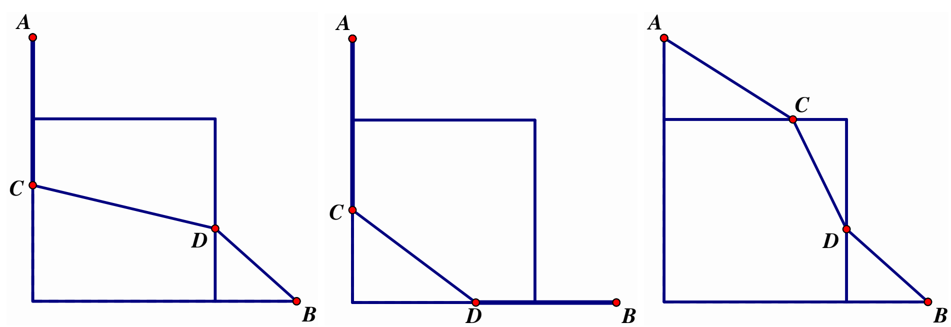
The Fastest Path Between Two Points, With a Symmetric Obstacle

Kathleen Bell, Shania Polson, and Tom Richmond
Western Kentucky University

Suppose you want to find the fastest path between two points at diagonally opposite corners $A(0,1)$ and $B(1,0)$ of a square $[0,1]\times[0,1]$. Assuming you can run at a constant speed, the direct path is the fastest. To complicate matters, suppose there is a swimming pool in the square $[0,a]\times[0,a]$, and your swimming speed is constant, but slower than running. What path will now be fastest? The analogous problem with the initial point on a straight seashore has been considered in [**5**], and the problem with initial and final points on adjacent edges of a rectangular pool is considered in [**3**] and [**4**].

If $a<\frac{1}{2}$, then the direct path from A to B misses the pool and remains optimal. Going out of your way to reach the pool, which is traversed at a slower rate, cannot reduce your time.

The candidates for optimal paths from A to B involve straight line segments through each medium, running from A to a point C on the north or west edge of the pool, swimming from C to a point D on the south or east edge of the pool, and running from D to B. Some such paths are shown in Figure 1.



(a) (b) (c)

Figure 1. Potential optimal paths.

We first show that the entry point C into the pool must be on the north edge of the pool. If C were on the west edge and D on the east edge, as in Figure 1(a), consider the path $AC'D'B$ where $C'=(0,a)$ and $\overline{C'D'}$ is parallel to $\overline{CD}$, as shown in Figure 2(a). Both $ACDB$ and $AC'D'B$ have the same swimming distance. Excluding the common running distance from A to C' and noting $C'C=D'D$, path $AC'D'B$ has the shorter remaining running distance $D'B$, compared to $D'DB$ for path $ACDB$. If C were on the west edge and D on the south edge, as in Figure 1(b), consider the path $AC'D'B$ where C' and D' are the images of C and D, respectively, reflected over the southeast diagonal of the pool, as shown in Figure 2(b). Both paths have the same swimming distance $CD=C'D'$. The running distance $D'B$ of path $AC'D'B$ is shorter than the running distance DB of path $ACDB$. This can be seen by drawing a circle of radius BD centered at B. Let D'' be the point on the circle due north of B. Now $\overline{DD''}$ lies entirely in the circle, and since D' lies on $\overline{DD''}$, it is inside the circle

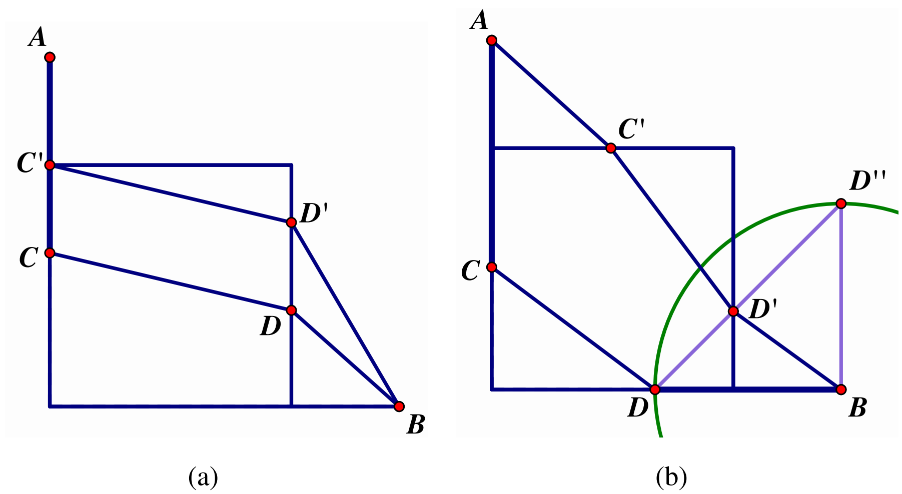


Figure 2. Paths to C on the west edge of the pool are not optimal.

and thus $D'B < DB$. Similarly, the running distance AC' of path $AC'D'B$ is shorter than the running distance AC of path $ACDB$. Thus, path $ACBD$ is not optimal if C lies on the west edge of the pool.

