Order ***Preprint***
https://doi.org/10.1007/s11083-020-09523-6

The Lattice of Functional Alexandroff Topologies

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Received: 28 February 2019 / Accepted: 12 March 2020

Abstract If $f : X \to X$ is a function, the associated functional Alexandroff topology on X is the topology P_f whose closed sets are $\{A \subseteq X : f(A) \subseteq A\}$. We present a characterization of functional Alexandroff topologies on a finite set X and show that the collection $FA(X)$ of all functional Alexandroff topologies on a finite set X, ordered by inclusion, is a complemented lattice.

Keywords Lattice of topologies · Alexandroff topology · Complemented lattice

In 1937, Alexandroff [1] studied topologies whose closed sets also form a topology. Such topologies, in which arbitrary intersections of open sets are open, are now called *Alexandroff topologies*. Every topology on a finite set is clearly an Alexandroff topology. Topologies on finite sets such as the computer digits of real numbers like π or the "points" (pixels) in the plane have driven much of the modern usage of Alexandroff topologies in computer science. Alexandroff topologies defined by functions were studied in [6], and in 2011 and 2012, Ayatollah Zadeh Shirazi and Golestani [5] and Echi [9], working independently, explicitly introduced a class of functional Alexandroff topologies on X. If $f : X \to X$ is a function and $x \in X$, taking the closure of x to be the orbit $\{f^n(x) : n \geq 0\}$ of x gives a topology P_f on X. A topology $\mathcal{T}$ on X is *functional Alexandroff* if it is P_f for some $f : X \to X$. Since their recent introduction, functional Alexandroff topologies have been further investigated in [4], [10], [8], [11], [15], [16], [17].

The lattice structure of topologies on a set X has been studied for over 50 years. Much attention has been given to showing that every topology in the lattice $T(X)$ of topologies on X has at least one complement, and when a certain type of topology has a certain type of complement. Anne Steiner [23] first showed that $T(X)$ is a complemented lattice by showing that certain topologies have complements which are Alexandroff topologies. It is known that the collection $A(X)$ of Alexandroff topologies on a set X is a complemented lattice. Other proofs that $T(X)$ is complemented [12], [22], [26], results on the number of complements [7], [21], [27], and results on types of complements [2], [3], [14], [18][20], [24], [25], [28] followed.

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In this paper, we will see that the collection $FA(X)$ of functional Alexandroff topologies on a set X need not be a lattice if X is infinite. If X is finite, we show that $FA(X)$ is a lattice, but generally is not a sublattice of $A(X) = T(X)$. As a subposet of $A(X)$, we investigate when a functional Alexandroff topology has a functional Alexandroff complement, showing that $FA(X)$ is a complemented lattice if X is finite. A special case of our complementation results framed in algebraic terminology was given in [13]. We start with a characterization of the functional Alexandroff topologies.

If X is a set, $T(X)$ represents the lattice of all topologies on X ordered by containment. The indiscrete topology $\mathcal{T}_I = \{\emptyset, X\}$ and the discrete topology $\mathcal{T}_D = \mathcal{P}(X)$ are the smallest and largest elements, respectively, of $T(X)$. The supremum of two topologies $\mathcal{T}, \mathcal{T}' \in T(X)$ has subbasis $\mathcal{T} \cup \mathcal{T}'$ and basis $\{U \cap V : U \in \mathcal{T}, V \in \mathcal{T}'\}$. The infimum $\mathcal{T} \wedge \mathcal{T}'$ is $\mathcal{T} \cap \mathcal{T}'$. A complement of $\mathcal{T} \in T(X)$ is a topology $\mathcal{T}' \in T(X)$ with $\mathcal{T} \vee \mathcal{T}' = \mathcal{T}_D$ and $\mathcal{T} \wedge \mathcal{T}' = \mathcal{T}_I$. $T(X)$ is a complemented lattice since every topology on X has a complement. A topology on X generally has many complements [7], [27].

The collection $A(X)$ of all Alexandroff topologies on X is a sublattice of $T(X)$ and a complete lattice [23]. It is an easy exercise to show that $A(X)$ is a complete sublattice of X if and only if $A(X) = T(X)$, which occurs if and only if X is finite. If $\mathcal{T}$ is an Alexandroff topology on X and $x \in X$, then $N(x) = \bigcap\{U : U \in \mathcal{T}, x \in U\}$ is the smallest neighborhood of x. Every Alexandroff topology $\mathcal{T}$ on X defines a quasiorder (that is, a reflexive, transitive relation) $\lesssim$ on X, called the *specialization quasiorder*, by taking $x \lesssim y$ if and only if $x \in cl\{y\}$, or equivalently, if and only if $y \in N(x)$. Conversely, every quasiorder on X defines an Alexandroff topology $\mathcal{T}$ on X through the same equivalent expressions. The one-to-one correspondence between quasiorders and Alexandroff topologies is widely used. See [19] for a survey of these connections. In a quasiordered set $(X, \lesssim)$, the *decreasing hull* of $A \subseteq X$ is $d(A) = \{x \in X : \exists a \in A, x \lesssim a\}$. A set is *decreasing* if $A = d(A)$. We write $d(x)$ for $d(\{x\})$. Increasing hulls $i(A)$ and increasing sets are defined dually. In the associated Alexandroff topology, $d(x) = cl\{x\}$ and $i(x) = N(x)$, the decreasing sets are the closed sets, and the increasing sets are the open sets. A quasiordered set will be called a *qoset*.

