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Finite-point order compactifications

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I

After the characterization of 1-point topological compactifications by Alexandroff in 1924, n-point topological compactifications by Magill [**4**] in 1965, and 1-point order compactifications by McCallion [**5**] in 1971, spaces that admit an n-point order compactification are characterized in Section 2. If X^* and X^{**} are finite-point order compactifications of X, $\sup\{X^*, X^{**}\}$ is given explicitly in terms of X^* and X^{**} in § 3. In § 4 it is shown that if an ordered topological space X has an m-point and an n-point order compactification, then X has a k-point order compactification for each integer k between m and n. The author is indebted to Professor Darrell C. Kent, who provided assistance and encouragement during the preparation of this paper.

Preliminaries. An *ordered topological space*, or simply an *ordered space*, is a triple (X, τ, θ) where τ is a topology on X and θ is the graph of a partial order on X. An *order compactification* of (X, τ, θ) is a pair $((X^*, \tau^*, \theta^*), \kappa)$ such that κ is an order embedding and $((X^*, \tau^*), \kappa)$ is a topological compactification of (X, τ). We will make no distinction between X and $\kappa(X)$. The property of a space being T_2-ordered ($T_{3\cdot5}$-ordered) is equivalent to the space having a closed order (completely regular order) as defined in [**7**]. For other properties of ordered spaces, the reader is referred to [**7**] and [**6**]. All compactifications mentioned will be assumed to be T_2, and all order compactifications will be assumed to be T_2-ordered. If (X^*, κ^*) and (X^{**}, κ^{**}) are order compactifications of ordered space X, then $X^* \geqslant X^{**}$ means there exists a continuous increasing function $f: X^* \to X^{**}$ such that $f \circ \kappa^* = \kappa^{**}$. An ordered space (X, τ, θ) is said to have a *convex topology* if τ has a subbase of open monotone sets. It is known that if X is $T_{3\cdot5}$-ordered, then X has a convex topology.

Magill [**4**] has shown that a locally compact T_2 space (X, τ) has an n-point compactification iff X can be partitioned into $K \cup \{G_i\}_{i=1}^n$ where K is compact, each G_i is open, and $K \cup G_i$ is not compact for any i. Such a collection $\{G_i\}_{i=1}^n$ of disjoint open subsets of X is called an *n-star on X*.

Throughout this paper, i, j, k, m, n, and p shall represent natural numbers, and $\mathscr{U}(x)$ shall represent the neighbourhood filter at x. Square brackets [ ] are used to denote the filter generated by the base enclosed. The supremum $\mathscr{F} \vee \mathscr{G}$ of filters $\mathscr{F}$ and $\mathscr{G}$ exists iff $\varnothing \notin \{F \cap G : F \in \mathscr{F}, G \in \mathscr{G}\}$.

II

If (X^*, τ^*) is an n-point topological compactification of a $T_{3\cdot5}$-ordered space X with $X^* = X \cup \{\omega_i\}_{i=1}^n$, we want to find intrinsic conditions on X which guarantee that the points $\{\omega_i\}_{i=1}^n$ can be ordered properly to make X^* into an order compactification.

Suppose (X,τ,θ) is a compact T_2-ordered space. For $x\in X$, put $\mathscr{U}^{\uparrow}(x) = [\{\text{open increasing neighbourhoods of } x\}]$, with $\mathscr{U}^{\downarrow}(x)$ defined dually. If $\omega_k \in X$ and $\{\omega_k\}\notin\tau$, put $\mathscr{G}_k^{\uparrow} = [\{N\backslash\{\omega_k\}: N\in\mathscr{U}^{\uparrow}(\omega_k)\}]$, and define $\mathscr{G}_k^{\downarrow}$ dually. Then $x\leqslant y$ iff $\mathscr{U}^{\uparrow}(x)\vee\mathscr{U}^{\downarrow}(y)$ exists, $x\leqslant\omega_i$ iff $\mathscr{U}^{\uparrow}(x)\vee\mathscr{G}_i^{\downarrow}$ exists, $\omega_i\leqslant x$ iff $\mathscr{G}_i^{\uparrow}\vee\mathscr{U}^{\downarrow}(x)$ exists, and $\omega_i\leqslant\omega_j$ iff $\mathscr{G}_i^{\uparrow}\vee\mathscr{G}_j^{\downarrow}$ exists.

2·1. THEOREM. *A $T_{3\cdot5}$-ordered space (X,τ,θ) has an n-point order compactification if and only if (X,τ) has an n-point topological compactification $X^* = X\cup\{\omega_i\}_{i=1}^n$ with corresponding n-star $\{G_i\}_{i=1}^n$, and*

$\forall i\in\{1,\ldots,n\}$, $\exists$ *filter $\mathscr{G}_i^{\uparrow}$ on X having a base of τ-open θ-increasing sets*
$\forall i\in\{1,\ldots,n\}$, $\exists$ *filter $\mathscr{G}_i^{\downarrow}$ on X having a base of τ-open θ-decreasing sets*
$\forall x\in X$, $\exists$ *filter $\mathscr{U}^{\uparrow}(x)$ on X having a base of τ-open θ-increasing sets*
$\forall x\in X$, $\exists$ *filter $\mathscr{U}^{\downarrow}(x)$ on X having a base of τ-open θ-decreasing sets*

such that the collection of all of these filters satisfies:

1. $\mathscr{G}_i^{\uparrow}\vee\mathscr{G}_i^{\downarrow} = \mathscr{G}_i \equiv [\{N\subseteq X: (K\cup G_i)\backslash N \text{ is compact}\}]$ *where* $K = X\backslash(\bigcup_{j=1}^n G_j)$, *and* $\mathscr{U}^{\uparrow}(x)\vee\mathscr{U}^{\downarrow}(x) = \mathscr{U}(x)\equiv$ *the neighbourhood filter at* x.
2. $\mathscr{U}^{\uparrow}(x)\vee\mathscr{U}^{\downarrow}(y)$ *exists* $\Leftrightarrow (x,y)\in\theta$.
3. (*a*) $\mathscr{G}_i^{\uparrow}\vee\mathscr{G}_j^{\downarrow}$ *exists* $\Rightarrow \mathscr{G}_i^{\uparrow}\leqslant\mathscr{G}_j$ *and* $\mathscr{G}_j^{\downarrow}\leqslant\mathscr{G}_i$.
   (*b*) $\mathscr{G}_i^{\uparrow}\vee\mathscr{U}^{\downarrow}(x)$ *exists* $\Rightarrow \mathscr{G}_i^{\uparrow}\leqslant\mathscr{U}(x)$ *and* $\mathscr{U}^{\downarrow}(x)\leqslant\mathscr{G}_i$.
   (*c*) $\mathscr{U}^{\uparrow}(x)\vee\mathscr{G}_j^{\downarrow}$ *exists* $\Rightarrow \mathscr{U}^{\uparrow}(x)\leqslant\mathscr{G}_j$ *and* $\mathscr{G}_j^{\downarrow}\leqslant\mathscr{U}(x)$.

The proof of Theorem 2·1 will be presented after Lemma 2·2.

If (X,τ,θ) is an ordered space with n-star $\{G_i\}_{i=1}^n$, a collection $\{\mathscr{G}_i^{\uparrow}, \mathscr{G}_i^{\downarrow}, \mathscr{U}^{\uparrow}(x), \mathscr{U}^{\downarrow}(x): i\in\{1,\ldots,n\}, x\in X\}$ of filters on X having bases of τ-open monotone sets as in Theorem 2·1 and satisfying the conditions 1–3 in Theorem 2·1 will be called an *n-point order compactifying family on X*. Thus, *(X,τ,θ) has an n-point order compactification iff there exists an n-point order compactifying family on X*. A short proof of the lemma below is given in ([**8**], lemma 2·10).

2·2. LEMMA. *Let X be a $T_{3\cdot5}$-ordered space with an n-point topological compactification $X^* = X\cup\{\omega_i\}_{i=1}^n$ corresponding to n-star $\{G_i\}_{i=1}^n$, and suppose $\{\mathscr{G}_i^{\uparrow}, \mathscr{G}_i^{\downarrow}, \mathscr{U}^{\uparrow}(x), \mathscr{U}^{\downarrow}(x)\}$ is an n-point order compactifying family on X. If $x,y,\in X$ and $x\nleq y$, then for each $i\in\{1,\ldots,n\}$, either (a) $\mathscr{U}^{\uparrow}(x)\vee\mathscr{G}_i$ does not exist, or (b) $\mathscr{U}^{\downarrow}(y)\vee\mathscr{G}_i$ does not exist.*

Proof of theorem 2·1. First suppose the $T_{3\cdot5}$-ordered space (X,τ,θ) has an n-point order compactification $X^* = X\cup\{\omega_i\}_{i=1}^n$. For $x\in X$, putting $\mathscr{U}^{\uparrow}(x) = [\{N\cap X: N \text{ is a } \tau^*\text{-open } \theta^*\text{-increasing neighbourhood of } x\}]$, and putting $\mathscr{G}_i^{\uparrow} = [\{N\cap X: N \text{ is a } \tau^*\text{-open } \theta^*\text{-increasing neighbourhood of } \omega_i\}]$, with $\mathscr{U}^{\downarrow}(x)$ and $\mathscr{G}_i^{\downarrow}$ defined dually, we can see that these filters form an n-point order compactifying family on X.

