4) BYWR	(7) BGWR	(10) BYGR
(2) BGWY	(5) BWRY	(8) BWRG	(11) BRYG
(3) BYGW	(6) BRYW	(9) BRGW	(12) BGRY

Observe that circuit (1) BWYG contains the edge GB and thus uses the blue tile with the green halfmoon on the left. Since there is only one such tile in the puzzle, no other row of a solution using circuit (1) may use that tile, or equivalently, no other circuit of a solution may use the edge GB. Thus, circuit (1) is incompatible with circuits (8) and (11). From the data in Table 1, we compile all incompatible circuits in the incompatibility graph of Figure 8(a), and taking the graph complement, we obtain the compatibility graph of Figure 8(b). Because we must pick one circuit from each color group $\{(1), (2), (3)\}$, $\{(4), (5), (6)\}$, $\{(7), (8), (9)\}$, and $\{(10), (11), (12)\}$, we know that any two circuits from the same color group are incompatible even if they do not share a common tile. To simplify the representation of Figure 8(a), we have not shown the duplicate-tile incompatibilities between circuits of the same color group.

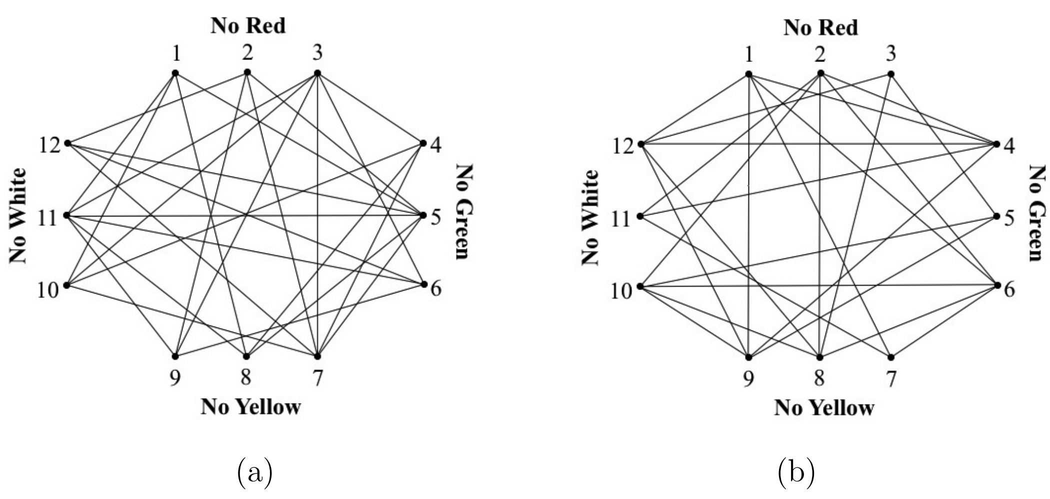


Figure 8: (a) The incompatibility graph. (b) The compatibility graph.

Now an ultimate solution consists of four compatible circuits, one from each color group. Thus, a solution must contain (1), (2), or (3) from the no-red color group.

If (1) is used, then (12) must be used, since (12) is the only no-white circuit compatible with (1). Then (4) is the only no-green circuit compatible with (1) and (12). Finally, (9) is the only no-yellow circuit compatible with (1), (12), and (4). So, the circuits (1), (4), (9), and (12) from Table 1 possibly form an ultimate solution.

Similarly, if (2) is used, then the only circuit of length 4 in the compatibility graph containing one vertex from each color group is (2), (6), (8), (10).

There are no solutions using (3) from the no-red color group since (5) and (8) are the only circuits from their respective color groups compatible with (3), but (5) and (8) are not compatible with each other.

Now the circuits (1),(4),(9), and (12) and the circuits (2),(6),(8), and (10) each correspond to four compatible rows which might produce an ultimate solution. If we place circuits (1), (4), (9), and (12) on the first, second, third, and fourth rows, respectively, then there is exactly one alignment of these cycles which gives distinct colors in each column. Similarly, if circuits (2), (6), (8), and (10) appear in that order from top to bottom, there is exactly one acceptable alignment. These alignments do correspond to ultimate solutions, and are shown in Table 2. Recall that the halfmoon on the right of each tile has the color of the tile so the halfmoon colors need not be shown in the solutions below.

Table 2: The two basic solutions to the ultimate challenge.