If $f : X \to X$ is a function, the associated functional Alexandroff topology P_f on X is the topology whose closed sets are those $A \subseteq X$ which satisfy $f(A) \subseteq A$. It is easy to see that in (X, P_f), the closure $cl\{x\}$ is the orbit $\mathcal{O}(x) = \{f^n(x) : n \in \mathbb{Z}, n \geq 0\}$ and the smallest neighborhood $N(x)$ is $\{y : \exists n \in \mathbb{Z}, n \geq 0 \text{ with } f^n(y) = x\}$. If $\{a, f(a), f^2(a), \ldots, f^n(a) = a\}$ has cardinality n, we call this set a *cycle* of length n.

1 Characterizing finite functional Alexandroff topologies

Theorem 1 *Suppose P_f is a functional Alexandroff topology on X. If P_f is a finite topology, then X is finite.*

Proof: If there exists $a \in X$ with $cl\{a\} = \{a, f(a), f^2(a), \ldots\}$ being infinite, then $cl\{a\}, cl\{f(a)\}, cl\{f^2(a)\}, \ldots$ give infinitely many distinct closed sets, contrary to P_f being finite. Thus, for $a \in X$, $cl\{a\}$ is finite, so a eventually maps into a cycle of finite length. Infinitely many such cycles would imply infinitely many closed sets of form $cl\{a\}$, so there must be only finitely many cycles. If a point x in one of the cycles has infinitely many predecessors (i.e., points $y \in i(x)$, or points y with $f^n(y) = x$ for some $n \geq 0$), the smallest neighborhoods $N(y)$ of those predecessors would give infinitely many open sets, a contradiction. Thus, each point of a cycle has only finitely many predecessors, and thus X is finite.

If $|X| = n$, the largest functional Alexandroff topology on X is $P_i = \mathcal{P}(X)$ where i is the identity function on X. It is easy to see that the next largest functional Alexandroff topology on $\{1, \ldots, n\}$ is generated by the function $f(j) = j$ for $1 \leq j < n-1$ and $j = n$, and $f(n-1) = f(n)$, and this gives $|P_f| = 2^{n-1} + 2^{n-2}$: Every subset of $\{1, \ldots, n-1\}$ is an open set excluding n, and the open sets including n have form $\{n-1, n\} \cup C$ where $C \subseteq \{1, \ldots, n-2\}$. A complete characterization of all k between 2 and 2^n which are realized as $|P_f|$ for some f on an n-element set is not known to us, but we present some partial results. By $A \subset B$ we mean $A \subseteq B$ and $A \neq B$.

Proposition 1 *If $|X| = n$ and $2 \leq k \leq 2n$, there exists $f : X \to X$ with $|P_f| = k$.*

Proof: Suppose $X = \{1, 2, \ldots, n\}$. If $2 \leq k \leq n+1$, define $f(i) = i+1$ for $i = 1, \ldots, n-1$ and $f(n) = k-1$. Then $P_f = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \ldots, \{1, 2, \ldots, k-2\}, X\}$ and $|P_f| = k$. For $2 \leq j \leq n$, define $g(i) = i+1$ for $i = 1, \ldots, n-2$, $f(n-1) = n-1$, and $f(n) = j$. It is easy to check that $|P_g| = n + j$, so the values of k between $n+2$ and $2n$ are realized as $|P_g|$.

If $|X| = n$, the example of a function $f : X \to X$ with $|P_f| = 2$ provided in the proof of Proposition 1 was a cycle. It is easy to see that the only way that $|P_f| = 2$ may occur is if X is finite and f is a cycle.

The next proposition shows that the example of a function $f : X \to X$ with $|P_f| = 3$ provided in the proof of Proposition 1 is the only way that $|P_f| = 3$ may occur.