The same argument, reflected over the diagonal $y = x$, shows that the point D where the optimal path leaves the pool must be on the east side of the pool. Thus, the optimal path must have the form shown in Figure 1(c).

Next, we will show that the optimal path must be symmetric around the diagonal $y = x$. We will argue by contradiction. Suppose $ACDB$ is an optimal path from A to B which is not symmetric about the diagonal $y = x$. Reflecting it about the diagonal gives another optimal asymmetric path $AD'C'B$ from A to B, shown in Figure 3.

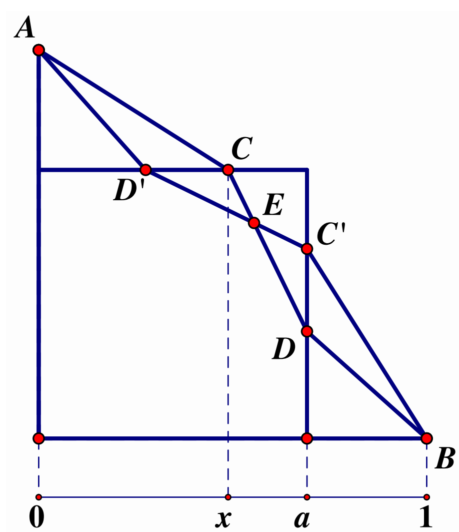


Figure 3. An asymmetric path and its reflection.

Let E be the point where paths $ACDB$ and $AD'C'B$ intersect the diagonal. Now path ACE must be an optimal path from A to E, for if there were a faster path from A to E, there would be a faster path from A to E and on to B. Likewise, $AD'E$ is an optimal path from A to E, and both EDB and $EC'B$ are optimal paths from E to B. Since ACE is the reflection of $EC'B$, each of these four paths from A or B to E require the same time to traverse. In particular, the path $ACEC'B$ requires the same time as the optimal path $ACEDB$, and thus is also optimal. But, the swimming part CEC' of this path must be the optimal path from C to C', that is, must be a straight line. But, CEC' is not a straight line since $ACDB$ was not symmetric. Thus, an asymmetric path $ACDB$ from A to B cannot be optimal.

Now we may find which symmetric path $ACC'B$ is optimal. We will assume the swimming speed s is 1 unit, and the running speed is $r > 1$. If x is the horizontal component of $\overline{AC}$, as shown in Figure 3, then the time to traverse a such a symmetric path is

$$T(x) = \frac{2\sqrt{(1-a)^2 + x^2}}{r} + \sqrt{2}(a - x),$$

for $x \in [0, a]$, and we have

$$T'(x) = \frac{2x}{r\sqrt{(1-a)^2 + x^2}} - \sqrt{2}.$$

It is easy to check that $T'(x)$ is always negative if $r \geq \sqrt{2}$, and for $1 < r < \sqrt{2}$, $T(x)$ decreases to the left of and increases to the right of the critical point $x_c = \frac{r(1-a)}{\sqrt{2-r^2}}$.

Thus, if $r \geq \sqrt{2}$, the optimal path involves no swimming, but running from A to the pool corner (a, a) and on to B. This is suggested by noting that the swimming portion of a symmetric path cuts off an isosceles right triangle from the corner of the pool, and comparing the swimming time along the hypotenuse $\sqrt{2}(a - x)/s = \sqrt{2}(a - x)$ to the running time $2(a - x)/r$ along the legs. This running time is smaller when $r > \sqrt{2}$.

If $1 < r < \sqrt{2}$, then the optimal path is the symmetric path determined by $C = (\frac{r(1-a)}{\sqrt{2-r^2}}, a)$.

A natural extension of this question is to consider the case of paths from A to B where A and B are diagonally opposite corners of a rectangle, with a rectangular pool obstructing the direct path. Results based on Fermat's principle and Snell's law (see [**2, 6**]) describe the angles of the running paths with the pool edges, but results in terms of the coordinates of points seems to be more complicated. The example below shows that simple transformations of the square solution will not generally yield solutions to the transformed rectangular case.

