

THE NUMBER OF CONVEX SETS IN A PRODUCT OF TOTALLY ORDERED SETS

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ABSTRACT. We give a formula for the number of convex sets in a product of two finite totally ordered sets with the product order and discuss related counting problems.

1. Introduction. The set $\{1, 2, \dots, n\}$, together with the natural order $1 < 2 < \dots < n$, will be denoted $[n]$. All products $[n] \times [m]$ will carry the product order (componentwise order), which is defined by $(a, b) \leq (x, y)$, if and only if $a \leq x$ and $b \leq y$. A set C in a poset X is *convex* if $x, z \in C$ and $x \leq y \leq z$ imply $y \in C$. Alternately, $C \subseteq X$ is convex if $C = i(C) \cap d(C)$ where $i(C) = \{y \in X : \exists c \in C \text{ with } y \geq c\}$ is the increasing hull of C , and $d(C) = \{y \in X : \exists c \in C \text{ with } y \leq c\}$ is the decreasing hull of C .

Our investigations were motivated by the classical questions in topology of finding the number $\text{Top}(n)$ of topologies on an n -element set. The values of $\text{Top}(n)$ are known for $n \leq 18$. The works of Ern e [3, 4] and the references therein are excellent sources. Moving to the category of ordered topological spaces, a very reasonable and commonly assumed compatibility condition between the topology τ and order \leq on set X is that each point have a neighborhood base of convex open sets. Such topologies are called *convex topologies*. The seminal work by Nachbin [6] provides basic details of ordered topological spaces. An enumeration of the convex topologies on a totally ordered set with cardinality $n \leq 10$ was given in [2]. While the goal would be an enumeration of convex topologies on $[n] \times [m]$, here we address more fundamental questions, counting classes of convex subsets of $[n] \times [m]$.

In Section 2, we determine the number of convex subsets of $[n] \times [m]$ which contain points from every row. In Section 3, we determine the

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number of such convex sets which, in addition, are of “full width.” The main result of the paper is the considerably more complicated determination in Section 4 of the number of convex subsets of $[n] \times [m]$.

Initial investigations on these topics were conducted in [1], the first author’s master’s thesis, which was directed by the third author. For the convenience of the reader, a limited amount of material from that thesis is included here.

2. Convex sets intersecting all rows. We start with the basic observation that *the number of nonempty convex sets in $[n]$ is $\binom{n+1}{2} = 1 + 2 + \dots + n$.* This follows since a convex set in $[n]$ is simply an interval, and there are $n + 1 - j$ positions for an interval of length j ($j = 1, \dots, n$) in $[n]$. Or, viewing a convex set in $[n]$ as interval $[a, b]$ with two distinct endpoints $a, b \in \{1, 2, \dots, n + 1\}$, there are $\binom{n+1}{2}$ ways to pick the endpoints.

Let $Cvx_m(n)$ denote the number of convex sets in $[n] \times [m]$ which contain points from each of the m rows. Then, the above remarks show $Cvx_1(n) = n(n + 1)/2$.



FIGURE 1. Some configurations of convex sets.

Suppose that C is a convex set in $[n] \times [2]$ which contains points in both rows. The restriction of C to the top row must be an interval $[i, j]$, and the restriction of C to the bottom row must be an interval $[k, l]$. To maintain convexity, we must have $1 \leq i \leq j \leq l \leq n$ and $i \leq k \leq l$. Figure 1 suggests some possible positions of i, j, k, l . Summing over all such configurations of endpoints i, j, l, k gives the expression for $Cvx_2(n)$, shown below.