Proposition 2 *If $\emptyset \subset A \subset X$ and $\{\emptyset, A, X\} = P_f$ for some $f : X \to X$, then X is a finite set, $|A| = 1$, and f is given by $f(a_i) = a_{i+1}$ for $i = 1, \ldots, n-1$ and $f(a_n) = a_2$ for some labeling $\{a_1, a_2, \ldots, a_n\}$ of the elements of X.*

Proof: Suppose $\emptyset \subset A \subset X$ and $\{\emptyset, A, X\} = P_f$. If $|A| \geq 2$, pick distinct elements $a_1, a_2 \in A$. Now $N(a_1) = i(a_1) = A = i(a_2) = N(a_2)$, so A must contain a cycle C containing $\{a_1, a_2\}$. For $b \in X - A$, $i(b) = N(b)$ must equal X. Now $a_1 \in X = i(b)$, so $b = f^n(a_1)$ for some $n \in \mathbb{N}$ and thus $b \in cl\{a_1\} = \mathcal{O}(a_1) = C \subseteq A$. This contradicts $b \notin A$. Thus, $|A| = 1$.

If $X - A$ is infinite, then either (a) it contains an infinite chain $c, f(c), f^2(c), \ldots$ and then $cl\{c\}$ and $cl\{f(c)\}$ are distinct proper open sets, or (b) $X - A$ contains (at least) two finite cycles, which give two nonempty proper closed sets. Both cases contradict $|P_f| = 3$, so $|A| = 1$ and $X - A$ is finite.

Finally, to see f has the form described, suppose $A = \{a_1\}$ and $b \in X - A$. If $cl\{b\} = \mathcal{O}(b) \neq X - A$, then there exists $c \in X - (A \cup cl\{b\})$ and $X - cl\{b\}$ is an open proper set containing $c \notin A$, so $X - cl\{b\} \notin \{\emptyset, A, X\}$.

Both papers introducing functional Alexandroff spaces ([5] as *functional Alexandroff spaces* and [9] using the terminology *primal spaces*) give characterizations for an Alexandroff topology to be functional Alexandroff, and both papers describe this as their main result. To have a self-contained development here, we present alternate proofs of the characterization in [5], slightly reworded and in less generality to suit our purposes.

Lemma 1 (cf. Theorem 3.5($\mathcal{C}_2$)[5]) *Suppose $\mathcal{T}$ is an Alexandroff topology on an arbitrary set X. The following are equivalent.*

(a) There exist distinct $a, b, c \in X$ with $N(a) = N(b) \subset N(c)$. (See Figure 1(a).)
(b) There exist distinct $a, b, c \in X$ with $cl\{c\} \subset cl\{b\} = cl\{a\}$.

Furthermore, if $\mathcal{T}$ satisfies these conditions, $\mathcal{T}$ is not functional Alexandroff.

Proof: First, we will show $N(b) \subseteq N(c)$ if and only if $cl\{b\} \supseteq cl\{c\}$. Suppose $N(b) \subseteq N(c)$. Now $x \in cl\{c\} \iff c \in N(x) \iff N(c) \subseteq N(x) \Rightarrow N(b) \subseteq N(x) \iff b \in N(x) \iff x \in cl\{b\}$. Conversely, suppose $cl\{b\} \supseteq cl\{c\}$. Now $x \in N(b) \iff b \in cl\{x\} \iff cl\{b\} \subseteq cl\{x\} \Rightarrow cl\{c\} \subseteq cl\{x\} \iff c \in cl\{x\} \iff x \in N(c)$.

Next, we will show that if $N(b) \subset N(c)$, then $cl\{b\} \neq cl\{c\}$. Indeed, under the hypotheses, $N(b) \neq N(c) \iff c \notin N(b) \iff b \notin cl\{c\} \Rightarrow cl\{b\} \neq cl\{c\}$. Similarly, if $cl\{b\} \supset cl\{c\}$, then $b \notin cl\{c\} \Rightarrow c \notin N(b) \Rightarrow N(b) \neq N(c)$. With the previous paragraph, this shows (a) and (b) are equivalent.

Now $N(a) = N(b)$ implies $b \in N(a)$, so $f^n(b) = a$ for some $n \in \mathbb{N}$, and similarly, $f^m(a) = b$ for some $m \in \mathbb{N}$. Thus, $a = f^n(b) = f^{n+m}(a)$, so a and b are in a cycle. Now $N(b) \subset N(c)$ implies $b \in N(c)$, so $c = f^k(b)$ for some $k \in \mathbb{N}$ and thus c is in the cycle with a and b. This implies $N(b) = N(c)$, contrary to $N(b) \subset N(c)$.

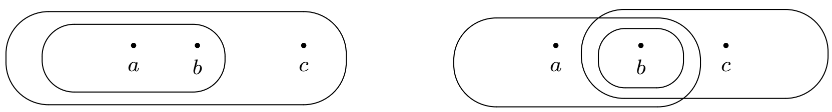
Lemma 2 (cf. Theorem 3.5($\mathcal{C}_1$)[5]) *Suppose $\mathcal{T}$ is an Alexandroff topology on an arbitrary set X. The following are equivalent.*

(a) There exist $a, b, c \in X$ with $N(a) \subset N(b), N(c)$, with $N(b)$ and $N(c)$ not nested (that is, with $N(b) \not\subseteq N(c)$ and $N(c) \not\subseteq N(b)$. (See Figure 1(b).)
(b) There exist $a, b, c \in X$ with $cl\{c\}, cl\{b\} \subset cl\{a\}$, with $cl\{b\}$ and $cl\{c\}$ not nested.

