THE AUTOHOMEOMORPHISM GROUP OF CONNECTED HOMOGENEOUS FUNCTIONALLY ALEXANDROFF SPACES

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Abstract. If $f : X \to X$ is a map, then the family $\{A \subseteq X : f(A) \subseteq A\}$ gives the closed sets of a topology $P(f)$ on X called a functionally Alexandroff topology. If $(X, P(f))$ is a connected homogeneous functionally Alexandroff space, we give an algebraic characterization and some basic properties of its autohomeomorphism group $\mathcal{H}(X)$.

Introduction

Alexandroff spaces were introduced in 1937 by P.S. Alexandroff in [1] (under the name Diskrete Räume "discrete space"). Alexandroff spaces are topological spaces in which arbitrary intersections of open sets are open. It is clear that a space is Alexandroff if and only if every point has a least neighborhood. Because every finite topology is Alexandroff, Alexandroff spaces are widely used in computer sciences and digital topology (see for instance [16, 14, 15]). For elementary properties of Alexandroff spaces we refer to [2, 3, 20].

Let X be a set and $f : X \to X$ be a function. We can define an Alexandroff topology $P(f)$ on X by taking $\{A \subseteq X : f(A) \subseteq A\}$ as the family of closed sets. This construction was first introduced by Shirazi and Golestani [22] in 2011 under the name *functional Alexandroff spaces*. Subsequently, many results and properties of such spaces have been given in [17, 12, 18]. Following the terminology of [17], in this paper, the topological space $(X, P(f))$ will be called a *functionally Alexandroff space*.

By the Alexandroff specialization theorem which characterizes an Alexandroff topology in terms of the quasi-order (preorder), the closure of $x \in (X, P(f))$ is the lower set $\downarrow\{x\} = \{f^n(x) : n \in \mathbb{N}\}$. The smallest open neighborhood of x, denoted by $V_f(x)$ is given by the corresponding upper set:

$$V_f(x) = \uparrow\{x\} = \{y \in X : \text{ there exists } n \in \mathbb{N} \text{ such that } f^n(y) = x\}.$$

A point $x \in (X, P(f))$ is called periodic of period $p \in \mathbb{N} - \{0\}$ if the points x, $f(x)$, $f^2(x)$, ..., $f^{p-1}(x)$ are distinct and $f^p(x) = x$, and the set $\{x, f(x), ..., f^{p-1}(x)\}$ is called a p-cycle of f.

A topological space Y is called homogeneous if for any points $a, b \in Y$ there exists a homeomorphism $\varphi : Y \to Y$ such that $\varphi(a) = b$. That is, homogeneous spaces are those spaces Y on which $Aut(Y)$ acts transitively.

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The following results from [17] give us a complete characterization of homogeneity of functionally Alexandroff spaces.

Theorem 0.1. *Let X be a set and f be a map from X to itself.*

1. *If f has a p-cycle, $(X, P(f))$ is homogeneous if and only if all elements are periodic with the same period p.*
2. *If f has no cycle, $(X, P(f))$ is homogeneous if and only if f is a κ-to-one function for some positive cardinal number κ.*

In the second case of the previous theorem, it is shown in [17] that the group $\mathcal{H}(X)$ of autohomeomorphisms of $(X, P(f))$ is exactly the subgroup of all permutations of X which commute with f.

Automorphisms of regular trees (rooted or not) have been investigated in many area of mathematics ranging from group theory to dynamical systems. For instance, authors like Serre, Tits, Burger, Trofimov, Znoiko, Möller, and Grigorchuk, among others, have investigated algebraic and topological properties of such groups. For more information see [21], [23], [9], [24], [26], [19] and [7].

We recall that if $\mathcal{T}$ is a locally finite tree with vertex set $V\mathcal{T}$, the automorphism group $Aut(\mathcal{T})$ of $\mathcal{T}$ can be equipped with a natural topology, called the permutation topology, which is defined by taking the collection $\{\mathcal{S}(x, y) : x, y \in V\mathcal{T}\}$ where

$$\mathcal{S}(x, y) = \{g \in Aut(\mathcal{T}) : g(x) = y\}$$

as a sub-basis for the open sets. It is well known that this topology renders $Aut(\mathcal{T})$ as a totally disconnected, locally compact and second countable (t.d.l.c.s.c) Hausdorff group. For more information about t.d.l.c. groups we refer to [25] and the proceedings [11].

Our intention in this paper is to describe the algebraic structure of the group $\mathcal{H}(X)$ for a connected homogeneous functionally Alexandroff space $(X, P(f))$ as a closed subgroup of $Aut(\mathcal{T})$ for a certain regular tree. For a further study about compactly generated, locally compact subgroups of $Aut(\mathcal{T})$ which satisfy Tits' independence property (P) we refer to [10].

In the first section, we present some preliminary terminology needed for the later development. The second section is devoted to the algebraic characterization and properties of $\mathcal{H}(X)$ and its major subgroups.

1. Preliminaries

Let $(X, P(f))$ be a functionally Alexandroff space. By [17, Proposition 2.3] $(X, P(f))$ is locally connected and the component of any $a \in X$ is given by $C_a = \bigcup_{n\geq 0} V_f(f^n(a))$.

Let I be an indexing set of the family $\{C_i : i \in I\}$ of components of X and let $f_i = f_{|C_i} : C_i \to C_i$ be the well defined restriction of f to C_i. Then, $X = \coprod_{i\in I} C_i$ and the topology $P(f)$ is the disjoint union topology, i.e. the space $(X, P(f))$ is homeomorphic to the disjoint union of its components $(C_i, P(f_i))$. In particular if $(X, P(f))$ is homogeneous, then all components are homeomorphic and consequently $X = \coprod_{i\in I} C$ where $(C, P(f_{|C}))$ is homogeneous.