Example. Consider the square case with $a = 0.9$, $r = 1.3$ feet per second, and $s = 1$ foot per second. Then the critical point is $x_c = \frac{r(1-a)}{\sqrt{2-r^2}} \approx .233487$, and the optimal path goes from $A = (0, 1)$ to $C = (x_c, a) \approx (.233487, .9)$ to $D = (a, x_c) \approx (.9, .233487)$ to $B = (1, 0)$, requiring about $T(.233487) = 1.33336$ seconds. If we scale this square case by 0.5 in the x-direction to obtain a rectangular problem, then 'scaled optimal path' is not the optimal path for the scaled rectangular problem, as seen in Figure 4. Scaling the optimal path $P1$ for the square case gives a path $P2$ running from $(0,1)$ to $(.5x_c, a)$, swimming on to $(.5a, x_c)$, and running on to $(.5, 0)$, which requires about 1.04711 seconds. With these specific dimensions, a numerical solution shows that the optimal path $P3$ requires about 1.02445 seconds, going approximately from $(0,1)$ to $(.152152, a)$, to $(.5a, .545073)$, to $(.5, 0)$.

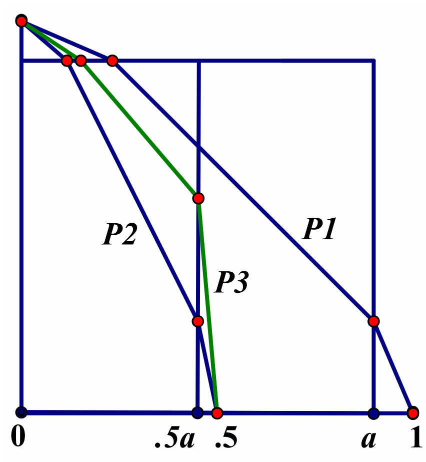
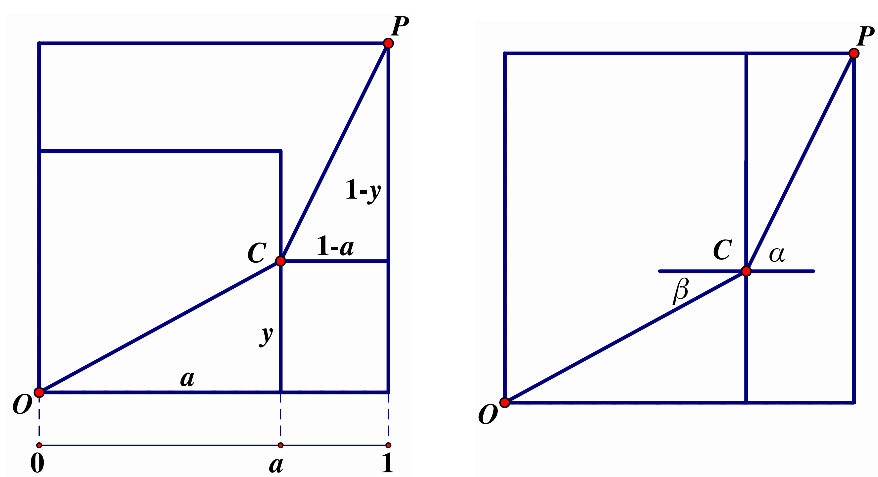


Figure 4. The optimal path for the square case scaled by 50% in the x-direction vs. the optimal path for the scaled course.

We conclude by considering the problem of the optimal path between the other pair of diagonally opposite corners. This path will follow segments $\overline{OC}$ and $\overline{CP}$ as shown in Figure 5(a), where $C = (a, y)$ for some $y \in [0, a]$, or will be the reflection of such a path over the line $y = x$.



(a) (b)

Figure 5. Optimal paths from O to P.

Assuming the swim speed is $s = 1$ and $r > s$, the time to traverse such a path is

$$T(y) = \sqrt{a^2 + y^2} + \frac{\sqrt{(1-a)^2 + (1-y)^2}}{r} \qquad (y \in [0, a]).$$

Setting $T'(y) = 0$ leads to a quartic equation in y with parameters a and r whose solution is lengthy. However, Snell's law provides the solution in terms of angles, as in [**1**]. Snell's law says that the optimal path from O to P swimming at speed s for $x < a$ and running at speed r for $x > a$ satisfies

$$\frac{\sin\alpha}{\sin\beta} = \frac{r}{s}$$

where α (respectively, β) is the angle between the running path (respectively, swimming path) and the horizontal. We note that the target point $C = (a, y)$ provided by Snell's law in Figure 5(b) has $y \leq a$ and thus provides a solution to the problem of Figure 5(a): Otherwise, we would have $\alpha < \pi/4 < \beta < \pi/2$, so $\sin\alpha < \sin\beta$, contrary to $\sin\alpha = \frac{r}{s}\sin\beta$ and our assumption that $r > s$.

Summary. If a square pool is positioned in a corner of a square courtyard, we find the fastest path from diagonally opposite corners of the courtyard, assuming the swimming speed through the pool is less than the running speed through the courtyard.

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