A convex set C in $[n] \times [3]$ which contains points in all rows will intersect the first and second rows, respectively, in intervals $[i, j]$ and $[k, l]$, as in the preceding paragraph, and will intersect the third row in

an interval $[p, q]$ where $l \leq q \leq n$ and $k \leq p \leq q$, leading to the expression for $Cvx_3(n)$:

$$\begin{aligned}
 Cvx_1(n) &= \sum_{i=1}^n \sum_{j=i}^n 1 = \frac{1}{2}(n)(n+1) \\
 Cvx_2(n) &= \sum_{i=1}^n \sum_{j=i}^n \sum_{l=j}^n \sum_{k=i}^l 1 = \frac{1}{12}n(n+1)^2(n+2) \\
 Cvx_3(n) &= \sum_{i=1}^n \sum_{j=i}^n \sum_{l=j}^n \sum_{k=i}^l \sum_{q=l}^n \sum_{p=k}^q 1 \\
 &= \frac{1}{144}n(n+1)^2(n+2)^2(n+3).
 \end{aligned}$$

The numerical coefficients 2, 12, 144 appearing in the formulae for $Cvx_m(n)$ above are $m!(m+1)!$. This leads to the conjecture [1] that

$$\begin{aligned}
 Cvx_m(n) &= \frac{n(n+1)^2(n+2)^2 \cdots (n+m-1)^2(n+m)}{m!(m+1)!} \\
 &= \frac{1}{m+1} \binom{n+m}{m} \binom{n+m-1}{m}.
 \end{aligned}$$

In order to prove the conjecture, we first develop a recursion for a related quantity.

Proposition 2.1. Define $c(m, u, v)$ for $m \geq 1, u \geq 0$ and $v \geq 0$ by

$$c(1, u, v) = 1$$

and, for $m \geq 2$,

$$c(m, u, v) = \sum_{r=0}^u \sum_{s=1}^{u+v-r} c(m-1, r, s).$$

Then:

(a) for all $m \geq 1, u \geq 0, v \geq 1$ and $n \geq u+v, c(m, u, v)$ is the number of convex subsets of $[n] \times [m]$ which contain points of each of the m rows and for which the bottom row is $(u, u+v)$.

(b) For all $m \geq 1$ and $n \geq 1, c(m+1, n, 0)$ is the number of convex subsets of $[n] \times [m]$ which contain points of each of the m rows.

Proof.

(a) The assertion is clear when $m = 1$. Suppose that $m \geq 2$, and the result holds for $m - 1$. Let C be a convex subset of $[n] \times [m]$ which contains points of each of the m rows and for which the bottom row is $(u, u + v]$. By convexity, the second-from-bottom row of C will be $(r, r + s]$, where $0 \leq r \leq u$ and $1 \leq s \leq u + v - r$. The first $m - 1$ rows of C constitute an arbitrary convex subset of the poset $[n] \times \{2, \dots, m\}$, which contains points of each of the $m - 1$ rows and for which the bottom row is $(r, r + s]$. By the inductive hypothesis applied to that poset, the number of such sets is $c(m - 1, r, s)$, and our result follows.

(b) We have

$$c(m + 1, n, 0) = \sum_{r=0}^n \sum_{s=1}^{n-r} c(m, r, s).$$

Note that the inner sum is empty when $r = n$. By part (a), the double sum enumerates the convex subsets of $[n] \times [m]$ which contain points of each of the m rows according to the possibilities for the bottom row. \square

In the following proposition, we make use of the standard summation formula $\sum_{k=s}^n \binom{k}{m} = \binom{n+1}{m+1} - \binom{s}{m+1}$.

Proposition 2.2. *For all $m \geq 1$, $u \geq 0$ and $v \geq 0$ with $m + u + v \geq 2$,*

$$c(m, u, v) = \frac{u + vm}{m(u + v + m - 1)} \binom{u + m - 1}{m - 1} \binom{u + v + m - 1}{m - 1}.$$

Proof. For $m = 1$ and $m + u + v \geq 2$, the formula gives the required value of 1. Suppose that $m \geq 2$ and the result holds for $m - 1$. Then,

$$\begin{aligned} c(m, u, v) &= \sum_{r=0}^u \sum_{s=1}^{u+v-r} c(m - 1, r, s) \\ &= \sum_{r=0}^u \sum_{s=1}^{u+v-r} \frac{r + s(m - 1)}{(m - 1)(r + s + m - 2)} \binom{r + m - 2}{m - 2} \binom{r + s + m - 2}{m - 2}. \end{aligned}$$