Now for an autohomeomorphism on X, the homeomorphic components $(C_i = C, i \in I)$ of X can be permuted among themselves and then we can apply an autohomeomorphism φ_i for each C_i. Thus, the autohomeomorphism group $\mathcal{H}(X)$ consists of pairs of local swirls and global swirls. Global swirls are permutations of the set I and local swirls are autohomeomorphisms of C. As known for the case of the group of repeated graphs, $\mathcal{H}(X)$ is in fact the unrestricted wreath product:

$$\mathcal{H}(X) = \mathcal{H}(C) \wr \mathcal{S}_I \tag{1}$$

where $\mathcal{S}_I$ denotes the symmetric group of the set I.

Let us briefly recall the unrestricted wreath product of a given group A by $\mathcal{S}_I$. Let $A^I = \prod_{i \in I} A$ be the external direct product of $|I|$ copies of A. Hence, each $\varphi \in \mathcal{S}_I$ defines a natural automorphism by permuting the coordinate positions of any $(a_i)_I \in A^I$. That is, the i-th coordinate of $(a_i)_I$ becomes the $\varphi(i)$-th coordinate of $(a_i)_I^{\varphi}$. So, the wreath product is the semidirect product of A^I and $\mathcal{S}_I$ with respect to φ

$$A \wr \mathcal{S}_I = A^I \rtimes_{\varphi} \mathcal{S}_I,$$

where the multiplication is defined by $((a_i)_I, \varphi_1).((a'_i)_I, \varphi_2) = ((a_i)_I.(a'_i)_I^{\varphi_1}, \varphi_1 \circ \varphi_2)$ and $(a'_i)_I^{\varphi_1} = (a'_{\varphi_1^{-1}(i)})_I$.

Remark 1.1. For the remainder of this paper, we assume that the homogeneous space $(X, P(f))$ is connected.

1.1. **Graphical notations and definitions.** The point of view that we adopt about groups acting on a tree is what Serre calls "action without inversion" which corresponds to morphisms that preserve a given orientation. Let us briefly recall Serre's notations for graphs and trees [21].

A graph X is a tuple $(VX, EX, o, t, ^-)$ consisting of a nonempty set of vertices VX, a set of edges EX and three mappings $o : EX \to VX$ (the origin), $t : EX \to VX$ (the terminus) and $^- : EX \to EX$ (the inverse) such that for every $e \in EX$ we have: $\overline{\overline{e}} = e$, $\overline{e} \neq e$ and $o(e) = t(\overline{e})$. A graph is locally finite if every vertex has a finite set of adjacent vertices.

A tree $\mathcal{T}$ is a connected graph without circuits. A geodesic $[uv]$ connecting vertices u and v is the unique reduced path from u to v. Hence, we define a graph metric on $\mathcal{T}$ by

$$d_{\mathcal{T}}(u, v) = \text{length of } [uv].$$

For $v \in V\mathcal{T}$ and $n \geq 0$, the closed n-ball around v is defined by:

$$B_n(v) = \{w \in V\mathcal{T} : d_{\mathcal{T}}(v, w) \leq n\}$$

An infinite reduced path $\gamma = (e_1, e_2, ...)$ in $\mathcal{T}$ is called a $\mathcal{T}$-ray. We can define an equivalence relation $\sim$ on $\mathcal{T}$-rays by declaring $\gamma \sim \gamma' \iff \gamma \cap \gamma'$ is a $\mathcal{T}$-ray. The equivalence class of a $\mathcal{T}$-ray is called an end of the tree $\mathcal{T}$.

Let G be a group. A tree $\mathcal{T}$ is called a G-tree if G acts on $\mathcal{T}$ by automorphisms on $V\mathcal{T}$ and $E\mathcal{T}$ without inversion of edges (i.e. $g.o(e) = o(g.e)$, $g.\overline{e} = \overline{g.e}$ and $g.e \neq \overline{e}$ for all $e \in E\mathcal{T}$ and $g \in G$).

For an automorphism g of a G-tree $\mathcal{T}$, we define $|g| = \min\{d_{\mathcal{T}}(u, g.u) : u \in V\mathcal{T}\}$. The automorphism g is called elliptic (or a rotation) if $|g| = 0$ (i.e. g fixes some vertex of $\mathcal{T}$); otherwise ($|g| > 0$) g is called hyperbolic (or a translation).

In our case, the graphical representation of the space $(X, P(f))$ will consist of the set X as the set of vertices, $EX = \{(x, f(x)), (f(x), x) : x \in X \text{ and } f(x) \neq x\}$ as the set of edges. From each pair of inverse edges $(e, \overline{e})$, we make the following choice of the orientations: the set of positively oriented edges is $EX^+ = \{e \in EX : t(e) = f(o(e))\}$ and the negatively oriented edges is $EX^- = \overline{EX^+} = \{e \in EX : o(e) = f(t(e))\}$. The vertex $f(x)$ is represented below x.

Since $(X, P(f))$ is connected and f is without cycles of length ≥ 2, the graph representation of $(X, P(f))$ is in fact a tree and if in addition f is κ-to-one, the tree is a $(\kappa+1)$-regular tree where each vertex v has κ incoming and 1 outgoing edges such that for each incoming edge e, we have $v = t(e)$ and for the outgoing edge e, we have $v = o(e)$. In the sequel, $\mathcal{T}_f^\kappa$ denotes the described tree of a connected homogenous space $(X, P(f))$ where f is a cycle-free, κ-to-one function. We note that if κ is countable, then $V\mathcal{T}_f^\kappa$ is countably infinite and $\mathcal{T}_f^\kappa$ is locally finite if and only if κ is finite. We retain the usual notation $\mathcal{T}^\kappa$ for the infinite complete rooted κ-ary tree (where each node has exactly κ children). The tree in Fig. 1 represents $\mathcal{T}_f^2$.

