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The Equal Area Zones Property

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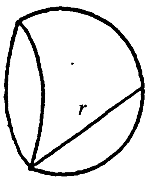
A classical exercise found in many calculus texts is the *zones of a sphere* problem: Verify that if a sphere is sliced by two parallel planes h units apart, then the surface area of the zone between the planes is dependent on h alone, and independent of the location of the planes. While this property of the sphere is usually surprising to students, it has long been known. It is an immediate consequence of the following result of Archimedes (see p. 185ff in [1]) illustrated by the figure below.

Proposition. *The surface area of any segment of a sphere is equal to the area of a circle whose radius is equal to the line drawn from the vertex of the segment to a point on the base circle of the segment of the sphere.*



Surface Area $= \pi r^2$

Viewing the sphere as a surface of revolution with axis of symmetry perpendicular to the parallel planes slicing it, it is natural to extend the routine exercise mentioned above by asking the question of which other surfaces of revolution enjoy this equal area zones property. Clearly the cylinder does. Are there others? Approaching this problem from different viewpoints, we present two solutions which are easily accessible to undergraduates.

To state the problem precisely, suppose $y = g(x)$ is a piecewise smooth nonnegative curve defined over $[a, b]$, and is revolved around the x-axis. A *zone of width* h $(h \leq b - a)$ of the resulting surface is the portion of the surface bounded between planes $x = x_0$ and $x = x_0 + h$, where $x_0 \in [a, b - h]$. Which surfaces of revolution have the *equal area zones* (*E.A.Z.*) *property* that for any width $h \in [0, b - a]$, the surface area

$$S(x,h) = \int_x^{x+h} 2\pi g(t)\sqrt{1 + g'(t)^2}\, dt \qquad (x \in [a, b - h])$$

of a zone of width h is a function of h alone, independent of x?

Two observations might be made about the two surfaces we know to have the E.A.Z. property, the sphere and the cylinder. First, one might observe that their generating curves, the circle and the line, are the only planar curves with nonnegative constant curvature. Secondly, in verifying that these two surfaces have the

E.A.Z property, one discovers that in both cases, the integrand $f(t) = 2\pi g(t)\sqrt{1 + g'(t)^2}$ is constant. This is clearly a sufficient condition for the E.A.Z. property. In our first solution, we will show that, in the case of a smooth curve $g(t)$, it is also necessary.

Suppose $g(t)$ is smooth and nonnegative over $[a, b]$. For a fixed h, the surface area integral $S(x, h)$ above has the form $S(x) = \int_x^{x+h} f(t)\,dt$ where $f(t) = 2\pi g(t)\sqrt{1 + g'(t)^2}$. If the surface generated by $g(t)$ has the E.A.Z. property, then for a fixed h, $S(x)$ is constant, so

$$\frac{d}{dx}S(x) = \frac{d}{dx}\int_x^c f(t)\,dt + \frac{d}{dx}\int_c^{x+h} f(t)\,dt = 0.$$

Applying the Fundamental Theorem of Calculus, we find that $f(x) = f(x + h)$ for any $x \in [a, b - h]$, so $f(x)$ is h-periodic over $[a, b]$. Since this argument holds for any $h \in [0, b - a]$, it follows that the integrand $f(t)$ must be constant.

Now to determine for which curves $y = g(x)$ the integrand $f(x)$ is constant, we must solve the differential equation

$$y\sqrt{1 + y'^2} = c.$$

If $c = 0$, then $g(x) \equiv 0$, so we may assume $c \neq 0$. It follows that $y = g(x) \neq 0$ for any x, and upon squaring both sides of the equation and solving for y', we get

$$y' = \frac{dy}{dx} = \pm\sqrt{\frac{c^2}{y^2} - 1}\,. \tag{1}$$

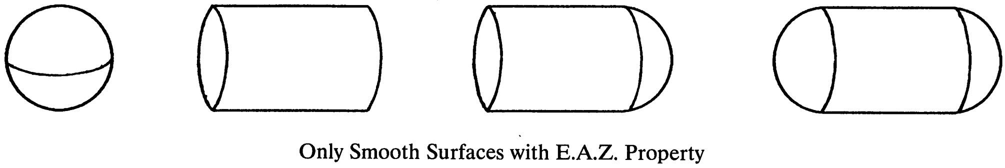
If $y \equiv c$ in some interval, we have $y = g(x)$ is constant, and the corresponding surface over that interval is a cylinder. If $g(x_0) \neq c$ for some $x_0 \in [a, b]$, by continuity, we have $g(x) \neq c$ in some interval I containing x_0, and separation of variables gives