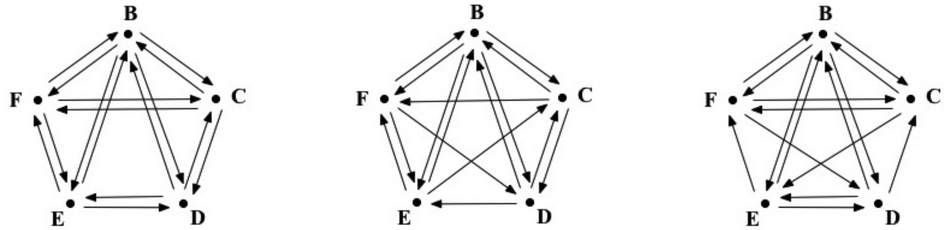
(1)	B	W	Y	G	(2)	B	G	W	Y
(4)	W	R	B	Y	(6)	Y	W	B	R
(9)	R	G	W	B	(8)	W	R	G	B
(12)	Y	B	G	R	(10)	R	B	Y	G

While these are essentially the only distinct solutions of the ultimate challenge, the order of the rows of each may be permuted in $4! = 24$ ways, giving rise to 48 solutions. Again, we do not distinguish between solutions obtained by rotating the puzzle through quarter turns. We also note that since the two solutions of Table 2 use different no-red circuits, their row permutations will be disjoint. Observe that the solution of Figure 6 shows the circuits (6), (8), (2), and (10) from top to bottom.

Seeking uniqueness

If we require 16 unique tiles, four with main color blue and three tiles of each of the four remaining colors, can an Instant Insanity II puzzle be produced with a *unique* ultimate solution, up to the 24 row permutations? Interestingly, the answer is "No." In the graph $\mathcal{G}$ of such a puzzle, a solution would use four circuits containing the vertex B, with each circuit omitting a different non-blue vertex. We may arrange this graph so that the first circuit is BDEF, omitting C, where CDEF is some permutation of RGYW. Now there must be a circuit which omits D and does not repeat any edge in BDEF. The three possibilities for such a circuit are BCFE, BFEC, and BFCE. In each case, we would need two more circuits, one omitting E and one omitting F, which do not repeat any existing edges. The resulting possible circuits are shown in Figure 9. (We note that two of these corresponded to the no-D circuit BCFE, and none corresponds to BFEC.)

In Figure 9(a), the only circuits excluding C, D, E, and F, respectively, are those listed and the same circuits traced in the opposite direction. The only combinations of these which are compatible are the four given in Figure 9(a), all traced in the order listed, and the same four all traced in the opposite direction. However, it is easily checked that these cycles (all in the forward order, or all in the reverse order) cannot be arranged into a solution.



BDEF	BDEF	BDEF
BCFE	BCFE	BFCE
BFCD	BFDC	BCFD
BEDC	BECD	BEDC
(a)	(b)	(c)

Figure 9: Possible halfmoon color $\rightarrow$ tile color graphs.

Now the graphs of Figure 9(b) and 9(c), with (C,D,E,F) = (W,Y,G,R) and (C,D,E,F) = (R,W,Y,G), respectively, are identical to Figure 7, which we know leads to two basic solutions. Thus, no such puzzle with 16 distinct tiles, four blue tiles, and three each of four other colors has a unique solution.

Summary. *Instant Insanity II* is a sliding mechanical puzzle whose solution requires the special alignment of 16 colored tiles. We count the number of solutions of the puzzle's classic challenge and show that the more difficult ultimate challenge has, up to row permutation, exactly two solutions, and further show that no similarly-constructed puzzle can have a unique ultimate solution.

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for your Mathematical Plays*, Vol. 4, 2nd Ed., A K Peters Ltd., Wellesley, MA, 2004.

[2] G. Chartrand, *Introductory Graph Theory*, Dover Publications, New York, 1985.

[3] J. Dénes and A. D. Keedwell, *Latin Squares and their Applications*, Academic Press, New York, 1974.

[4] ———, *Latin squares: New developments in the theory and applications*, Annals of Discrete Mathematics. 46, Academic Press, Amsterdam, 1991.

[5] W. W. Johnson and W. E. Story, Notes on the 15 Puzzle, *American Journal of Mathematics* **2** (4), (1879), 397404, doi:10.2307/2369492

[6] R. Johnsonbaugh, *Discrete Mathematics*, 7th Edition, Pearson Education, 2009.

[7] C. F. Laywine and G. L. Mullen, *Discrete Mathematics using Latin squares*, John Wiley & Sons, Inc.. New York, 1998.

[8] A. Levitin and M. Levitin, *Algorithmic Puzzles*, Oxford University Press, Oxford, 2011.

[9] I. Peterson, Averting Instant Insanity, *Ivars Peterson's MathTrek*, Mathematical Association of America, August 9, 1999, `http://www.maa.org/mathland/mathtrek_8_9_99.html`.

[10] J. Slocum and D. Sonneveld, *The 15 Puzzle Book*, Slocum Puzzle Foundation, Beverly Hills, CA, 2006.