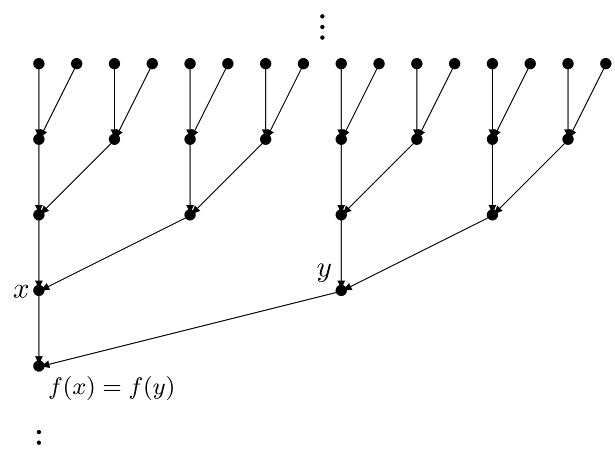

FIGURE 1. The tree $\mathcal{T}_f^2$ of a connected homogeneous functionally Alexandroff space where f is 2-to-one

Applying Theorem 0.1, we deduce the following:

Proposition 1.2. *If f is κ-to-one and has no cycles then, $\mathcal{T}_f^\kappa$ is a $\mathcal{H}(X)$-tree.*

Proof. It is clear that any $q \in \mathcal{H}(X)$ is an automorphism which is compatible with the structure of the tree $\mathcal{T}_f^\kappa$. If q is an inversion then, for some $x \in X$ we get $q(x) = f(x)$ and $q(f(x)) = x$. Since q commutes with f then $f^2(x) = x$, which is impossible. □

1.2. **Bass-Serre theory of groups acting on trees.**

In this section, we briefly recall the theory of Bass-Serre [21] which gives a complete characterization in terms of amalgamated products and HNN extensions of groups acting without inversions on trees. If G is a group acting on a tree X without inversions, we can associate the graph $(\mathcal{G}, Y)$ of groups (for more information we refer to [6, Ch. 7, Definition 11]). The graph Y is the factor (quotient) graph $G\backslash X$ defined as the graph with set of vertices $\{\mathcal{O}(v) : v \in VX\}$ and the set of edges $\{\mathcal{O}(e) : e \in EX\}$ where $\mathcal{O}(v)$ (resp. $\mathcal{O}(e)$) denotes the orbit of v (resp. e) with respect to this action, provided that:

(1) $\mathcal{O}(v)$ is the origin of $\mathcal{O}(e)$ if $\exists g \in G$ such that $g(v)$ is the origin of e.
(2) $\overline{\mathcal{O}(e)} = \mathcal{O}(\bar{e})$.

The vertex groups and edges groups of the the graph $(\mathcal{G}, Y)$ are the stabilizers in G of selected set of edges and vertices of X with $\mathcal{G}_e = \mathcal{G}_{\overline{e}}$ and a monomorphism $\varphi_e : \mathcal{G}_e \to \mathcal{G}_{o(e)}$ for every $e \in EY$.

The fundamental group $\pi_1(\mathcal{G}, Y, T)$ is constructed from the vertex groups and edges groups of $(\mathcal{G}, Y)$ with respect to a maximal subtree T in terms of amalgamated products and HNN extensions. This yields to the structure theorem of groups acting (without inversions) on trees:

$$G \approxeq \pi_1(\mathcal{G}, Y, T).$$

For instance, let suppose that G acts without inversions on a tree X such that the factor graph is a loop (see Fig. 2), where $e = (P, Q)$ is an edge and $G_P, G_e = G_{\overline{e}}$ are the stabilizers of P and e in G (i.e. $G_e = G_P \bigcap G_Q$). Choose $h \in G$ such that $h(P) = Q$. Let $G'_e = h^{-1} G_e h$ and let $v : G_e \to G'_e$ denote the corresponding isomorphism induced by conjugation by h. G_e and G'_e are isomorphic subgroups of G_P. By the Bass-Serre structure theorem, the homomorphism:

$$\Phi : \left\langle G_P, t \mid t^{-1} a t = v(a),\ a \in G_e \right\rangle \to G,$$

where Φ is the identity on G_P and $\Phi(t) = h$, is an isomorphism.

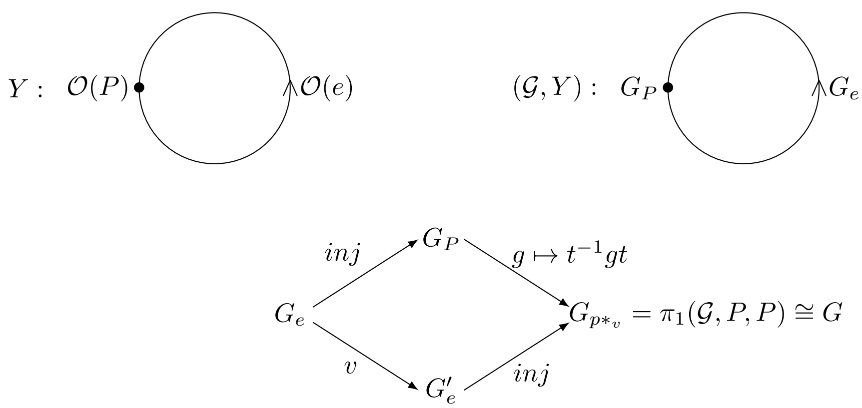

FIGURE 2. A loop of groups

2. The algebraic description and properties of $\mathcal{H}(X)$

For each $x \in X$ we denote by $\mathbb{S}t(x)$ the stabilizer of x, which consist of autohomeomorphisms $g \in \mathcal{H}(X)$ such that $g(x) = x$. By Proposition 1.2 the group $\mathcal{H}(X)$ acts without inversions on the tree $\mathcal{T}_f^{\kappa}$ and we can see, due to the homogeneity of $(X, P(f))$, that the factor graph consists of a single loop. Let $e = (P, f(P))$ be an edge in $\mathcal{T}_f^{\kappa}$ and choose $h \in \mathcal{H}(X)$ such that $h(P) = f(P)$. For each $g \in \mathbb{S}t(P)$ we have $g(f(P)) = f(g(P)) = f(P)$, so g is a stabilizer of the edge e. Thus, $\mathbb{S}t(P) = \mathbb{S}t(e)$. Hence, the following proposition gives the characterization of $\mathcal{H}(X)$.