Furthermore, if $\mathcal{T}$ satisfies these conditions, $\mathcal{T}$ is not functional Alexandroff.

Proof: The equivalence of (a) and (b) follows from the equivalence of $N(b) \subseteq N(c)$ and $cl\{b\} \supseteq cl\{c\}$ and the corresponding statement for strict inclusions given in the proof of Lemma 1. Note that the conditions (a) and (b) each imply that the points a, b, c are distinct.

Suppose $\mathcal{T}$ satisfies (a). Now $a \in N(b)$ implies $b = f^n(a)$ for some $n \in \mathbb{N}$, and similarly $a \in N(c)$ implies $c = f^m(a)$ for some $m \in \mathbb{N}$. If $n \leq m$, say $m = n + k$, then $c = f^m(a) = f^k(f^n(a)) = f^k(b)$, so $b \in N(c)$, giving the contradiction that $N(b) \subseteq N(c)$. A similar contradiction follows if $m < n$.



(a) (b)

Fig. 1 Minimal neighborhood configurations which imply the space is not functional Alexandroff, as characterized in (a) Lemma 1 and (b) Lemma 2.

While the previous lemmas hold for arbitrary sets X, in the case of finite sets X they provide the only ways a topology may fail to be functional Alexandroff. The characterization of functional Alexandroff spaces in Theorem 3.5 of [5] is stronger than the one below. It contains an extra condition which does not require the assumption that X be finite.

Theorem 2 (cf. Theorem 3.5[5]) *A topology $\mathcal{T}$ on a finite set X is functional Alexandroff if and only if (a) there are no distinct points $a, b, c \in X$ with $N(a) = N(b) \subset N(c)$ and (b) $N(a) \subset N(b), N(c)$ implies $N(b)$ and $N(c)$ are nested.*

Again we note that this says the conditions of Lemmas 1 and 2 are the only things that can prevent a topology on a finite set from being functional Alexandroff. We also note that the condition (a) could be replaced by the equivalent (a)$'$ if $a \neq b$ and $N(a) = N(b)$, then $N(a) \not\subset N(c)$ for any $c \in X$.
Proof: Lemmas 1 and 2 show that if $\mathcal{T}$ is functional Alexandroff, then (a) and (b) hold. Conversely, suppose (a) and (b) hold. We give an algorithm to construct a function f for which $\mathcal{T} = P_f$.

Iterative Step: Let $\mathcal{N} = \{N(x) : x \in X$ and $f(x)$ has not been defined$\}$, ordered by set inclusion. Pick $a \in X$ such that $N(a)$ is minimal in $\mathcal{N}$.

If $N(a)$ contains $b \neq a$: the the minimality of $N(a)$ implies $N(b) = N(a)$. Now $N(a) = \{a_1, \ldots, a_k\}$ where $N(a_i) = N(a)$ for $i = 1, \ldots, k$. By (a), $N(a) \not\subset N(c)$ for any $c \in X$. Define $f(a_i) = a_{i+1}$ for $i = 1, \ldots, k-1$ and $f(a_k) = a_1$. Return to the Iterative Step.

If $N(a) = \{a\}$: Suppose $N(a) \not\subset N(b)$ for any $b \in X$. Then define $f(a) = a$ and return to the Iterative Step. If $N(a) = \{a\} \subset N(b)$ for some $b \in X$, by (b), $N(a) \subset N(b), N(c)$ implies $N(b), N(c)$ are nested, so there exists $b^* \in X$ such that $N(b)$ is minimal among the members of $\mathcal{N}$ which strictly contain $N(a)$. Define $f(a) = b^*$.

If there exists $b' \neq b^*$ with $N(b') = N(b^*)$, then by (a) there is no c with $N(b^*) \subset N(c)$. Let $\{x : N(x) = N(b^*)\} = \{b_1, b_2, \ldots, b_k\}$ and define $f(b_i) = b_{i+1}$ for $i = 1, \ldots, k$ and $f(b_k) = b_1$. Return to the Iterative Step.

If $N(b') \neq N(b^*)$ for any $b' \neq b^*$ and $N(b) \not\subset N(c)$ for any $c \in X$, return to the Iterative Step.

If $N(b') \neq N(b^*)$ for any $b' \neq b^*$ and there exists $c \in X$ with $N(b) \subset N(c)$, by (b) there exists $c^* \in X$ such that $N(c^*)$ is minimal among the members of $\mathcal{N}$ which strictly contain $N(b)$. Define $f(b^*) = c^*$.