$$\left[\left(\frac{c}{y}\right)^2 - 1\right]^{-1/2} dy = \pm dx.$$

Substituting $u = c/y$ and integrating gives

$$\int \frac{c\,du}{u^2\sqrt{u^2 - 1}} = \pm\int dx.$$

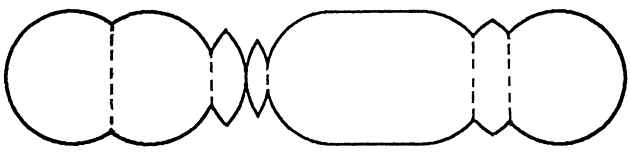
The trigonometric substitution $u = \csc\theta$ yields $cu^{-1}\sqrt{u^2 - 1} = \pm x + k$, or $y^2 + (\pm x + k)^2 = c^2$. From this form, it is clear that the corresponding curve over I is a portion of a semicircle of radius c centered at $(\pm k, 0)$. This, together with the smoothness of g implies that if $g(x_1) = c = g(x_2)$ for $x_1 < x_2$, then $g(x) = c$ for all $x \in [x_1, x_2]$. We conclude that revolving a nonnegative smooth curve yields a surface with the E.A.Z. property iff the surface is a sphere, a cylinder, a "silo", or a "capsule."



Only Smooth Surfaces with E.A.Z. Property

These four surfaces can all be described as zones of a capsule, where a capsule is a cylinder of height $l \geq 0$ with a hemisphere attached to each end. Alternately, the differential equation (1) could be solved by the substitution $u = y^2$, which leads to a separable differential equation which can be integrated using only the power rule. The solutions of (1) above also illustrate the fact that the envelope of a family of solutions to a first order differential equation is again a solution to the differential equation. (e.g., see **[2]**.) The envelope of the spheres of radius c centered at $(\pm k, 0)$ is the cylinder of radius c.

It follows that if $g(x)$ is a continuous nonnegative piecewise smooth curve that generates a surface of revolution with the E.A.Z. property, then the surface must be a "string of beads" where every bead is a zone of a capsule of fixed radius c.



A String of Zones of a Capsule

Rather than a direct proof as presented above, with enough intuition one might pursue a uniqueness argument to show that the zones of a capsule are the only smooth surfaces with the E.A.Z. property. Such an argument follows.

Suppose g is a smooth nonnegative curve over $[a, b]$ generating a surface of revolution with the E.A.Z. property, and with the same zone surface area function $S(h) = 2\pi rh$ as the sphere generated by $c(x) = \sqrt{r^2 - x^2}$. If $g(x_0) > r$, then $2\pi g(x_0)\sqrt{1 + g'(x_0)^2} > 2\pi r = 2\pi c(x_0)\sqrt{1 + c'(x_0)^2}$, and by continuity, there exists $\varepsilon > 0$ such that $2\pi g(x)\sqrt{1 + y'(x)^2} > 2\pi c(x)\sqrt{1 + c'(x)^2}$ for all $x \in I = [x_0 - \varepsilon, x_0 + \varepsilon] \cap [a, b]$. This gives the contradiction that the surface area of any zone generated by g within the interval I has greater area than a zone of equal width generated by $c(x)$. Thus, $g(x) \leq r$ for all x, and therefore, for any x_0, there exists z_0 such that $g(x_0) = c(z_0)$ and $g'(x_0)$ and $c'(z_0)$ do not have opposite signs. If $g'(x_0) \neq c'(z_0)$, then as above we conclude that there exists $\varepsilon > 0$ such that

$$\left|2\pi g(x_0 + \delta)\sqrt{1 + g'(x_0 + \delta)^2} - 2\pi c(z_0 + \delta)\sqrt{1 + c'(z_0 + \delta)^2}\right| > 0 \quad \text{for } |\delta| < \varepsilon,$$

contradicting that g and c generate surfaces with zones of equal area. Thus $g'(x_0) = c'(z_0)$ whenever $g(x_0) = c(z_0)$. It follows that on any interval on which g is invertible, that is, on which $g'(x) \neq 0$, or equivalently, $g(x) \neq r$, g agrees with a horizontal translation of a portion of the curve c. Using a smoothness argument as in the previous solution, we conclude that the surface generated by g must be a zone of a capsule.

REFERENCES

1. E. J. Dijksterhuis, *Archimedes*, Princeton University Press, Princeton, N.J., 1987.
2. Ray Redheffer, *Differential Equations, Theory and Applications*, Jones and Bartlett Publishers, Boston, 1991.

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