Proposition 2.1.

$$\begin{aligned} \mathcal{H}(X) &\approxeq HNN(\mathbb{S}t(P), \mathbb{S}t(P), v) \\ &\approxeq \left\langle \mathbb{S}t(P), t \mid t^{-1} g t = h^{-1} g h,\ g \in \mathbb{S}t(P) \right\rangle. \end{aligned}$$

Remark 2.2. By Britton's lemma [8, Ch.2, Sec.14], each element $\varphi \in \mathcal{H}(X)$ has a unique representation $(g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, ..., t^{\epsilon_n}, g_n)$ in a normal form. For this, we need to select coset representatives for $A := \mathbb{S}t(P)$ in $\mathbb{S}t(P)$ and for $B := v(\mathbb{S}t(P))$ in $\mathbb{S}t(P)$. For A, the system of representatives of cosets consists of the identity element. For B, let $f^{-1}(\{P\}) = \{y_i : i \in I\}$ where $|I| = \kappa$ and fix some $x \in f^{-1}(\{P\})$. We choose a system of coset representatives $\{k_i \in \mathcal{H}(X) : k_i(x) = y_i,\ i \in I\}$ for B. Hence, φ can be represented in a unique normal form

$$\varphi = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} ... t^{\epsilon_n} g_n$$

where

(1) $g_0 \in \mathbb{S}t(P)$ is arbitrary,
(2) if $\epsilon_i = -1$ then, $g_i = id$,
(3) if $\epsilon_i = 1$ then, $g_i \in \{k_i :\ i \in I\}$,
(4) there is no subsequence of the form t, id, t^{-1} or t^{-1}, id, t.

In order to give a presentation of $\mathcal{H}(X)$ as a semidirect product, we need the following lemma.

Lemma 2.3. *The smallest normal subgroup of $\mathcal{H}(X)$ which contains $\mathbb{S}t(P)$ is*

$$\langle \mathbb{S}t(P) \rangle^{\lhd}_{\mathcal{H}(X)} = \varinjlim \mathbb{S}t(f^n(P)) = \bigcup_{x \in X} \mathbb{S}t(x).$$

Proof. Let $\alpha = g^{-1}hg$ for some $h \in \mathbb{S}t(P)$ and $g \in \mathcal{H}(X)$. Then, there exist $y \in X$ such that $g(y) = P$ and consequently $\alpha(y) = y$ (i.e. $\alpha \in \mathbb{S}t(y)$). Pick $m, n \in \mathbb{N}$ such that $f^n(y) = f^m(P)$. Then, $\mathbb{S}t(y) \subseteq \mathbb{S}t(f^n(y)) = \mathbb{S}t(f^m(P))$.

Now if $\psi \in \langle \mathbb{S}t(P) \rangle^{\lhd}_{\mathcal{H}(X)}$, $\psi = \alpha_1 \alpha_2 ... \alpha_n$ and each $\alpha_i \in \mathbb{S}t(f^{m_i}(P))$. Hence $\psi \in \bigcup_{n \geq 0} \mathbb{S}t(f^n(P))$, so,

$$\langle \mathbb{S}t(P) \rangle^{\lhd}_{\mathcal{H}(X)} \subseteq \bigcup_{n \geq 0} \mathbb{S}t(f^n(P)).$$

Conversely, let $\psi \in \mathbb{S}t(f^n(P))$ and choose $g \in \mathcal{H}(X)$ such that $g^{-1}(P) = f^n(P)$. Then, $g\psi g^{-1}(P) = P$ so $\psi \in \langle \mathbb{S}t(P) \rangle^{\lhd}_{\mathcal{H}(X)}$. The family $\{\mathbb{S}t(f^n(P)) : n \in \mathbb{N}\}$ is a directed collection of subgroups, so the group $\bigcup_{n \geq 0} \mathbb{S}t(f^n(P))$ is in fact the direct limit $\varinjlim \mathbb{S}t(f^n(P))$ and we conclude that

$$\langle \mathbb{S}t(P) \rangle^{\lhd}_{\mathcal{H}(X)} = \varinjlim \mathbb{S}t(f^n(P)).$$

Finally, every elliptic autohomeomorphism must belong to some $\mathbb{S}t(f^n(P))$, so $\bigcup_{x \in X} \mathbb{S}t(x) = \varinjlim \mathbb{S}t(f^n(P))$ which is the smallest normal subgroup of $\mathcal{H}(X)$ containing $\mathbb{S}t(P)$. □

We denote by $E_{\mathcal{H}(X)} = \bigcup_{x \in X} \mathbb{S}t(x)$ the normal subgroup of all elliptic autohomeomorphisms of $\mathcal{H}(X)$. Now, we are in a position to deduce the following theorem.