If there exists $c' \neq c^*$ with $N(c') = N(c^*)$, then by (a) there is no d with $N(c^*) \subset N(d)$. Let $\{x : N(x) = N(c^*)\} = \{c_1, c_2, \ldots, c_k\}$ and define $f(c_i) = c_{i+1}$ for $i = 1, \ldots, k$ and $f(c_k) = c_1$. Return to the Iterative Step.

From this construction, it is clear that f is a well-defined function on X with $P_f = \mathcal{T}$.

2 Lattice properties of functional Alexandroff topologies

The set $A(X)$ of Alexandroff topologies on X is a sublattice of $T(X)$ and is a complete lattice, but is not a complete sublattice unless $T(X) = A(X)$ (that is, unless X is finite). We will consider the subposet $FA(X)$ of functional Alexandroff spaces.

Proposition 3 *The indiscrete topology on X is functional Alexandroff if and only if X is finite.*

Proof: If X is finite and f is any cyclic permutation of X, $P_f = \{\emptyset, X\}$. If X is infinite, suppose $P_f = \{\emptyset, X\}$. Pick $a \in X$. Now $cl\{a\} = \{a, f(a), f^2(a), \ldots\} = X$, so X is countable and $f^n(a) \neq a$ for any $n \in \mathbb{N}$. Now $cl\{f(a)\} = \{f(a), f^2(a), f^3(a), \ldots\}$ is a nonempty closed set not containing a, contrary to $P_f = \{\emptyset, X\}$.

$FA(X)$ need not be a lattice. For example, for $X = \mathbb{Z}$, define $f(n) = n+1$ and $g(n) = n-1$. Now P_f is the left ray topology $P_f = \{(-\infty, m) : m \in \mathbb{Z}\} \cup \{\emptyset, \mathbb{Z}\}$ and P_g is the right ray topology. In $T(X)$, the only topology coarser than both P_f and P_g is the indiscrete topology $\mathcal{T}_I$, so in $T(X)$ and $A(X)$, $P_f \wedge P_g = \mathcal{T}_I$. By Proposition 3, the indiscrete topology on $\mathbb{Z}$ is not functional Alexandroff, so P_f and P_g have no lower bounds in $FA(\mathbb{Z})$.

The main result of this section gives several lattice properties of $FA(X)$.

Theorem 3 *(a) $FA(X)$ is a $\vee$-semilattice and $P_f \vee P_g$ in $FA(X)$ agrees with $P_f \vee P_g$ in $T(X)$.*
(b) If X is finite, $FA(X)$ is a lattice.
(c) $FA(X)$ is a sublattice of $A(X)$ if and only if $|X| \leq 2$.

Proof: (a) Suppose $P_f, P_g \in FA(X)$ have associated quasiorders $\lesssim_f, \lesssim_g$. Then $\lesssim_f \cap \lesssim_g$ is a quasiorder $\lesssim$, and it is easy to verify that the associated Alexandroff topology is $P_f \vee P_g$. It remains to show that the topology associated with $\lesssim_f \cap \lesssim_g$ is functional Alexandroff. Define $h : X \to X$ by $h(x) = f^k(x)$ where $k \in \mathbb{N}$ is the smallest natural number such that $f^k(x) \in \{g(x), g^2(x), g^3(x), \ldots\}$, or $h(x) = x$ if there is no such k. Clearly h is a well-defined function. To show $P_h = P_f \vee P_g$, it suffices to show $N_h(z) = N_f(z) \cap N_g(z)$ for any $z \in X$, or equivalently (since $x \in cl\{z\}$ if and only if $z \in N(x)$), $cl_h\{x\} = cl_f\{x\} \cap cl_g\{x\}$ for all $x \in X$. Suppose x is given. Since $h(x) \in cl_f\{x\} \cap cl_g\{x\}$, it follows that $cl_h\{x\} \subseteq cl_f\{x\} \cap cl_g\{x\}$. Suppose $z \in cl_f\{x\} \cap cl_g\{x\}$. If $z = x$, then $z \in cl_h\{x\}$, so we may assume $z = f^{k'}(x) = g^{n'}(x)$, where $k' > 0$. If $h(x) = f^{k_1}(x) = g^{n_1}(x)$, iterating f we get an increasing sequence $k_1 < k_2 < k_3 < \cdots$ such that $f^{k_i}(x) \in \{g(x), g^2(x), g^3(x), \ldots\}$ and $f^j(x) \notin \{g(x), g^2(x), g^3(x), \ldots\}$ for $k_i < j < k_{i+1}$. Now when $k_i = k'$ we have $z = f^{k_i}(x) = h^i(x)$, so $z \in cl_h\{x\}$.

(b) If X is finite, $FA(X)$ has a least element $P_f = \{\emptyset, X\}$ where f is any cyclic permutation of X. By (a) finite (and thus arbitrary) suprema exist in $FA(X)$, so $FA(X)$ is a (complete) lattice.