Conversely, let $X^* = X\cup\{\omega_i\}_{i=1}^n$ be the n-point topological compactification of X whose existence is guaranteed by the existence of the n-point order compactifying family $\{\mathscr{G}_i^{\uparrow}, \mathscr{G}_i^{\downarrow}, \mathscr{U}^{\uparrow}(x), \mathscr{U}^{\downarrow}(x)\}$. Define a relation $\leqslant^*$ on X^* by putting $x\leqslant^* y$ iff $\mathscr{U}^{\uparrow}(x)\vee\mathscr{U}^{\downarrow}(y)$ exists, $x\leqslant^*\omega_i$ iff $\mathscr{U}^{\uparrow}(x)\vee\mathscr{G}_i^{\downarrow}$ exists, $\omega_i\leqslant^* x$ iff $\mathscr{G}_i^{\uparrow}\vee\mathscr{U}^{\downarrow}(x)$ exists, and $\omega_i\leqslant^*\omega_j$ iff $\mathscr{G}_i^{\uparrow}\vee\mathscr{G}_j^{\downarrow}$ exists (for $x,y\in X$ and ω_i, $\omega_j\in X^*\backslash X$). We claim that $\leqslant^*$ is a partial order. *Reflexivity* is immediate. *Antisymmetry*: If $x\leqslant^* y, y\leqslant^* x$ for $x,y\in X$, then $x\leqslant y$ and $y\leqslant x$ in X, and thus $x=y$. If $\omega_i\leqslant^*\omega_j$, $\omega_j\leqslant^*\omega_i$, then $\mathscr{G}_i^{\uparrow}\vee\mathscr{G}_j^{\downarrow}$ and $\mathscr{G}_j^{\uparrow}\vee\mathscr{G}_i^{\downarrow}$ exist, so $\mathscr{G}_i^{\uparrow}\leqslant\mathscr{G}_j$, $\mathscr{G}_j^{\downarrow}\leqslant\mathscr{G}_i$, $\mathscr{G}_j^{\uparrow}\leqslant\mathscr{G}_i$, and $\mathscr{G}_i^{\downarrow}\leqslant\mathscr{G}_j$. Thus, $\mathscr{G}_i^{\uparrow}\vee\mathscr{G}_i^{\downarrow} = \mathscr{G}_i\leqslant\mathscr{G}_j$ and $\mathscr{G}_j^{\uparrow}\vee\mathscr{G}_j^{\downarrow} = \mathscr{G}_j\leqslant\mathscr{G}_i$ so that $\mathscr{G}_i = \mathscr{G}_j$. That X^* is T_2 implies $\omega_i = \omega_j$. A similar argument shows that $\omega_i\leqslant^* x, x\leqslant^*\omega_i$ would imply $\mathscr{U}(x) = \mathscr{G}_i$, which is impossible since $\mathscr{U}(x)$ has a base of compact sets and $\mathscr{G}_i$ does not. *Transitivity*: *Case* 1: $x \leqslant^* y$, $y \leqslant^* \omega_i$. Then the existence of $\mathscr{U}^\uparrow(x) \vee \mathscr{U}^\downarrow(y)$ and $\mathscr{U}^\uparrow(y) \vee \mathscr{G}_i^\downarrow$ implies that both $\mathscr{U}^\uparrow(x)$ and $\mathscr{G}_i^\downarrow$ are contained in $\mathscr{U}(y)$, so that $\mathscr{U}^\uparrow(x) \vee \mathscr{G}_i^\downarrow$ exists, i.e. $x \leqslant^* \omega_i$.

Case 2: $x \leqslant^* \omega_i, \omega_i \leqslant^* y$. By antisymmetry, $x \neq y$. Since $x \leqslant^* \omega_i \Rightarrow \mathscr{U}^\uparrow(x) \vee \mathscr{G}_i$ exists and $\omega_i \leqslant^* y \Rightarrow \mathscr{U}^\downarrow(y) \vee \mathscr{G}_i$ exists, $x \nless y$ would contradict Lemma 2·2.

Case 3: $x \leqslant^* \omega_i, \omega_i \leqslant^* \omega_j$. Then $\mathscr{U}^\uparrow(x) \vee \mathscr{G}_i^\downarrow$ and $\mathscr{G}_i^\uparrow \vee \mathscr{G}_j^\downarrow$ exist, which imply, respectively, that $\mathscr{U}^\uparrow(x) \leqslant \mathscr{G}_i$ and $\mathscr{G}_j^\downarrow \leqslant \mathscr{G}_i$, so that $\mathscr{U}^\uparrow(x) \vee \mathscr{G}_j^\downarrow$ exists, i.e. $x \leqslant^* \omega_j$. The remaining cases are similar. Thus $\leqslant^*$ is a partial order.

We will denote the graph of $\leqslant^*$ by θ^*. From the definition of $x \leqslant^* y$ and condition 2 in the definition of order compactifying family, it is immediate that the order θ^*, when restricted to X, agrees with the original order θ. All that remains is to show that (X^*, τ^*, θ^*) is T_2-ordered. Suppose $\alpha, \beta \in X^*$ and $\alpha \nleqslant^* \beta$. To eliminate some repetitive cases, define $\mathscr{U}_\alpha^\uparrow = \mathscr{U}^\uparrow(\alpha)$ if $\alpha \in X^* \cap X$ and $\mathscr{U}_\alpha^\uparrow = \mathscr{G}_i^\uparrow$ if $\alpha = \omega_i \in X^* \backslash X$, with $\mathscr{U}_\alpha^\downarrow$ and $\mathscr{U}_\alpha$ defined analogously. Now $\alpha \nleqslant^* \beta \Rightarrow \mathscr{U}_\alpha^\uparrow \vee \mathscr{U}_\beta^\downarrow$ does not exist $\Rightarrow \exists$ increasing $N_\alpha^\uparrow \in \mathscr{U}_\alpha^\uparrow$, decreasing $N_\beta^\downarrow \in \mathscr{U}_\beta^\downarrow$ such that $N_\alpha^\uparrow \cap N_\beta^\downarrow = \varnothing$. Observe that $i(N_\alpha^\uparrow \cup \{\alpha\})$ is a θ^*-increasing neighbourhood of α and $d(N_\beta^\downarrow \cup \{\beta\})$ is a θ^*-decreasing neighbourhood of β (monotone hulls taken in X^*). These neighbourhoods are disjoint as required, for

$$\begin{aligned} &i(N_\alpha^\uparrow \cup \{\alpha\}) \cap d(N_\beta^\downarrow \cup \{\beta\}) \\ &\quad = [i(N_\alpha^\uparrow) \cap d(N_\beta^\downarrow)] \cup [i(N_\alpha^\uparrow) \cap d(\beta)] \cup [i(\alpha) \cap d(N_\beta^\downarrow)] \cup [i(\alpha) \cap d(\beta)], \end{aligned}$$

and it is easily shown that each of these last four bracketed terms must be empty. Thus X^* is T_2-ordered. ∎

We now characterize ordered spaces for which *every* order compactification is a finite-point compactification. For $A, B \subseteq X$, we will say A and B are *monotonically separated* if there exists a monotone continuous function $f: X \to \mathbf{R}$ such that $f(A) = 0$ and $f(B) = 1$. $\beta_o X$ denotes the largest order compactification of X, known as the Stone-Čech-, or the Nachbin-order compactification of X.