Theorem 2.4. *Let P be a point of X and choose $h \in \mathcal{H}(X)$ such that $h(P) = f(P)$. Then*

$$\mathcal{H}(X) \approxeq E_{\mathcal{H}(X)} \rtimes \langle h \rangle .$$

Proof. By the Bass-Serre structure theory of groups acting on a tree [21, Ch.1, Sec.5] $\mathcal{H}(X)$ is none other than the semi-direct product of the infinite cyclic group $\langle h\rangle$ with the normal subgroup R generated by all the conjugates $h^n\mathbb{S}t(P)h^{-n}$ for $n\in\mathbb{Z}$. By the previous lemma, the normal subgroup R is indeed the group $E_{\mathcal{H}(X)}$. Then we conclude that $\mathcal{H}(X)$ is the semi-direct product of the group of elliptic autohomeomorphisms and the infinite cyclic group generated by some fixed hyperbolic autohomeomorphism. □

Corollary 2.5. If f is 1-to-one then

$$\mathcal{H}(X)\cong(\mathbb{Z},+).$$

Proof. The tree representation of the connected space $(X,P(f))$ is a bi-infinite line order-isomorphic to $\mathbb{Z}$, so $E_{\mathcal{H}(X)}=\{id\}$, and by Theorem 2.4

$$\mathcal{H}(X)=\{id\}\rtimes\langle h\rangle\cong\langle h\rangle\cong(\mathbb{Z},+).$$

□

Remark 2.6. Let us consider the case when the homogeneous space $(X,P(f))$ is not necessarily connected and f is a bijection. By (1) the group $\mathcal{H}(X)=\mathcal{H}(C)\wr\mathcal{S}_I$. Two cases arises:

- If f has a periodic point with period n, then by [17, Proposition 2.3] we have $X=\coprod_{i\in I}C_n$ where C_n is a cycle of length n, and thus $\mathcal{H}(X)=\mathcal{H}(C_n)\wr\mathcal{S}_I$. Since C_n is equipped with the indiscrete topology, then any permutation of the points of C_n belongs to $\mathcal{H}(C_n)$. Therefore:
$$\mathcal{H}(X)=\mathcal{S}_n\wr\mathcal{S}_I.$$
- If f is without periodic points, then:
$$\begin{aligned}\mathcal{H}(X)&\cong\mathbb{Z}\wr\mathcal{S}_I\\&\cong(\{(k,s):s\in\mathcal{S}_I\text{ and }k\in\mathbb{Z}^I\},\bullet)\end{aligned}$$
where the composition law $\bullet$ is defined by:
$$(k_1,s_1)\bullet(k_2,s_2)=(k_1+k_2\circ s_1^{-1},s_1\circ s_2).$$
The identity element is $(\overline{0},id_I)$ where $\overline{0}$ denote the null function. The inverse of (k,s) is $(-k\circ s,s^{-1})$.

We recall that a group G is said to be complete if it is centerless and every automorphism of G is inner. For $\kappa=1$, $\mathcal{H}(X)\cong(\mathbb{Z},+)$ and consequently is not complete. For any cardinal $\kappa>2$, we have the following result.

Proposition 2.7. *$\mathcal{H}(X)$ is complete for any cardinal number $\kappa>2$.*

Proof. First, it is clear that $\mathcal{H}(X)$ is centerless. Now, we prove that each automorphism is an inner automorphism.

Fix $\phi\in Aut(\mathcal{H}(X))$. By Proposition 2.1, $\phi(\mathcal{H}(X))$ is an HNN extension of the group $\phi(\mathbb{S}t(P))$ with the associated subgroups $\phi(\mathbb{S}t(P))$ and $\phi(h)^{-1}\phi(\mathbb{S}t(P))\phi(h)$. According to the relationship between trees and HNN extensions [8, Ch.2, Sec.15] there exists a tree $\mathcal{T}$ (which is isomorphic to the tree $\mathcal{T}_f^\kappa$) such that $\mathcal{H}(X)$ acts on $\mathcal{T}$ without inversion of edges. Moreover, there exists a segment $\widetilde{Y}$ in $\mathcal{T}$ with vertices $\phi(\mathbb{S}t(P))$ and $\phi(t)\phi(\mathbb{S}t(P))$. Let Q and M be vertices of $\mathcal{T}_f^\kappa$ such that

$$\begin{cases}\mathbb{S}t(Q)=\phi(\mathbb{S}t(P))\\ \mathbb{S}t(M)=\phi(t)\phi(\mathbb{S}t(P))\phi(t)^{-1}\end{cases}$$

where $\phi(\mathbb{S}t(P))$ is the stabilizer of the segment $\widetilde{Y}$. For each $g \in \mathbb{S}t(Q)$, g stabilizes the segment $\widetilde{Y}$ so, $g \in \mathbb{S}t(M)$ and consequently the point $M = f(Q)$.

We also have $\phi(t\mathbb{S}t(P)t^{-1}) = \phi(\mathbb{S}t(f(P)))$. So in summary, for any point $P \in X$, there exists $Q \in X$ such that

$$\begin{cases} \phi(\mathbb{S}t(P)) = \mathbb{S}t(Q) \\ \phi(\mathbb{S}t(f(P))) = \mathbb{S}t(f(Q)). \end{cases} \tag{2}$$

Since ϕ is an automorphism of $\mathcal{H}(X)$, one can check easily that ϕ induce a permutation of the set $\{\mathbb{S}t(x),\ x \in X\}$. Consequently, we can define a permutation $\overline{\phi} : X \to X$ by: $\forall x \in X$, $\overline{\phi}(x) = \overline{x}$ is defined by $\phi(\mathbb{S}t(x)) = \mathbb{S}t(\overline{x})$. By (2) we get $\phi(\mathbb{S}t(f(x))) = \mathbb{S}t(f(\overline{x}))$. Then, $\forall x \in X$, $\overline{\phi}(f(x)) = f(\overline{\phi}(x))$ and thus $\overline{\phi} \circ f = f \circ \overline{\phi}$, which means that $\overline{\phi} \in \mathcal{H}(X)$.