We shall use $r + s(m - 1) = (r + s)(m - 1) - r(m - 2)$ to break the

sum into two parts. We first have

$$\begin{aligned}
 & \sum_{r=0}^u \sum_{s=1}^{u+v-r} \frac{(r+s)(m-1)}{(m-1)(r+s+m-2)} \binom{r+m-2}{m-2} \binom{r+s+m-2}{m-2} \\
 &= \sum_{r=0}^u \binom{r+m-2}{m-2} \sum_{s=1}^{u+v-r} \frac{r+s}{r+s+m-2} \binom{r+s+m-2}{m-2} \\
 &= \sum_{r=0}^u \binom{r+m-2}{m-2} \sum_{s=1}^{u+v-r} \binom{r+s+m-3}{m-2} \\
 &= \sum_{r=0}^u \binom{r+m-2}{m-2} \left[\binom{u+v+m-2}{m-1} - \binom{r+m-2}{m-1} \right] \\
 &= \binom{u+m-1}{m-1} \binom{u+v+m-2}{m-1} - \sum_{r=0}^u \binom{r+m-2}{m-2} \binom{r+m-2}{m-1}.
 \end{aligned}$$

In the calculation of the second part, the reader who does not wish to contemplate the negative lower binomial arguments, cf., [5], that occur when $m = 2$ may instead simply observe that, in that case, the expression obtained on the last line has the required value of zero.

$$\begin{aligned}
 & \sum_{r=0}^u \sum_{s=1}^{u+v-r} \frac{r(m-2)}{(m-1)(r+s+m-2)} \binom{r+m-2}{m-2} \binom{r+s+m-2}{m-2} \\
 &= \sum_{r=0}^u \frac{r}{m-1} \binom{r+m-2}{m-2} \sum_{s=1}^{u+v-r} \frac{m-2}{r+s+m-2} \binom{r+s+m-2}{m-2} \\
 &= \sum_{r=0}^u \binom{r+m-2}{m-1} \sum_{s=1}^{u+v-r} \binom{r+s+m-3}{m-3} \\
 &= \sum_{r=0}^u \binom{r+m-2}{m-1} \left[\binom{u+v+m-2}{m-2} - \binom{r+m-2}{m-2} \right] \\
 &= \binom{u+m-1}{m} \binom{u+v+m-2}{m-2} \\
 &\quad - \sum_{r=0}^u \binom{r+m-2}{m-1} \binom{r+m-2}{m-2}.
 \end{aligned}$$

When we subtract the double sums, the sums over r cancel. The resulting difference may readily be manipulated into the required form, and we are done by induction. \square

Theorem 2.3. *The number of convex sets in $[n] \times [m]$ which intersect all m rows is*

$$Cvx_m(n) = \frac{1}{m+1} \binom{n+m}{m} \binom{n+m-1}{m},$$

that is, $Cvx_m(n) = N(n+m, m+1)$, where $N(j, k)$ is the (j, k) th Narayana number, appearing as [7, A001263].

Proof. Combining the preceding two propositions, we obtain

$$\begin{aligned} Cvx_m(n) &= c(m+1, n, 0) \\ &= \frac{n}{(m+1)(n+m)} \binom{n+m}{m} \binom{n+m}{m} \\ &= \frac{1}{m+1} \binom{n+m}{m} \binom{n+m-1}{m}, \end{aligned}$$

as required. \square

3. Convex subsets of full width. We define the *width* w of a set $C \subseteq [n] \times [m]$ by

$$w = \max\{x : \exists (x, y) \in C\} - \min\{x : \exists (x, y) \in C\} + 1,$$

that is, the width w of C is the width of the convex hull of the projection of C onto $[n]$. A set $C \subseteq [n] \times [m]$ is said to have *full width* if its width is n . We are interested in finding the number $FWCvx_m(n)$ of full-width convex sets in $[n] \times [m]$ which intersect all m rows. Such sets have a determined top left point $(1, m)$ and a determined bottom right point $(n, 1)$, thereby eliminating two of the sums of the calculations given before Proposition 2.1. The relationship between convex subsets of full width and arbitrary convex subsets is explored in [1], where one may also find the first few formulae for $FWCvx_m(n)$, three of which are given below

$$FWCvx_1(n) = 1,$$

$$\begin{aligned}
 \text{FWCvx}_2(n) &= \sum_{j=1}^n \sum_{k=1}^n 1 = n^2, \\
 \text{FWCvx}_3(n) &= \sum_{j=1}^n \sum_{l=j}^n \sum_{k=1}^l \sum_{p=k}^n 1 \\
 &= \frac{1}{24}(n)(n+1)(5n^2+5n+2).
 \end{aligned}$$