2·3. Theorem. *For a $T_{3\cdot 5}$-ordered space X, the following are equivalent*:

(*a*) *$\beta_o X \backslash X$ has n points.*

(*b*) *n is the largest cardinal number for which X has an n-point order compactification.*

(*c*) *There exist n closed, non-compact sets in X which are monotonically separated in pairs, but no such collection of $n+1$ sets.*

Proof. Clearly (*a*) and (*b*) are equivalent. After observing that if $a_1, \ldots, a_n$ are elements of a compact T_2-ordered space, then there exist compact neighbourhoods U_i of a_i $(i = 1, \ldots, n)$ which are monotonically separated in pairs, the equivalence of (*a*) and (*c*) is a direct generalization of Firby's proof of the corresponding topological result ([**3**], theorem 2·2). ∎

Note that the embedding of X into the order compactification X^* generated from the order compactifying family is id_X. If the order on a space is discrete, an n-point order compactifying family can be obtained by taking $\mathscr{G}_i^\uparrow = \mathscr{G}_i^\downarrow = \mathscr{G}_i$ and $\mathscr{U}^\uparrow(x) = \mathscr{U}^\downarrow(x) = \mathscr{U}(x)$, and the results of 2·1 reduce to Magill's results on n-point topological compactifications. Also, ω_i is minimal in X^* iff we may take the filter $\mathscr{G}_i^\downarrow$ from the corresponding order compactifying family to be equal to $\mathscr{G}_i$.

III

Blatter [1] has shown that every non-empty set of T_2-ordered compactifications of a $T_{3\cdot5}$-ordered space has a supremum (unique up to equivalence) with respect to the usual ordering. Here we give an explicit description of the supremum of two finite-point order compactifications of X. We say two n-stars on X are *equivalent* if they generate equivalent compactifications of X. Recall that for an n-star $\{G_i\}_{i=1}^n$, $\mathscr{G}_i$ is defined to be $\{N \subseteq X : (K \cup G_i)\backslash N \text{ is compact}\}$ where $K = X\backslash(\bigcup_{j=1}^n G_j)$. We define two stars $\{G_i\}_{i=1}^n$ and $\{H_j\}_{j=1}^m$ to be a *reduced pair* if $\bigcup_{i=1}^n G_i = \bigcup_{j=1}^m H_j$ and $G_i \cap H_j \neq \varnothing$ iff $\mathscr{G}_i \vee \mathscr{H}_j$ exists.

3·1. PROPOSITION. *Given any n-star $\{G_i\}_{i=1}^n$ on X and any m-star $\{H_i\}_{i=1}^m$ on X, there exists a reduced pair of stars $\{\hat{G}_i\}_{i=1}^n$ and $\{\hat{H}_i\}_{i=1}^m$ on X such that $\{\hat{G}_i\}_{i=1}^n$ is equivalent to $\{G_i\}_{i=1}^n$ and $\{\hat{H}_i\}_{i=1}^m$ is equivalent to $\{H_i\}_{i=1}^m$.*

Proof. Let $K_1 = X\backslash(\bigcup_{i=1}^n G_i)$ and $K_2 = X\backslash(\bigcup_{i=1}^m H_i)$. If $K_1 \neq K_2$, put $K = K_1 \cup K_2$, $\hat{G}_i = G_i\backslash K$, and $\hat{H}_i = H_i\backslash K$. If $\hat{G}_i = \varnothing$, then $G_i \subseteq K_2 \Rightarrow \mathrm{cl}(G_i)$ is compact. But $X\backslash(K_1 \cup G_i)$ is open, so that $\mathrm{cl}(G_i) \subseteq K_1 \cup G_i$, contrary to $K_1 \cup G_i = K_1 \cup \mathrm{cl}(G_i)$ being non-compact. Thus, each $\hat{G}_i$ is non-empty. Observe that K is compact, $X = K \cup \{\hat{G}_i\}_{i=1}^n$, and $\{\hat{G}_i\}_{i=1}^n$ is a collection of disjoint open sets. Since $K_1 \cup G_i$ is not compact but $K_1 \cup G_i \cup \{\omega_i\}$ is (where ω_i is the compactification point corresponding to G_i in Magill's compactification), there is an ultrafilter $\mathscr{F} \to \omega_i$ with $K \cup \hat{G}_i \in \mathscr{F}$, so $K \cup \hat{G}_i$ is not compact. Thus, $\{\hat{G}_i\}_{i=1}^n$ is an n-star. Similarly, $\{\hat{H}_i\}_{i=1}^m$ is an m-star, and $X\backslash(\bigcup_{i=1}^n \hat{G}_i) = X\backslash(\bigcup_{i=1}^m \hat{H}_i) = K$. The n-stars $\{G_i\}_{i=1}^n$ and $\{\hat{G}_i\}_{i=1}^n$ are equivalent if and only if $[\{N \subseteq G_i : (K_1 \cup G_i)\backslash N \text{ is compact}\}] = [\{N \subseteq G_i : (K \cup \hat{G}_i)\backslash N \text{ is compact}\}]$, and this is easily verified.

