COLLECTIONS OF MUTUALLY DISJOINT CONVEX SUBSETS OF A TOTALLY ORDERED SET

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Abstract. We present a combinatorial proof of an identity for F_{2n+1} by counting the number of collections of mutually disjoint convex subsets of a totally ordered set of n points. We discuss how the problem is motivated by counting certain topologies on finite sets.

Theorem. *Given a totally ordered set X of n points, the number $C(n)$ of collections of mutually disjoint convex subsets of X is given by*

$$C(n) = 1 + \sum_{p=1}^{n} \sum_{j=1}^{p} \binom{n-p+j}{j}\binom{p-1}{j-1} = F_{2n+1}.$$

Proof. For any natural number k, let $\underline{k}$ denote the set $\{1, 2, \ldots, k\}$ with the usual total order $1 < 2 < \cdots < k$. Note that a convex subset of $\underline{k}$ is simply an interval in $\underline{k}$. Suppose $\mathcal{C}$ is a collection of mutually disjoint convex subsets of $X = \underline{n}$. We will call the members of $\mathcal{C}$ *blocks*. If $\mathcal{C}$ has j blocks ($j = 0, \ldots, n$) and $|\bigcup \mathcal{C}| = p$, these p elements may be divided into j convex blocks in $\binom{p-1}{j-1}$ ways by inserting $j-1$ dividers into the $p-1$ gaps between the p points. Now we may totally order the $n-p$ points and j blocks, by choosing which of the $n-p+j$ items will be blocks, in $\binom{n-p+j}{j}$ ways. Summing as p goes from 1 to n and as j goes from 1 to p, and adding the one exceptional case corresponding to $j = 0$, we have

$$C(n) = 1 + \sum_{p=1}^{n} \sum_{j=1}^{p} \binom{n-p+j}{j}\binom{p-1}{j-1}. \tag{1}$$

We may also find a recursive formula for $C(n)$. For any collection $\mathcal{C}$ of mutually disjoint convex subsets of $\underline{n}$, consider the point $n \in \underline{n}$. Now $n \notin \bigcup \mathcal{C}$ if and only if $\mathcal{C}$ is one of the $C(n-1)$ collections of mutually disjoint convex subsets of $\underline{n-1}$. Furthermore, $n \in \{j+1, \ldots, n\} \in \mathcal{C}$ where, for now, $j \in \{1, 2, \ldots, n-1\}$, if and only if $\mathcal{C} \setminus \{\{j+1, \ldots, n\}\}$ is one of the $C(j)$ collections of mutually disjoint convex subsets of $\underline{j}$. If $j = 0$, that is, if $n \in \{1, 2, \ldots, n\} \in \mathcal{C}$, then $\mathcal{C} = \{\underline{n}\}$ is the unique acceptable collection, and for this reason, we adopt the convention that $C(0) = 1$. Now summing over all cases $n \notin \bigcup \mathcal{C}$ and $n \in \{j+1, \ldots, n\} \in \mathcal{C}$ for $j = 0, 1, \ldots, n-1$, we have

$$C(n) = C(n-1) + \sum_{j=0}^{n-1} C(j). \tag{2}$$

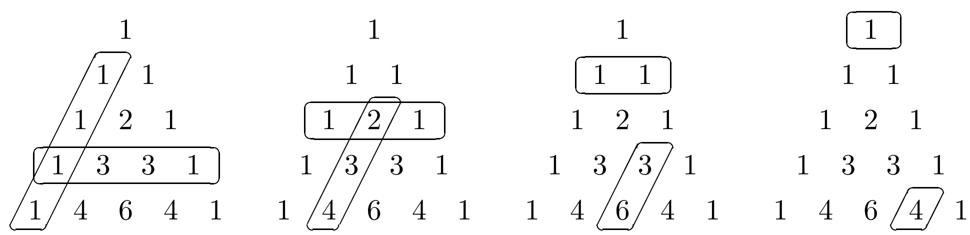
From either formula (1) or (2), we find the initial values of the sequence $\{C(n)\}_{n=0}^{\infty}$ to be $1, 2, 5, 13, 34, 89, \ldots$, which agree with the values of F_{2n+1}. Suppose $C(n) = F_{2n+1}$ for $n = 1, 2, \ldots, k-1$. From the recurrence formula (2) we have