(c) If $X = \{a\}$, the unique topology on X is generated by the unique function $f : X \to X$. If $X = \{a, b\}$, define $f(a) = b, f(b) = a$, and for $x \in \{a, b\}$, $g(x) = a, h(x) = b$, and $i(x) = x$. Now the four topologies on X are realized as P_f, P_g, P_h, P_i.

If $|X| \geq 3$, pick three distinct elements $x_1, x_2, x_3 \in X$ and define $f(x_1) = x_2, f(x_2) = x_3$, $g(x_2) = x_1, g(x_1) = x_3$, and for $x \in X - \{x_1, x_2\}$, $f(x) = g(x) = x$. Figure 2 shows f, g, and the associated topologies P_f, P_g consisting of the increasing sets from the quasiorders shown. In $A(X)$, $P_f \wedge P_g$ has basis of minimal neighborhoods $\{\{x_1, x_2\}, \{x_1, x_2, x_3\}\} \cup \{\{x\} : x \in X - \{x_1, x_2, x_3\}\}$. By Lemma 1, this topology is not in $FA(X)$. In $FA(X)$, $P_f \wedge P_g$ has basis $\{\{x_1, x_2, x_3\}\} \cup \{\{x\} : x \in X - \{x_1, x_2, x_3\}\}$.

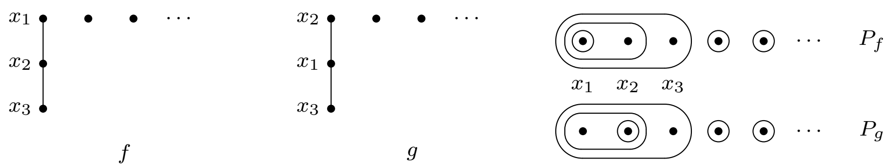


Fig. 2 Topologies whose infima in $A(X)$ and $FA(X)$ differ.

For the remainder of this section, we present some specific examples of infima in $FA(X)$.

While Proposition 2 gave an explicit characterization of the three-element functional Alexandroff topologies, the next result shows that every three-element topology on a finite set is the infimum of functional Alexandroff topologies.

Proposition 4 *If X is finite, $\mathcal{T} \in T(X)$ and $|\mathcal{T}| = 3$, then $\mathcal{T} = P_f \wedge P_g$ for some $f, g : X \to X$.*

Proof: Suppose $\mathcal{T} = \{\emptyset, A, X\}$ where, after relabeling, $A = \{1, 2, \dots, k\}$ and $X = \{1, 2, \dots, n\}$. Define $f, g : X \to X$ by

$$f(j) = \begin{cases} j+1 & j = 1, \dots, n-1 \\ k+1 & j = n \end{cases} \qquad g(j) = \begin{cases} k+1 & j = 1 \\ n & j = k+1 \\ j-1 & j \in X - \{1, k+1\}. \end{cases}$$

It is easy to check that $P_f \wedge P_g = \mathcal{T}$.

If $f : X \to X$ is injective, the components (in the graph theoretic sense) of the specialization quasiorder tree can be order isomorphic to $\mathbb{Z}, \mathbb{N}$, or a finite cycle. Note that finite chains leading into a cycle are not possible. If f is bijective, then components order isomorphic to $\mathbb{N}$ are not possible since $1 \notin f(\mathbb{N})$. If f is bijective and every $x \in X$ is part of a finite cycle, then $P_f = P_{f^{-1}} = \mathcal{P}(\{\mathcal{O}(a) : a \in X\}) = \{C \subseteq X : C \text{ is a cycle of } f\}$. The result below is more general.

Proposition 5 *If $f : X \to X$ is bijective, then $P_f \wedge P_{f^{-1}} = \mathcal{P}(\{D : D$ is a component of the $\leq_f$ qoset$\})$, and $P_f \vee P_{f^{-1}}$ has basis $\{\{x\} : x$ is not in any cycle of $f\} \cup \{C : C$ is a finite cycle of $f\}$.*

Corollary 1 *If $f : X \to X$ is bijective, $P_f \wedge P_{f^{-1}} = \{\emptyset, X\}$ if and only if X has one component (a cycle, or order isomorphic to $\mathbb{Z}$).*

Corollary 2 *If f^{-1} exists, P_f is the complement of $P_{f^{-1}}$ in $A(X)$ if and only if X is order isomorphic to $\mathbb{Z}$. (Observe that in this case, $P_f \wedge P_{f^{-1}} \notin FA(X)$.)*

The examples below suggest useful techniques for achieving a desired infimum of P_f, P_g.

Example 1 If $f : X \to X$ is bijective and has three components in the $\lesssim_f$ qoset isomorphic to $\mathbb{Z}$ and two which are cycles as suggested by Figure 3, then there exists $g : X \to X$, as in Figure 3 with $P_f \wedge P_g = \{\emptyset, X\}$. Slight modifications of this example would show that if the $\lesssim_f$ qoset has a finite number of components, then there exists a function g with $P_f \wedge P_g = \{\emptyset, X\}$.