Let C denote a convex subset of $[n] \times [m]$ which intersects all m rows. Suppose that the width of C is k . Then, the top left point of C will be (j, m) , and the bottom right point will be $(j+k-1, 1)$, where $1 \leq j \leq n-k+1$. It follows that

$$\text{Cvx}_m(n) = \sum_{k=1}^n (n-k+1)\text{FWC}_m(k).$$

This relationship may be inverted. With the convention that $\text{Cvx}_m(n) = 0$ when $m \geq 1$ and $n \leq 0$, we obtain that, for $m, n \geq 1$,

$$\text{FWC}_m(n) = \text{Cvx}_m(n) - 2\text{Cvx}_m(n-1) + \text{Cvx}_m(n-2).$$

It is possible to proceed as in the previous section, developing a related recursion, and thence, a formula. We record the two results.

Proposition 3.1. *For all $m, n \geq 1$, the number of convex sets of $[n] \times [m]$, which are of full width and intersect each of the m rows, is*

$$\begin{aligned}
 &\text{FWCvx}_m(n) \\
 &= \frac{1}{m+1} \binom{n+m}{m} \binom{n+m-1}{m} \\
 &\quad - \frac{2}{m+1} \binom{n+m-1}{m} \binom{n+m-2}{m} \\
 &\quad + \frac{1}{m+1} \binom{n+m-2}{m} \binom{n+m-3}{m} \\
 &= \binom{n+m-2}{m-1}^2 - \binom{n+m-1}{m+1} \binom{n+m-3}{m-3}.
 \end{aligned}$$

In the second formula, those binomials with negative lower argument which occur for $m = 1$ and $m = 2$ are to be interpreted as zero; this accords with [5]. For $m \geq 3$, the formula may be seen by factorial manipulation to be equivalent to the preceding one.

4. The number of convex sets in $[n] \times [m]$. We shall now turn to counting the number $Cvx(n, m)$ of nonempty convex subsets in $[n] \times [m]$. Theorem 4.4 below gives our formula for $Cvx(n, m)$. While the first proposition is only slightly more complicated than the corresponding one (Proposition 2.1) for convex sets which intersect every row, the inductive proof of the formula for $Cvx(n, m)$ is much more complicated than that for $Cvx_m(n)$.

It is worth noting that, as the “transpose” of a convex set is convex, we have that $Cvx(n, m) = Cvx(m, n)$. We shall not make use of this fact in our arguments, but it is handy when performing computations.

Proposition 4.1. *Define $g(m, u, v)$ for $m \geq 1$, $u \geq 0$ and $v \geq 0$ by*

$$g(1, u, v) = 1,$$

and, for $m \geq 2$,

$$g(m, u, v) = g(m-1, u, 0) + \sum_{r=0}^u \sum_{s=1}^{u+v-r} g(m-1, r, s).$$

Then:

- (a) $g(m, 0, 0) = 1$ for all $m \geq 1$.
- (b) For all $m \geq 1$, $u \geq 0$, $v \geq 0$ and $n = u + v \geq 1$, $g(m, u, v)$ is the number of convex subsets of $[n] \times [m]$ whose bottom row is (u, n) .
- (c) For all $m \geq 1$ and $n \geq 1$, $g(m+1, n, 0)$ is the number of convex subsets of $[n] \times [m]$.

Proof.

(a) When $u = 0 = v$, the double sum is empty, so $g(m, 0, 0) = g(m-1, 0, 0)$ for all $m \geq 2$.

(b) We use induction on m . When $m = 1$, $g(m, u, v) = 1$, and $C = (u, n] \times [1]$ is the unique such convex subset. Note that, when $v = 0$, C is empty.