$$C(k) = F_{2k-1} + \sum_{j=0}^{k-1} F_{2j+1}.$$

Applying the identity $\sum_{j=0}^{m} F_{2j+1} = F_{2m+2}$ (Identity #2 in [2], noting their convention that $f_n = F_{n+1}$), we have $C(k) = F_{2k-1} + F_{2k} = F_{2k+1}$. With the initial cases, this shows that $C(n) = F_{2n+1}$ for all natural numbers n. $\square$

The second half of the proof above, showing that $C(n) = F_{2n+1}$, can also be accomplished using a tiling argument of Anderson and Lewis [1] which allows tiles of any length. Think of a convex subset of $\underline{k}$ as a white tile on a $1 \times k$ strip. Then a collection of mutually disjoint convex subsets of $\underline{k}$ may be represented by a tiling of a $1 \times k$ strip by white tiles of various lengths and red squares in any remaining gaps, and the number of such tilings is $C(k)$. Having tiled a $1 \times k$ strip, we may obatain a suitable tiling of a $1 \times (k+1)$ strip either by appending a red square in the $k+1^{st}$ position (producing $C(k)$ tilings), appending a white square in the $k+1^{st}$ position (producing $C(k)$ tilings), or, if the tile covering the k^{th} slot is white, it may be expanded to cover the $k+1^{st}$ slot. To count these expansions easily, expand the tile covering the k^{th} slot, red or white, to cover the $k+1^{st}$ slot (in $C(k)$ ways), then remove those $C(k-1)$ ending in a red domino (and leaving a suitable tiling of a $1 \times (k-1)$ strip). Thus, $C(k+1) = 3C(k) - C(k-1)$. This recurrence relation is satisfied by F_{2n+1} (see Identity #7 in [2]), and since the initial terms agree, we conclude that $C(n) = F_{2n+1}$ for all natural numbers n. The authors are grateful to the referee for pointing out this tiling argument.

For a fixed p, the second factors $\binom{p-1}{j-1}$ in the double sum of the theorem constitute the $(p-1)^{st}$ row of Pascal's triangle, while the values of the first factors $\binom{n-p+j}{j} = \binom{n-p+j}{n-p}$ are a subset of the $(n-p)^{th}$ diagonal. Thus, the double-sum formula for $F_{2n+1} - 1$ can be viewed as the sum of dot products of vectors in Pascal's triangle, as illustrated below for $n = 4$.



The sum of the dot products of the circled pairs of vectors is $F_{2(4)+1} - 1$.

Our motivation for this problem arose from counting certain finite topologies, as described below. If j is any point in a finite topological space, let $N(j)$ be the intersection of all open sets containing j.

Corollary. *Let $\mathcal{T}$ be the set of topologies τ on $\underline{n}$ such that the basis $\{N(j) : j \in \underline{n}\}$ consists of a collection $\mathcal{C}$ of mutually disjoint convex subsets of $\underline{n}$, or such a collection $\mathcal{C}$ together with $\underline{n}$. Then $|\mathcal{T}| = F_{2n+1} - 1$.*

The corollary follows from the almost one-to-one correspondence between the topologies of $\mathcal{T}$ and the collections $\mathcal{C}$ counted by $C(n)$, where for $j \in \underline{n} \setminus \bigcup \mathcal{C}$, we take $N(j) = \underline{n}$.

However, the collection having no blocks generates the same topology—namely the indiscrete topology—as the collection having a single block containing all the points.

References

[1] P. G. Anderson and R. H. Lewis, *Board Tilings of the Second Kind,* Proceedings of the Thirteenth International Conference on Fibonacci Numbers and Their Applications, Patras, Greece, July 2008. Florian Luca (Ed.), Utilitas Mathematica Publishing, Inc. *(to appear).*

[2] A. Benjamin and J. Quinn, *Proofs that Really Count*, Dolciani Mathematical Expositions no. 27, Mathematical Association of America, Washington, DC, 2003.

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