Example 2 If $f : X \to X$ is bijective and has a countably infinite number of components isomorphic to $\mathbb{Z}$ in the $\lesssim_f$ qoset and a countable or finite number of components which are cycles, then there exists a $g : X \to X$ with $P_f \wedge P_g = \{\emptyset, X\}$. Figure 4 suggests f and a function g with the desired properties.

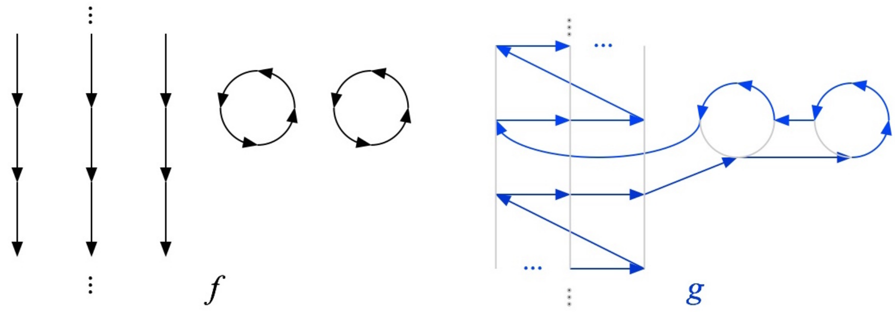


Fig. 3 $P_f \wedge P_g = \{\emptyset, X\}$

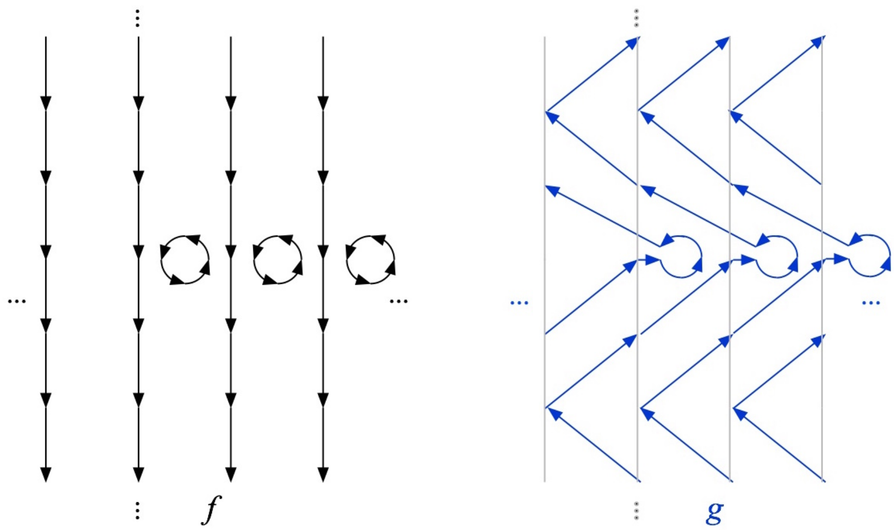


Fig. 4 $P_f \wedge P_g = \{\emptyset, X\}$

3 Complementation in $FA(X)$.

This section is devoted to describing a constructive algorithm to produce a functional Alexandroff complement to any functional Alexandroff topology on a finite set X. This will prove the following result.

Theorem 4 *If X is finite, the lattice $FA(X)$ of functional Alexandroff topologies on X is complemented.*

Proof: Suppose X is finite and P_f is a functional Alexandroff topology on X. We will construct a function g on X so that P_g is a complement of P_f. Let Q_0 be the qoset diagram for the quasiorder $\lesssim_f$ (defined by $x \lesssim_f y$ if and only if $x \in cl\{y\}$, if and only if $x = f^n(y)$ for some $n \geq 0$). Loosely speaking, the points at the top of the qoset for f should be at the bottom of the qoset for g. Let $C_0^1, C_0^2, \ldots, C_0^k$ be the components of Q_0 which have no maximal element. Since X is finite, each C_0^i is a cycle of f. From each C_0^i $(i = 1, \ldots, k)$, pick a representative $c_i \in C_0^i$.

Let $M_0 = \{x \in X : x \text{ is maximal in } Q_0\} \cup \{c_i\}_{i=1}^k$.

Since M_0 contains all $\lesssim_f$-maximal points and a point of each cycle C_i^0 not having a maximal element, it follows that every $x \in X$ is in the orbit of some point of M_0. In particular, $cl_f(M_0) = X$.

Define g to be a cycle through all the points of M_0, and pick a point $m_0 \in M_0$.

Define $g(x) = x$ for each $x \in C_0 \equiv \bigcup_{i=1}^{k} C_0^i - M_0$. (Thus, every point of a cycle C_0^i is fixed, except the representative point c_i.)