Suppose that $m \geq 2$ and that the result holds for $m-1$. We first assert that $g(m-1, u, 0)$ is the number of convex subsets C of $[n] \times [m]$ whose bottom row is $(u, n]$ and second-from-bottom row is empty. When $u = 0$, the bottom row is $[n]$, and there is exactly one such subset, as required. Now, suppose that $u > 0$. Then, the inductive hypothesis, applied to the poset $[u] \times \{2, \dots, m\}$, gives that there are $g(m-1, u, 0)$ possibilities for the first u columns of C . By convexity, columns $u+1, \dots, u+v$ of C (if any) must be empty except for the entry in the bottom row. Thus, the assertion is correct for all u .

We next assert that the double sum enumerates those convex subsets C of $[n] \times [m]$ whose bottom row is $(u, n]$ and second-from-bottom row is nonempty. The number r of initial empty positions of the second-from-bottom row of C satisfies $0 \leq r \leq u$. The number s of points that follow the r initial empty positions satisfies $1 \leq s \leq u+v-r$. By convexity, columns $r+s+1, \dots, u+v$ of C (if any) must be empty except for the entry in the bottom row. For each such r, s there are, by the inductive hypothesis applied to the poset $[r+s] \times \{2, \dots, m\}$, $g(m-1, r, s)$ possibilities for the first $r+s$ columns of C . Thus, the number of convex subsets of $[n] \times [m]$ with bottom row $(u, n]$ and second-from-bottom row nonempty is $\sum_{r=0}^u \sum_{s=1}^{u+v-r} g(m-1, r, s)$. Note that, when $u = n$ and $r = u$, there is, as required, an empty inner sum. Thus, our result holds for m . By induction, it holds for all $m \geq 1$.

(c) Follows from (b), using the bijection between convex subsets of $[n] \times [m+1]$ whose bottom row is empty and convex subsets of $[n] \times [m]$. □

In order to facilitate working with the recursion for $g(m, u, v)$ in a systematic manner, we introduce two auxiliary quantities.

Let $u \geq 0, v \geq 0$. We define $h(1, u, v) = 0 = k(1, u)$, and, for $m \geq 2$,

$$h(m, u, v) = \sum_{r=0}^{u-1} g(m-1, r, u+v-r)$$

$$k(m, u) = \sum_{r=0}^{u-1} \sum_{s=1}^{u-r} g(m-1, r, s).$$

Note that, since the corresponding sums are empty, we have the

initial ($u = 0$) values

$$\begin{aligned} h(m, 0, v) &= 0 \quad \text{for } m \geq 1, v \geq 0 \\ k(m, 0) &= 0 \quad \text{for } m \geq 1. \end{aligned}$$

We record some relationships among these quantities. For $m \geq 2$, $u \geq 1$ and $v \geq 0$, we have

$$\begin{aligned} h(m, u, v) &= \sum_{r=0}^{u-2} g(m-1, r, (u-1) + (v+1) - r) \\ &\quad + g(m-1, u-1, v+1), \end{aligned}$$

so

$$(R1) \quad h(m, u, v) = h(m, u-1, v+1) + g(m-1, u-1, v+1).$$

For $m \geq 2$ and $u \geq 1$, we have

$$k(m, u) = \sum_{r=0}^{u-2} \sum_{s=1}^{u-r-1} g(m-1, r, s) + \sum_{r=0}^{u-1} g(m-1, r, u-r),$$

so

$$(R2) \quad k(m, u) = k(m, u-1) + h(m, u, 0).$$

For $m \geq 2$ and $u \geq 0$, we have by definition that

$$g(m, u, 0) = g(m-1, u, 0) + \sum_{r=0}^u \sum_{s=1}^{u-r} g(m-1, r, s).$$

Since the inner sum is empty when $r = u$, we have

$$(R3) \quad g(m, u, 0) = g(m-1, u, 0) + k(m, u).$$

For $m \geq 2$, $u \geq 0$ and $v \geq 1$, we have

$$\begin{aligned}
 g(m, u, v) &= g(m - 1, u, 0) \\
 &+ \sum_{r=0}^u \sum_{s=1}^{u+(v-1)-r} g(m - 1, r, s) \\
 &+ \sum_{r=0}^u g(m - 1, r, u + v - r) \\
 &= g(m, u, v - 1) \\
 &+ \sum_{r=0}^{u-1} g(m - 1, r, u + v - r) + g(m - 1, u, v),
 \end{aligned}$$