Now, for any $g \in \mathcal{H}(X)$ and $x \in X$, we have

$$\begin{cases} g\mathbb{S}t(x)g^{-1} = \mathbb{S}t(g(x)) \\ \phi(\mathbb{S}t(g(x))) = \mathbb{S}t(\overline{\phi} \circ g(x)). \end{cases} \tag{3}$$

Then, by replacing g in (3) by $\phi(g)$ and x by $\overline{\phi}(x)$ we get

$$\phi(g\mathbb{S}t(x)g^{-1}) = \phi(g)\mathbb{S}t(\overline{\phi}(x))\phi(g^{-1}) = \mathbb{S}t(\phi(g)(\overline{\phi}(x))).$$

By (2), we also have

$$\phi(g\mathbb{S}t(x)g^{-1}) = \phi(\mathbb{S}t(g(x))) = \mathbb{S}t(\overline{\phi}(g(x)).$$

We conclude that, for all $x \in X$, $\phi(g)(\overline{\phi}(x)) = \overline{\phi}(g(x))$. So, for all $g \in \mathcal{H}(X)$,

$$\phi(g) = \overline{\phi} \circ g \circ \overline{\phi}^{-1},$$

which completes the proof. □

Problem 2.8. If $\kappa = 2$, is $\mathcal{H}(X)$ complete?

2.1. The vertex stabilizer subgroups of $\mathcal{H}(X)$.

The pioneering work of Jacques Tits on automorphism groups of a tree [23] answers the question asked by J. P. Serre of whether the subgroup $Aut^+(\mathcal{T}_n)$ of $Aut(\mathcal{T}_n)$ generated by the vertex stabilizers of an n-regular tree is simple. More generally, Tits gives criteria for a group acting on a tree to be simple. This result is known as Tits' simplicity theorem. It is known that $Aut^+(\mathcal{T}_n)$ is a maximal, normal and simple subgroup of $Aut(\mathcal{T}_n)$ of index 2 (with $n \geq 3$). Furthermore, one may naturally split the set $V\mathcal{T}_n$ into two equivalence classes T_0 and T_1 where the equivalence relation is defined by $x \sim y \iff d_{\mathcal{T}_n}(x, y) \equiv 0\ (mod\ 2)$. Then, it is clear that the action of $Aut(\mathcal{T}_n)$ on $V\mathcal{T}_n$ is not primitive because T_0 is a non-trivial block of imprimitivity. However, the action of $Aut^+(\mathcal{T}_n)$ on T_0 is primitive [19].

We start by giving the following result.

Theorem 2.9. $\left|\mathcal{H}(X) : E_{\mathcal{H}(X)}\right| = \aleph_0$.

Proof. First, we recall that the conjugation of a rotation $g \in E_{\mathcal{H}(X)}$ with a translation $t \in \langle h \rangle$ (where h is the translation defined in Theorem 2.4) is also a rotation. Now two elements (g_1, t_1) and (g_2, t_2) are in the same coset of $E_{\mathcal{H}(X)}$ if:

$$((g_2^{-1})^{t_2^{-1}}(g_1)^{t_2^{-1}}, t_2^{-1}t_1) \in E_{\mathcal{H}(X)}$$

which is equivalent to say that $t_1 = t_2$. So, $\{(id, t),\ t \in \langle h \rangle\}$ is a family of representatives for the cosets of $E_{\mathcal{H}(X)}$ in $\mathcal{H}(X)$. Thus, $\left|\mathcal{H}(X) : E_{\mathcal{H}(X)}\right| = |\langle h \rangle| = \aleph_0$. □

Consequently, we deduce that $E_{\mathcal{H}(X)}$ is not maximal in $\mathcal{H}(X)$. Now, Depending on the cardinal number κ, we get the following result:

Theorem 2.10. *Let x be in X. Then*

$$\left|E_{\mathcal{H}(X)} : \mathbb{S}t(x)\right| = \begin{cases} \aleph_0 & \text{if } 2 \leq \kappa \leq \aleph_0 \\ \kappa & \text{if } \aleph_0 < \kappa. \end{cases}$$

Proof. Let y and z be two points in X. We say that y and z are in the same level in the tree $\mathcal{T}_f^\kappa$ if $\exists n \in \mathbb{N}$ such that $f^n(y) = f^n(z)$. Let us define a basic bi-infinite line

$$\mathcal{L} = \{..., v_{-2}, v_{-1}, v_0, v_1, v_2, ...\}$$

such that $v_{-n} = f^n(v_0)$ and $v_n \in f^{-1}(\{v_{n-1}\})$ $(\forall n \geq 1)$ where v_0 is an arbitrary point of X. Since $(X, P(f))$ is connected, then $X = \coprod_{i\in\mathbb{Z}} L_i$ where

$$L_i = \{y \in X : y \text{ and } v_i \text{ are in the same level}\}.$$

We denote by $L_{i(x)}$ the level containing x. Its not difficult to note that for any rotation g and for any point $y \in X$, we have y and $g(y)$ are in the same level. Now, for any $g_1, g_2 \in E_{\mathcal{H}(X)}$, one can see easily that: g_1, g_2 are in the same right coset of $\mathbb{S}t(x)$ if and only if $g_1(x) = g_2(x)$. For each y in $L_{i(x)}$ we associate a rotation $\varphi_y \in \mathbb{S}t(f(x))$ such that $\varphi_y(x) = y$. Then, the family $\left\{\varphi_y,\ y \in L_{i(x)}\right\}$ is a system of representatives for the right cosets of $\mathbb{S}t(x)$ in $E_{\mathcal{H}(X)}$. Thus:

$$\left|E_{\mathcal{H}(X)} : \mathbb{S}t(x)\right| = \left|L_{i(x)}\right| = \left|\bigcup_{n\geq 0} f^{-n}(\{f^n(x)\})\right|.$$

So, if $\kappa \leq \aleph_0$ then, $\left|E_{\mathcal{H}(X)} : \mathbb{S}t(x)\right| = \aleph_0$. If $\kappa > \aleph_0$ then, each set $f^{-n}(\{f^n(x)\})$ has cardinality κ and the the union of $\aleph_0$ sets of cardinality $\kappa > \aleph_0$ has cardinality κ [13, Ch.9]. Figure 3 gives a detailed illustration of the level partition of X. □

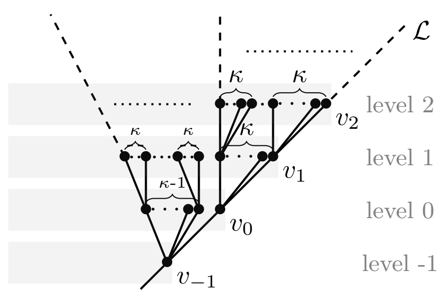

FIGURE 3. The level partition of X.