Now assume that the original stars $\{G_i\}_{i=1}^n$ and $\{H_i\}_{i=1}^m$ satisfied $\bigcup_{i=1}^n G_i = \bigcup_{i=1}^m H_i$. We claim that if G_i is replaced by any $\tilde{G}_i \subseteq G_i$ such that $(K \cup G_i)\backslash\tilde{G}_i$ is compact, the resulting collection is an equivalent n-star. Since $(K \cup G_i)\backslash\tilde{G}_i$ is compact, $[(K \cup G_i)\backslash\tilde{G}_i] \cup [K \cup (\bigcup_{j\neq i} G_j)]$ is closed. Now $\tilde{G}_i$, being the compliment of this set, is open. Clearly $\tilde{G}_i$ is disjoint from G_j for $j \neq i$. If $K \cup \tilde{G}_i$ were compact, then $(K \cup \tilde{G}_i) \cup [(K \cup G_i)\backslash\tilde{G}_i]$ would be compact. But this union is $K \cup G_i$, which is not compact, so $K \cup \tilde{G}_i$ is not compact. Thus, replacing G_i by $\tilde{G}_i$ does give a second n-star. Since G_i and $\tilde{G}_i$ are both punctured neighbourhoods of ω_i, the two n-stars are equivalent.

Now if $\mathscr{G}_i \vee \mathscr{H}_j$ does not exist, then $\exists \tilde{G}_i \in \mathscr{G}_i$, $\tilde{H}_j \in \mathscr{H}_j$ such that $\tilde{G}_i \cap \tilde{H}_j = \varnothing$. Replacing G_i and H_j by $\tilde{G}_i$ and $\tilde{H}_j$ respectively gives an equivalent pair of stars satisfying the condition that $\mathscr{G}_i \vee \mathscr{H}_j$ exists iff $\tilde{G}_i \cap \tilde{H}_j \neq \varnothing$. ∎

It is easily seen that if $\{G_i\}_{i=1}^n$ and $\{H_i\}_{i=1}^m$ are a reduced pair of stars on X, then $\mathscr{C}' = \{G_i \cap H_j : G_i \cap H_j \neq \varnothing\}$ is a p-star on X where p, the cardinality of $\mathscr{C}'$, satisfies $\max\{m, n\} \leqslant p \leqslant mn$.

3·2. THEOREM. *Suppose X has an n-point order compactifying family $\{\mathscr{G}_i^\uparrow, \mathscr{G}_i^\downarrow, \mathscr{U}_n^\uparrow(x), \mathscr{U}_n^\downarrow(x)\}$ and m-point order compactifying family $\{\mathscr{H}_i^\uparrow, \mathscr{H}_i^\downarrow, \mathscr{U}_m^\uparrow(x), \mathscr{U}_m^\downarrow(x)\}$ associated with an n-star $\{G_i\}_{i=1}^n$ and m-star $\{H_i\}_{i=1}^m$, where $\{G_i\}_{i=1}^n$ and $\{H_i\}_{i=1}^m$ form a reduced pair. Then the collection*

$$\mathscr{C} \equiv \{\mathscr{G}_i^\uparrow \vee \mathscr{H}_j^\uparrow, \mathscr{G}_i^\downarrow \vee \mathscr{H}_j^\downarrow, \mathscr{U}_n^\uparrow(x) \vee \mathscr{U}_m^\uparrow(x), \mathscr{U}_n^\downarrow(x) \vee \mathscr{U}_m^\downarrow(x) : \mathscr{G}_i \vee \mathscr{H}_j \text{ exists}\}$$

is a p-point order compactifying family on X, with $\max\{m, n\} \leqslant p \leqslant mn$.

Proof. Since $\mathscr{C}$ has p elements, let us name them $\{J_i\}_{i=1}^p$. If $J_k = G_i \cap H_j$, define $\mathscr{J}_k = \mathscr{G}_i \vee \mathscr{H}_j$, $\mathscr{J}_k^\uparrow = \mathscr{G}_i^\uparrow \vee \mathscr{H}_j^\uparrow$, $\mathscr{U}^\uparrow(x) = \mathscr{U}_n^\uparrow(x) \vee \mathscr{U}_m^\uparrow(x)$, with $\mathscr{J}_k^\downarrow$ and $\mathscr{U}^\downarrow(x)$ defined dually. Observe that $\mathscr{U}^\uparrow(x) \vee \mathscr{U}^\downarrow(x) = \mathscr{U}(x)$, and $\mathscr{U}^\uparrow(x) \vee \mathscr{U}^\downarrow(y)$ exists iff $x \leqslant y$. Thus, we must only show

(1) $\mathscr{J}_k \equiv \mathscr{G}_i \vee \mathscr{H}_j = [\{N \subseteq X : [K \cup (G_i \cap H_j)]\backslash N \text{ is compact}\}]$,
(2) $\mathscr{J}_i^\uparrow \vee \mathscr{J}_k^\downarrow$ exists $\Rightarrow \mathscr{J}_i^\uparrow \leqslant \mathscr{J}_k$ and $\mathscr{J}_k^\downarrow \leqslant \mathscr{J}_i$,
(3) $\mathscr{J}_i^\uparrow \vee \mathscr{U}^\downarrow(x)$ exists $\Rightarrow \mathscr{J}_i^\uparrow \leqslant \mathscr{U}(x)$ and $\mathscr{U}^\downarrow(x) \leqslant \mathscr{J}_i$,
(4) $\mathscr{U}^\uparrow(x) \vee \mathscr{J}_i^\downarrow$ exists $\Rightarrow \mathscr{U}^\uparrow(x) \leqslant \mathscr{J}_i$ and $\mathscr{J}_i^\downarrow \leqslant \mathscr{U}(x)$.