so

$$(R4) \quad g(m, u, v) = g(m - 1, u, v) + g(m, u, v - 1) + h(m, u, v).$$

We used (R1)–(R4) recursively to express $g(m, u, v)$, $h(m, u, v)$ and $k(m, u)$, for small values of u , as sums of “terms” that were functions of m with “coefficients” that did not depend on m . This led, empirically, to representations of the form

$$(F1) \quad h(m, u, v) = \sum_{r=0}^{u-1} \sum_{s=0}^r h_{rs}^{u+v} \binom{m+v+r-1}{v+r+s+1}$$

$$(F2) \quad k(m, u) = \sum_{r=0}^{u-1} \sum_{s=0}^r g_{rs}^u \binom{m+r-1}{r+s+1}$$

$$(F3) \quad g(m, u, v) = \binom{m+v-1}{v} + \sum_{r=0}^{u-1} \sum_{s=0}^r g_{rs}^{u+v} \binom{m+v+r}{v+r+s+2}.$$

We observed certain simple relationships that the coefficients satisfied. We shall now state those relationships and then show that, with the unique coefficients satisfying those relationships, the formulae (F1)–(F3) hold. The proof, which, besides the coefficient relationships, uses only initial conditions for h, k, g and the relations (R1)–(R4), implicitly shows how the representations were obtained.

Theorem 4.2. *For integers t, r, s , define quantities g_{rs}^t and h_{rs}^t as follows. Set $g_{rs}^t = 0 = h_{rs}^t$ if (t, r, s) is not in $D = \{(t, r, s) : 0 \leq s \leq r < t\}$. For $(t, r, s) \in D$, set*

$$g_{rs}^t = \begin{cases} t - r & \text{if } s = 0, \\ g_{rs}^{t-1} + h_{rs}^t & \text{otherwise,} \end{cases}$$

$$h_{rs}^t = \begin{cases} 1 & \text{if } s = 0, \\ h_{r-1\ s}^t + g_{r-1\ s-1}^t & \text{otherwise.} \end{cases}$$

With these quantities, the formulae (F1)–(F3) hold for all $m \geq 1, u \geq 0$ and $v \geq 0$.

Proof. Let $(t, r, s) \in D$. If $s = 0$, g_{rs}^t and h_{rs}^t are given explicitly. Otherwise, if g_{rs}^t, h_{rs}^t are known when $t + r + s$ is smaller, the recursive clauses first give us h_{rs}^t and then g_{rs}^t . Thus, the above conditions determine unique quantities g_{rs}^t and h_{rs}^t .

We first consider the case $m = 1$. By definition, $h(1, u, v) = 0 = k(1, u)$ and $g(1, u, v) = 1$, as given by (F1), (F2) and (F3), respectively.

We proceed by induction on u . Suppose that $u = 0$. As previously observed, $h(m, 0, v) = 0 = k(m, 0)$; thus, (F1) and (F2) follow immediately. For (F3), we are reduced to showing $g(m, 0, v) = \binom{m+v-1}{v}$. First, suppose that $v = 0$. By Proposition 4.1 (a), $g(m, 0, 0) = 1$ for all $m \geq 1$, so (F3) holds for $u = 0 = v$.

We continue by induction on $m + v$. We may assume that $m \geq 2$ and $v \geq 1$ so that $m + v \geq 3$, and that (F3) holds for smaller values of $m + v$. By (R4) and the induction hypothesis, we have that

$$g(m, 0, v) = g(m - 1, 0, v) + g(m, 0, v - 1) + h(m, 0, v)$$

$$= \binom{m + v - 2}{v} + \binom{m + v - 2}{v - 1} = \binom{m + v - 1}{v}.$$

Thus, the inductive step for $m + v$ holds, and (F1), (F2) and (F3) hold when $u = 0$.