Let $Y_0 = C_0 \cup M_0 = \{x \in X : g(x)$ has been defined$\}$. Since Y_0 was taken from the top of the qoset for f, this should be at the bottom of the qoset for g. In future iterations, no further points of X will be fixed by g. All remaining points of X will appear above m_0 in the qoset diagram for g.

> If $x \in M_0 - \{c_i\}_{i=1}^{k}$, then $N_f(x) = \{x\}$. If $x \in C_0$, then $N_g(x) = \{x\}$ (since nothing will subsequently map to x in future iterations of the algorithm). If $x = c_i = C_0^i \cap M_0$, then $N_f(x) = C_0^i$ and $N_g(x) \cap C_0^i = \{x\}$. In all cases, $N_f(x) \cap N_g(x) = \{x\}$.

Now we are set to start an inductive argument.

Suppose $g(x)$ has been defined on Y_j and m_j has been defined. (**)

Let $Q_{j+1} = X - Y_j$ (considered as a qoset diagram) be the set of points of X for which $g(x)$ has not yet been defined.

Let $M_{j+1} = \{x \in Q_{j+1} : x$ is maximal in $Q_{j+1}\}$, and

$C_{j+1} = \{x \in Q_{j+1} : x$ is in a component of Q_{j+1} which has no maximal element$\}$. Thus, C_{j+1} consists of the points of the cycles having no "stem" leading into them.

For $x \in C_{j+1}$, define $g(x) = m_j$.

Linearly order the elements $a_1 < a_2 < \cdots < a_i$ of M_{j+1} in any manner, and define

$$g(a_k) = a_{k+1} \text{ for } k = 1, \ldots, i-1$$
$$g(a_i) = m_j.$$

Now let $Y_{j+1} = Y_j \cup M_{j+1} \cup C_{j+1} = \{x \in X : g(x)$ has been defined$\}$ and let $m_{j+1} = a_1$ (which is the maximal point of the main branch of the qoset diagram thus far defined for g). Iterate from (**) until all points of X are exhausted.

> If $x \in C_{j+1}$, $N_g(x) = \{x\}$ (since nothing will map to x in future iterations of the algorithm). From the definition of M_{j+1} as containing the maximal elements in the f-qoset $X - Y_j$ at which g has not been previously defined, it follows that for $x = a_k \in M_{j+1}$, $N_f(x) \subseteq \{x\} \cup Y_j$. Since $N_g(x) \subseteq M_{j+1} \cup (X - Y_{j+1})$, we have $N_f(x) \cap N_g(x) = \{x\}$. Thus, $P_f \wedge P_g = \mathcal{P}(X)$.
>
> Finally, from the construction, note that for every $x \in X - C_0$, $cl_g(x) = \{x, g(x), g^2(x), \ldots\}$ eventually contains the cycle M_0. Since $cl_f(M_0) = X$, the only $P_f \vee P_g$-closed set containing x is X. If $x \in C_0^i - \{c_i\} = C_0^i - M_0$, then $c_i \in cl_f(x) = C_0^i$, and since $cl_g(c_i) = M_0$ and $cl_f(M_0) = X$, again we have that the only $P_f \vee P_g$-closed set containing x is X.

The example below illustrates the algorithm.

Example 3 For $X = \{a, b, \ldots, l\}$, let f be the function whose qoset diagram is shown at the top of Figure 5(a). Pick j as the representative element of the cycle. Then $M_0 = \{a, b, e, g, j\}$ and $C_0 = \{k, l\}$. We pick $m_0 = a$. The algorithm produces the partial qoset shown at the bottom of Figure 5(a). For the next iteration, we have $M_1 = \{c, f\}$ and $C_1 = \{h, i\}$. We linearly order M_1 by $c < f$ and thus $m_1 = f$. Figure 5(b) show the result of this iteration. The final iteration is show in Figure 5(c).

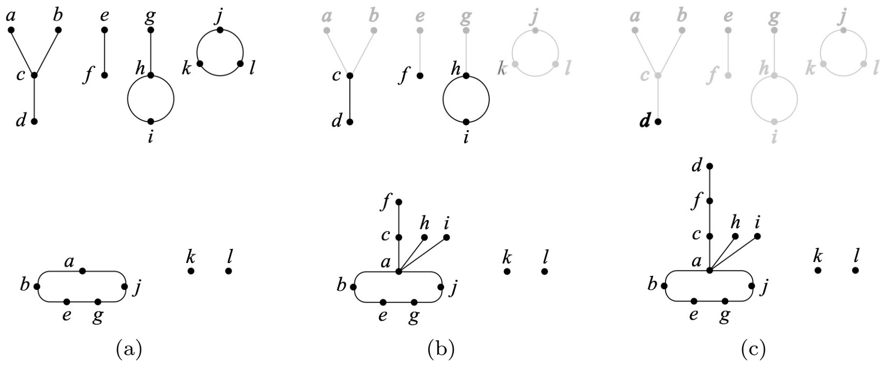


Fig. 5 Iterations of the complementation algorithm.

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