For (1), suppose first that $N \in \mathscr{J}_k$ is a basic set, i.e. $N = N_g \cap N_h$ where $(K \cup G_i)\backslash N_g$ and $(K \cup H_j)\backslash N_h$ are compact. Because $K \cup H_j$ and $K \cup G_i$ are closed,

$$[K \cup (G_i \cap H_j)]\backslash N = [\{(K \cup G_i)\backslash N_g\} \cap (K \cup H_j)] \cup [\{(K \cup H_j)\backslash N_h\} \cap (K \cup G_i)]$$

is compact, and thus $\mathscr{J}_k \leqslant [\{N \subseteq X : [K \cup (G_i \cap H_j)]\backslash N \text{ is compact}\}]$. Conversely, suppose $[K \cup (G_i \cap H_j)]\backslash N$ is compact. Put $N_g = N \cup (G_i\backslash H_j)$ and $N_h = N \cup (H_j\backslash G_i)$. Now $N_g \cap N_h = N$ and $(K \cup G_i)\backslash N_g = [K \cup (G_i \cap H_j)]\backslash N$, which is compact. Similarly, $(K \cup H_j)\backslash N_h$ is compact. This completes the proof of (1).

For (2), we must show that $\mathscr{J}_i^\uparrow \vee \mathscr{J}_k^\downarrow$ exists $\Rightarrow \mathscr{J}_i^\uparrow \leqslant \mathscr{J}_k$ and $\mathscr{J}_k^\downarrow \leqslant \mathscr{J}_i$. Renumber indices (possibly introducing some repetition in the collection $\{G_i\}_{i=1}^p$, since $p \geqslant n$) so that $\mathscr{J}_i = \mathscr{G}_i \vee \mathscr{H}_i$ and $\mathscr{J}_k = \mathscr{G}_k \vee \mathscr{H}_k$. Now $\mathscr{J}_i^\uparrow \vee \mathscr{J}_k^\downarrow$ exists $\Rightarrow \mathscr{G}_i^\uparrow \vee \mathscr{G}_k^\downarrow$ exists $\Rightarrow$ $\mathscr{G}_i^\uparrow \leqslant \mathscr{G}_k$. Thus $\mathscr{G}_i^\uparrow \leqslant \mathscr{G}_k \vee \mathscr{H}_k = \mathscr{J}_k$ and similarly, $\mathscr{G}_k^\downarrow \leqslant \mathscr{J}_i, \mathscr{H}_i^\uparrow \leqslant \mathscr{J}_k$ and $\mathscr{H}_k^\downarrow \leqslant \mathscr{J}_i$, whence $\mathscr{J}_i^\uparrow = \mathscr{G}_i^\uparrow \vee \mathscr{H}_i^\uparrow \leqslant \mathscr{J}_k$ and $\mathscr{J}_k^\downarrow \leqslant \mathscr{J}_i$. This completes the proof of (2). Similar arguments work for (3) and (4). ∎

In X', the order compactification generated from the order compactifying family in Theorem 3·2, define μ_{ij} to be the compactification point having punctured neighbourhood filter $\mathscr{G}_i \vee \mathscr{H}_j$. Thus, $X' = X \cup \{\mu_{ij} : \mathscr{G}_i \vee \mathscr{H}_j \text{ exists}\}$, and the compactification point μ_{ij} in X' is associated with the compactification points ω_i in X^* and ν_j in X^{**}. With this notation, we will see in Theorem 3·4 that $X' = \sup\{X^*, X^{**}\}$.

3·3. Lemma. *If (X^*, τ^*, θ^*) and $(X^{**}, \tau^{**}, \theta^{**})$ are finite-point order compactifications of (X, τ, θ), and if (X', τ', θ') is the order compactification generated from X^* and X^{**} as in Theorem* 3·2, *with $X' = X \cup \{\mu_{ij} : \mathscr{G}_i \vee \mathscr{H}_j$ exists$\}$, then*

$$\begin{aligned} x <' \mu_{ij} &\Leftrightarrow x <^* \omega_i \quad \textit{and} \quad x <^{**} \nu_j, \\ \mu_{ij} <' x &\Leftrightarrow \omega_i <^* x \quad \textit{and} \quad \nu_j <^{**} x, \\ \mu_{ij} <' \mu_{kl} &\Leftrightarrow \omega_i <^* \omega_k \quad \textit{and} \quad \nu_j <^{**} \nu_l. \end{aligned}$$

Proof. We will prove only the last statement above. Considering the order compactifying family as in Theorem 3·2, $\mu_{ij} \leqslant' \mu_{kl} \Leftrightarrow \mathscr{G}_i^\uparrow \vee \mathscr{H}_j^\uparrow \vee \mathscr{G}_k^\downarrow \vee \mathscr{H}_l^\downarrow$ exists $\Rightarrow$ $\mathscr{G}_i^\uparrow \vee \mathscr{G}_k^\downarrow$ and $\mathscr{H}_j^\uparrow \vee \mathscr{H}_l^\downarrow$ exist $\Leftrightarrow \omega_i <^* \omega_k$ and $\nu_j <^{**} \nu_l$.