Remark 2.11. By the non-maximality of $\mathbb{S}t(x)$ in $E_{\mathcal{H}(X)}$ we deduce that the action of $E_{\mathcal{H}(X)}$ on any subset S of X is not primitive. In fact, a transitive action of a group G on a set S is primitive if and only if for each $s \in S$ the stabilizer G_s is a maximal subgroup of G. Since $\mathbb{S}t_{E_{\mathcal{H}(X)}}(s) \lneq \mathbb{S}t_{E_{\mathcal{H}(X)}}(f(s)) \lneq E_{\mathcal{H}(X)}$ the result follow.

Now we turn our attention to prove the simplicity of $E_{\mathcal{H}(X)}$. For this we need to recall the following definition.

Definition 2.12 (Tits' independence property (P))**.** Let G a group acting on a tree $\mathcal{T}$. Let C be a geodesic (finite or infinite) in $\mathcal{T}$. Let $\pi : \mathcal{T} \to C$ denote the projection to the unique closest point on C. So, for any $c \in C$, $\pi^{-1}(c)$ is a sub-tree which can be seen as a branch of $\mathcal{T}$ with root c. Let F be the subgroup of G which fixes all elements of C (i.e. $F = \{g \in G : \forall c \in C,\ g(c) = c\}$). It is clear that $\forall c \in C$, $\pi^{-1}(c)$ is invariant under the action of the group F. Then, for each $c \in C$ let F_c be the group of automorphisms of the tree $\pi^{-1}(c)$ obtained by restricting the action of F to the set $\pi^{-1}(c)$, i.e. $F_c = \left\{g_{|\pi^{-1}(c)} : g \in F\right\}$. Hence, we get a monomorphism

$$\varphi_C : F \hookrightarrow \prod_{c\in C} F_c.$$

That is, a group G acting on the tree $\mathcal{T}$ satisfies Tits' independence property (P) if for any geodesic C (finite or infinite) the morphism φ_C is an isomorphism. The point is that F acts independently on each branch $\pi^{-1}(c)$ of the geodesic C.

It is clear that $\mathcal{H}(X)$ as a subgroup of $Aut(\mathcal{T}_f^\kappa)$ does not stabilize any nonempty proper sub-tree of $\mathcal{T}_f^\kappa$ and does not fix any end of $\mathcal{T}_f^\kappa$. Thus, if $E_{\mathcal{H}(X)}$ is not trivial then by [23, Theorem 4.5] $E_{\mathcal{H}(X)}$ is simple whenever $\mathcal{H}(X)$ satisfies Tits' independence property (P). So, we get the following theorem.

Theorem 2.13.

(1) $E_{\mathcal{H}(X)}$ *is a simple group for* $\kappa \geq 2$.
(2) $E_{\mathcal{H}(X)} = \{id\}$ *when* $\kappa = 1$.

Proof. If $\kappa > 1$, then it is clear that $E_{\mathcal{H}(X)}$ is not trivial.

Now, if φ_C is an isomorphism (as defined above) for all edges C, then this defines the weaker independence property. It is known that the weaker independence property is equivalent to Tits' independence property (P) for closed subgroups of automorphisms of a tree (see [4] or [5] for a generalization). So, let us first show that $\mathcal{H}(X)$ is a closed subgroup of $Aut(\mathcal{T}_f^\kappa)$.

Let $\{g_n; n \in D, \geq\}$ be a net of elements of $\mathcal{H}(X)$ which converges to g. If $\mathcal{S}(x, g(x))$ is a neighborhood of g, then $\exists m \in D$ such that if $n \in D, n \geq m$, then $g_n \in \mathcal{S}(x, g(x))$. Similarly, $\mathcal{S}(f(x), g(f(x)))$ is a neighborhood of g and $\exists p \in D$ such that if $n \in D, n \geq p$, then $g_n \in \mathcal{S}(f(x), g(f(x)))$. But, there exist $q \in D$ such that $q \geq p, q \geq m$, so $g_q \in \mathcal{H}(X)$ and $g_q \in \mathcal{S}(x, g(x)) \bigcap \mathcal{S}(f(x), g(f(x)))$. Since $g_q \circ f(x) = f \circ g_q(x)$, we get $g \circ f(x) = f \circ g(x)$. This is true for any $x \in X$ so g commutes with f, which is equivalent to saying that $g \in \mathcal{H}(X)$.

Now, using the same terminology of the previous definition, it is not difficult to check that for any $(g_1, g_2) \in F_x \times F_{f(x)}$ the associated map $g \in \mathcal{S}_X$ defined by

$$\begin{cases} g_{|\pi^{-1}(x)} = g_1 \\ g_{|\pi^{-1}(f(x))} = g_2 \end{cases}$$

is a well defined permutation of X which commutes with f. Thus, $g \in \mathcal{H}(X)$ and consequently the monomorphism $\varphi_{(x,f(x))}$ is surjective. □

A group G is said to be Hopfian if every epimorphism from G to itself is an isomorphism, or equivalently, if G is not isomorphic to any of its proper quotients. G is said to be co-Hopfian if every monomorphism is an isomorphism, or equivalently, if G is not isomorphic to any of its proper subgroups.

For $\kappa > 1$, the following proposition gives a short look into the algebraic structure of the group $\mathbb{S}t(x)$.