Now, suppose that $u \geq 1$, and that (F1), (F2) and (F3) hold for $u - 1$ for all $m \geq 1$ and $v \geq 0$. We may assume that $m \geq 2$. Then, by (R1)

and the inductive hypothesis, we have

$$\begin{aligned}
 h(m, u, v) &= h(m, u - 1, v + 1) + g(m - 1, u - 1, v + 1) \\
 &= \sum_{r=0}^{u-2} \sum_{s=0}^r h_{rs}^{u+v} \binom{m+v+r}{v+r+s+2} + \binom{m+v-1}{v+1} \\
 &\quad + \sum_{r=0}^{u-2} \sum_{s=0}^r g_{rs}^{u+v} \binom{m+v+r}{v+r+s+3} \\
 &= \sum_{r=1}^{u-1} \sum_{s=0}^{r-1} h_{r-1s}^{u+v} \binom{m+v+r-1}{v+r+s+1} + \binom{m+v-1}{v+1} \\
 &\quad + \sum_{r=1}^{u-1} \sum_{s=1}^r g_{r-1s-1}^{u+v} \binom{m+v+r-1}{v+r+s+1}.
 \end{aligned}$$

When $s = r$, $(u + v, r - 1, s)$ is not in D ; thus, $h_{r-1s}^{u+v} = 0$. When $s = 0$, $(u + v, r - 1, s - 1)$ is not in D , so $g_{r-1s-1}^{u+v} = 0$. Thus, we may expand the ranges of the inner sums, obtaining

$$\begin{aligned}
 h(m, u, v) &= \binom{m+v-1}{v+1} + \sum_{r=1}^{u-1} \sum_{s=0}^r [h_{r-1s}^{u+v} + g_{r-1s-1}^{u+v}] \binom{m+v+r-1}{v+r+s+1} \\
 &= \binom{m+v-1}{v+1} + \sum_{r=1}^{u-1} \sum_{s=0}^r h_{rs}^{u+v} \binom{m+v+r-1}{v+r+s+1} \\
 &= \sum_{r=0}^{u-1} \sum_{s=0}^r h_{rs}^{u+v} \binom{m+v+r-1}{v+r+s+1},
 \end{aligned}$$

as $h_{00}^{u+v} = 1$. Hence, (F1) holds for u .

Using (R2), the inductive hypothesis and (F1) for u , we have

$$\begin{aligned}
 k(m, u) &= k(m, u - 1) + h(m, u, 0) \\
 &= \sum_{r=0}^{u-2} \sum_{s=0}^r g_{rs}^{u-1} \binom{m+r-1}{r+s+1} + \sum_{r=0}^{u-1} \sum_{s=0}^r h_{rs}^u \binom{m+r-1}{r+s+1}.
 \end{aligned}$$

Since $(u - 1, u - 1, s)$ is not in D , $g_{u-1\ s}^{u-1} = 0$, and we have

$$\begin{aligned} k(m, u) &= \sum_{r=0}^{u-1} \sum_{s=0}^r [g_{rs}^{u-1} + h_{rs}^u] \binom{m+r-1}{r+s+1} \\ &= \sum_{r=0}^{u-1} \sum_{s=0}^r g_{rs}^u \binom{m+r-1}{r+s+1}. \end{aligned}$$

Thus, (F2) holds for u .

To complete the inductive step for u , it remains to show (F3). We first consider the case $v = 0$. We use induction on m and recall that the case $m = 1$ is known. Suppose that $m \geq 2$, and that (F3) holds for $g(m - 1, u, 0)$. Using (R3), the inductive hypothesis and (F2) for u , we obtain that

$$\begin{aligned} g(m, u, 0) &= g(m - 1, u, 0) + k(m, u) \\ &= \binom{m-2}{0} + \sum_{r=0}^{u-1} \sum_{s=0}^r g_{rs}^u \binom{m+r-1}{r+s+2} \\ &\quad + \sum_{r=0}^{u-1} \sum_{s=0}^r g_{rs}^u \binom{m+r-1}{r+s+1}, \end{aligned}$$

and we readily obtain the required $\binom{m-1}{0} + \sum_{r=0}^{u-1} \sum_{s=0}^r g_{rs}^u \binom{m+r}{r+s+2}$. By our induction on m , we thus have that, when $v = 0$, (F3) holds for u for all $m \geq 1$.