Conversely, if $\omega_i <^* \omega_k$ and $\nu_j <^{**} \nu_l$, then $\mathscr{G}_i^\uparrow \vee \mathscr{G}_k^\downarrow$ and $\mathscr{H}_j^\uparrow \vee \mathscr{H}_l^\downarrow$ exist. Thus $\mathscr{G}_k^\downarrow \leqslant \mathscr{G}_i \leqslant \mathscr{G}_i \vee \mathscr{H}_j$ (and $\mathscr{G}_i \vee \mathscr{H}_j$ must exist since this is the punctured neighbourhood filter of μ_{ij}) and $\mathscr{H}_l^\downarrow \leqslant \mathscr{H}_j \leqslant \mathscr{G}_i \vee \mathscr{H}_j$. Also, since $\mathscr{G}_i^\uparrow \leqslant \mathscr{G}_i \leqslant \mathscr{G}_i \vee \mathscr{H}_j$ and $\mathscr{H}_j^\uparrow \leqslant \mathscr{H}_j \leqslant \mathscr{G}_i \vee \mathscr{H}_j$, we have $\mathscr{G}_i^\uparrow \vee \mathscr{H}_j^\uparrow \vee \mathscr{G}_k^\downarrow \vee \mathscr{H}_l^\downarrow$ exists, so $\mu_{ij} \leqslant' \mu_{kl}$.

The other statements can be proved similarly, using the fact that $\mathscr{U}_m(x) = \mathscr{U}_n(x) = \mathscr{U}(x)$. ∎

3·4. Theorem. *If X^* and X^{**} are finite-point order compactifications of X, and X' is the order compactification of X corresponding to the order compactifying family described in Lemma* 3·3, *then $X' = \sup\{X^*, X^{**}\}$.*

Proof. First we will show that $X' \geqslant X^*$. Since both X^* and X' are order compactifications generated by order compactifying families, the associated embeddings κ^* and κ' of X into the compactification spaces are simply id_X. Now recalling that $X' = X \cup \{\mu_{ij}\}$ and $X^* = X \cup \{\omega_i\}_{i=1}^n$, define $f: X' \to X^*$ by $f(x) = x$ for $x \in X' \cap X$ and $f(\mu_{ij}) = \omega_i$. Then $f \circ \kappa' = \kappa^* = \mathrm{id}_X$, and a routine exercise verifies that f is continuous and increasing. Similarly, $X' \geqslant X^{**}$.

Suppose (X'', κ'') is any other upper bound of X^* and X^{**}. Since $X'' \geqslant X^*$ and $X'' \geqslant X^{**}$, there exist continuous increasing maps $\phi^*: X'' \to X^*$ and $\phi^{**}: X'' \to X^{**}$ such that $\phi^* \circ \kappa'' = \kappa^* = \mathrm{id}_X$ and $\phi^{**} \circ \kappa'' = \kappa^{**} = \mathrm{id}_X$. Define $\phi: X'' \to X'$ by $\phi(x) = \kappa''^{-1}(x)$ for $x \in \kappa''(X)$, and

$$\phi(\alpha) = \mu_{ij} \quad \text{where} \quad \begin{cases} \phi^*(\alpha) = \omega_i \\ \phi^{**}(\alpha) = \nu_j \end{cases} \quad \text{for} \quad \alpha \in X'' \backslash \kappa''(X).$$

Observe that for $x \in X$, $\phi \circ \kappa''(x) = \kappa''^{-1} \circ \kappa''(x) = \mathrm{id}_X(x)$, so that $\phi \circ \kappa'' = \kappa' = \mathrm{id}_X$ as needed. It remains to show that ϕ is continuous and increasing.

ϕ is continuous. If N is an open neighbourhood of $x \in X' \cap X$ such that $N \subseteq X' \cap X$, then $\phi^{-1}(N) = \kappa''(N)$ which is open in X'' since κ'' is a homeomorphism. If $\mu_{ij} \in X' \backslash X$, a basic open neighbourhood N of μ_{ij} is of the form $(G_i \cap H_j) \cup \{\mu_{ij}\}$ where G_i is an open set in $\mathscr{G}_i$ and H_j is an open set in $\mathscr{H}_j$. Thus, $\phi^{-1}(N) = [\phi^{-1}(G_i) \cap \phi^{-1}(H_j)] \cup \phi^{-1}(\mu_{ij})$. Since $\phi^{-1}(G_i) = \kappa''(G_i)$ is an open neighbourhood of each of its points, it remains only to show that $\phi^{-1}(N)$ is a neighbourhood of each point of $\phi^{-1}(\mu_{ij})$. Since $\phi^* \circ \kappa'' = \kappa^* = \mathrm{id}_X$, we have $\kappa'' = \phi^{*-1}$ so that $\phi^{-1}(G_i) = \kappa''(G_i) = \phi^{*-1}(G_i)$. Also, noticing that $\phi^{-1}(\mu_{ij}) = \alpha$ where $\phi^*(\alpha) = \omega_i$, we see that $\phi^{-1}(\mu_{ij}) = \phi^{*-1}(\omega_i) = \alpha$. Continuity of ϕ^* implies that $\phi^{*-1}(G_i \cup \omega_i) = \phi^{*-1}(G_i) \cup \phi^{*-1}(\omega_i) = \phi^{-1}(G_i) \cup \phi^{-1}(\mu_{ij})$ is an open neighbourhood of $\phi^{-1}(\mu_{ij})$. Similarly, $\phi^{-1}(H_j) \cup \phi^{-1}(\mu_{ij})$ is an open neighbourhood of $\phi^{-1}(\mu_{ij})$ in X''.