Proposition 2.14. *If $\kappa > 1$ then, for each x in X, $\mathbb{S}t(x)$ is a non-Hopfian and non-Co-Hopfian group.*

Proof. Let $(x, f(x))$ be a geometric edge and let $\mathcal{T}_f^\kappa - \{(x, f(x)), (f(x), x)\}$ be the graph obtained by deleting the edges $(x, f(x))$ and $(f(x), x)$ from $\mathcal{T}_f^\kappa$. Let $\mathcal{T}_{(x,f(x))}$ be the component (tree) of $\mathcal{T}_f^\kappa - \{(x, f(x)), (f(x), x)\}$ which contains x and $\mathcal{T}_{(f(x),x)}$ be the other tree which contains $f(x)$.

In fact, $\mathcal{T}_{(x,f(x))}$ (resp. $\mathcal{T}_{(f(x),x)}$) can be viewed as rooted tree $\mathcal{T}^\kappa$ with root x (resp. $f(x)$). Hence, we get

$$\mathbb{S}t(x) \approxeq Aut(\mathcal{T}_{(x,f(x))}) \times Aut^*(\mathcal{T}_{(f(x),x)})$$

where $Aut^*(\mathcal{T}_{(f(x),x)})$ is the subgroup of $Aut(\mathcal{T}_{(f(x),x)})$ which fixes the infinite path $(f(x), f^2(x), ...f^n(x), ...)$. We can identify $Aut(\mathcal{T}_{(x,f(x))}) = Aut(\mathcal{T}^\kappa)$.

To simplify the expression, let $\mathcal{T}^{(1)}$ be the tree $\mathcal{T}_{(f(x),x)}$. Then

$$Aut^*(\mathcal{T}^{(1)}) = Aut(\mathcal{T}^{(1)}_{(f(x),f^2(x))}) \times Aut^*(\mathcal{T}^{(1)}_{(f^2(x),f(x))})$$

where $Aut^*(\mathcal{T}^{(1)}_{(f^2(x),f(x))})$ is the subgroup of $Aut(\mathcal{T}^{(1)}_{(f^2(x),f(x))})$ which fixes the infinite path $(f^2(x), f^3(x), ...f^n(x), ...)$. So, with similar notation as before, we get

$$Aut^*(\mathcal{T}^{(1)}) = Aut(\mathcal{T}^{(1)}_{(f(x),f^2(x))}) \times Aut^*(\mathcal{T}^{(2)}).$$

This decomposition can be inductively repeated for every $i \geq 1$. That is,

$$\begin{aligned} Aut^*(\mathcal{T}^{(i)}) &\approxeq Aut(\mathcal{T}^{(i)}_{(f^i(x),f^{i+1}(x))}) \times Aut^*(\mathcal{T}^{(i)}_{(f^{i+1}(x),f^i(x))}) \\ &\approxeq Aut(\mathcal{T}^{(i)}_{(f^i(x),f^{i+1}(x))}) \times Aut^*(\mathcal{T}^{(i+1)}) \end{aligned}$$

where $\mathcal{T}^{(i)}$ is the tree $\mathcal{T}^{(i-1)}_{(f^i(x),f^{i-1}(x))}$ and $Aut^*(\mathcal{T}^{(i)}_{(f^{i+1}(x),f^i(x))})$ is the subgroup of $Aut(\mathcal{T}^{(i)}_{(f^{i+1}(x),f^i(x))})$ which fixes the infinite path $(f^{i+1}(x), f^{i+2}(x), ...f^n(x), ...)$.

By homogeneity of $(X, P(f))$ the trees $\mathcal{T}^{(i)}_{f^i((x),f^{i+1}(x))}$ are isomorphic. In summary, we have constructed a surjective inverse system $\{Aut(\mathcal{T}^{(i)}_{f^i((x),f^{i+1}(x))}),\ id,\ \mathbb{N}^*\}$ such that $Aut^*(\mathcal{T}^{(1)})$ is its inverse limit:

$$Aut^*(\mathcal{T}^{(1)}) = \varprojlim Aut(\mathcal{T}^{(i)}_{f^i((x),f^{i+1}(x))}).$$

For simplicity, we consider two cases.

- If κ is finite, then the rooted tree $\mathcal{T}^{(i)}_{(f^i(x),f^{i+1}(x))}$ has root $f^i(x)$ with only $(\kappa - 1)$ children and each other vertex has κ children. So

$$Aut(\mathcal{T}^{(i)}_{(f^i(x),f^{i+1}(x))}) \approxeq Aut(\mathcal{T}^\kappa) \wr \mathcal{S}_{\kappa-1}$$

  and consequently,

$$\mathbb{S}t(x) \approxeq Aut(\mathcal{T}^\kappa) \times \prod_{\mathbb{N}} \left(Aut(\mathcal{T}^\kappa) \wr \mathcal{S}_{\kappa-1}\right).$$

- If κ is an infinite cardinal number, then $\left|f^{-1}(\{x\})\right| = \left|f^{-1}(\{f(x)\}) - \{x\}\right| = \kappa$. That is, each vertex in the trees $\mathcal{T}^{(i)}_{f^i((x),f^{i+1}(x))}$ has κ children. So, the trees $\mathcal{T}_{(x,f(x))}$ and $\mathcal{T}^{(i)}_{f^i((x),f^{i+1}(x))}$ are homeomorphic as subspaces of the space $(X, P(f))$ [17, Theorem. 2.7]. So

$$\mathbb{S}t(x) \approxeq \prod_{\mathbb{N}} Aut(\mathcal{T}^{\kappa}).$$

Hence, if κ is finite (resp. infinite) we can embedded $Aut(\mathcal{T}^{\kappa}) \wr \mathcal{S}_{\kappa-1}$ as a normal subgroup A of $\mathbb{S}t(x)$ (resp. $Aut(\mathcal{T}^{\kappa})$). Thus we get

$$\mathbb{S}t(x) \approxeq \mathbb{S}t(x) \times A$$

where A is a non trivial normal subgroup of $\mathbb{S}t(x)$.

Consequently, we deduce that $\mathbb{S}t(x)$ is a non-Hopfian and non-co-Hopfian group. $\square$

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