ϕ is increasing. Suppose $\alpha \leqslant'' \beta$ in X''. We consider only the case where $\alpha, \beta \in X'' \backslash X$. Then $\phi(\alpha) = \mu_{ij}$ and $\phi(\beta) = \mu_{kl}$ where $\phi^*(\alpha) = \omega_i$, $\phi^{**}(\alpha) = \nu_j$, $\phi^*(\beta) = \omega_k$, and $\phi^{**}(\beta) = \nu_l$. Since ϕ^* and ϕ^{**} are increasing and $\alpha \leqslant'' \beta$, we have $\omega_i \leqslant^* \omega_k$ and $\nu_j \leqslant^{**} \nu_l$, so by Lemma 3·3, $\mu_{ij} \leqslant' \mu_{kl}$. ∎

A simple induction argument shows that the results of 3·3 and 3·4 generalize to any finite set of finite-point order compactifications.

Let topsup$\{X^*, X^{**}\}$ represent the supremum with respect to the ordering of T_2 topological compactifications of X, while sup$\{X^*, X^{**}\}$ represents the supremum with respect to the ordering of T_2-ordered order compactifications of X.

3·5. THEOREM. *If (X^*, τ^*, θ^*) and $(X^{**}, \tau^{**}, \theta^{**})$ are finite-point order compactifications of (X, τ, θ), and $(X', \tau', \theta') = \sup\{(X^*, \tau^*, \theta^*), (X^{**}, \tau^{**}, \theta^{**})\}$, then $(X', \tau') = \operatorname{topsup}\{(X^*, \tau^*), (X^{**}, \tau^{**})\}$.*

Proof. Clearly $(X', \tau', \theta') \geqslant (X^*, \tau^*, \theta^*)$, $(X^{**}, \tau^{**}, \theta^{**})$ implies $(X', \tau') \geqslant (X^*, \tau^*)$, (X^{**}, τ^{**}). If (X'', τ'') is any other upper bound of (X^*, τ^*) and (X^{**}, τ^{**}), then $\phi: X'' \to X'$ as defined in the proof of 3·4 guarantees that $X'' \geqslant X'$. ∎

Thus, if (X^*, τ^*) and (X^{**}, τ^{**}) are finite-point topological compactifications of (X, τ, θ) that admit orders θ^* and θ^{**} that make (X^*, τ^*, θ^*) and $(X^{**}, \tau^{**}, \theta^{**})$ T_2-ordered compactifications of X, then topsup$\{(X^*, \tau^*), (X^{**}, \tau^{**})\}$ also admits an order that makes it into a T_2-ordered compactification of X.

IV

If a topological space has an n-point compactification, then it has a k-point compactification for each $k \in [1, n]$ but $\mathbf{R}$ with the usual topology and order shows that this statement is false for order compactifications.

Let (X^*, κ^*) and (X', κ') be two finite-point order compactifications of X, with $X^* = X \cup \{\omega_i\}_{i=1}^n$, and $X' = X \cup \{\mu_i\}_{i=1}^m$, and with $X' \geqslant X^*$. Then there exists a continuous increasing function $f: X' \to X^*$ such that $f \circ \kappa' = \kappa^*$. If f is one-to-one, then $n = m$. Otherwise, there must be an element $\omega \in X^* \backslash X$ such that $f^{-1}(\omega)$ has at least two elements. Observe that $f^{-1}(\omega) \subseteq X' \backslash \kappa'(X)$. Let μ_1 be maximal among $f^{-1}(\omega)$ and let μ_2 be maximal among $f^{-1}(\omega) \backslash \{\mu_1\}$. Identify μ_1 and μ_2 into a single point μ_0 and represent this quotient space by $(Q(X'), \tau'')$. Let

$$
\begin{aligned}
\theta'' = {} & [\theta' \cap (X' \backslash \{\mu_1, \mu_2\})^2] \\
& \cup \{(\mu_0, \alpha) : \alpha \in X' \backslash \{\mu_1, \mu_2\}, \text{ and } (\mu_i, \alpha) \in \theta' \text{ for } i = 1 \text{ or } 2\} \\
& \cup \{(\alpha, \mu_0) : \alpha \in X' \backslash \{\mu_1, \mu_2\}, \text{ and } (\alpha, \mu_i) \in \theta' \text{ for } i = 1 \text{ or } 2\} \\
& \cup \{(\alpha, \beta) : \alpha, \beta \in X' \backslash \{\mu_1, \mu_2\}, (\alpha, \mu_i) \in \theta', \text{ and } (\mu_j, \beta) \in \theta' \text{ for } i, j = 1 \text{ or } 2\} \\
& \cup \{(\mu_0, \mu_0)\}.
\end{aligned}
$$

A somewhat lengthy but routine argument will give the result stated below. Details may be found in ([**8**], proposition 4·6).

4·1. LEMMA. *$(Q(X'), \tau'', \theta'')$ is an $(m-1)$-point order compactification of (X, τ, θ) and $Q(X') \geqslant X^*$.*

4·2. THEOREM. *If (X, τ, θ) has an m-point order compactification and an n-point order compactification, then (X, τ, θ) has a k-point order compactification for each integer k between m and n.*

Proof. Suppose X^* and X^{**} are m- and n-point order compactifications of X, respectively, with $m \leqslant n$. Starting with X^* and $\sup\{X^*, X^{**}\}$, repeatedly apply Lemma 4·1. ∎

This result is not valid if m or $n = \infty$, since $\mathbf{R}$ with the usual topology and discrete order has 1-point, 2-point, and infinite-point order compactifications, but no k-point order compactification for $3 \leqslant k < \infty$.

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