We continue by induction on $m + v$. We may assume that $m \geq 2$ and $v \geq 1$ so that $m + v \geq 3$, and that (F3) holds for u for smaller values of $m + v$. By (R4), the induction hypothesis and (F1) for u , we have

$$\begin{aligned} g(m, u, v) &= g(m - 1, u, v) + g(m, u, v - 1) + h(m, u, v) \\ &= \binom{m+v-2}{v} + \sum_{r=0}^{u-1} \sum_{s=0}^r g_{rs}^{u+v} \binom{m+v+r-1}{v+r+s+2} \end{aligned}$$

$$\begin{aligned}
 &+ \binom{m+v-2}{v-1} + \sum_{r=0}^{u-1} \sum_{s=0}^r g_{rs}^{u+v-1} \binom{m+v+r-1}{v+r+s+1} \\
 &+ \sum_{r=0}^{u-1} \sum_{s=0}^r h_{rs}^{u+v} \binom{m+v+r-1}{v+r+s+1}.
 \end{aligned}$$

As $g_{rs}^{u+v-1} + h_{rs}^{u+v} = g_{rs}^{u+v}$, we readily obtain (F3) for $m+v$. By the induction on $m+v$, (F3) holds for u . This completes the induction step for u , and we have, by that induction, that (F1), (F2) and (F3) hold for all $m \geq 1, u \geq 0$ and $v \geq 0$. \square

Next, we establish formulae for g_{rs}^t and h_{rs}^t .

Proposition 4.3. *For all $(t, r, s) \in D$, we have*

$$\begin{aligned}
 g_{rs}^t &= \frac{t-r}{t+1} \binom{r}{s} \binom{t+s+1}{s+1}, \\
 h_{rs}^t &= \frac{(s+1)t-rs+1}{(t+1)(t+s+1)} \binom{r}{s} \binom{t+s+1}{s+1}.
 \end{aligned}$$

Proof. Let $(t, r, s) \in D$. If $s = 0$, then the formulae for g_{rs}^t and h_{rs}^t give the required values of $t-r, 1$, respectively. We shall argue by induction on $t+r+s$. We may assume that $1 \leq s \leq r < t$ so that $t+r+s \geq 4$, and that the formulae apply when $t+r+s$ is smaller.

We first consider the formula for h_{rs}^t . Note that $(t, r-1, s-1) \in D$. First, suppose that $r = s$. Then

$$h_{rs}^t = h_{rr}^t = h_{r-1\ r}^t + g_{r-1\ r-1}^t = g_{r-1\ r-1}^t = \frac{t+1-r}{t+1} \binom{t+r}{r},$$

which agrees with the formula. When $s < r$, we have $(t, r-1, s) \in D$ as well, and thus,

$$\begin{aligned}
 h_{rs}^t &= h_{r-1\ s}^t + g_{r-1\ s-1}^t \\
 &= \frac{(s+1)t - (r-1)s + 1}{(t+1)(t+s+1)} \binom{r-1}{s} \binom{t+s+1}{s+1} \\
 &\quad + \frac{t+1-r}{t+1} \binom{r-1}{s-1} \binom{t+s}{s}
 \end{aligned}$$

which simplifies to the required

$$\frac{(s+1)t - rs + 1}{(t+1)(t+s+1)} \binom{r}{s} \binom{t+s+1}{s+1}.$$

Now, we consider the formula for g_{rs}^t . When $t = r + 1$, we have

$$g_{rs}^t = g_{rs}^{r+1} = g_{rs}^r + h_{rs}^{r+1} = h_{rs}^{r+1} = \frac{1}{r+2} \binom{r}{s} \binom{r+s+2}{s+1},$$

as is given by the formula. When $t > r + 1$, we have $(t - 1, r, s) \in D$, and thus,

$$\begin{aligned} g_{rs}^t &= g_{rs}^{t-1} + h_{rs}^t \\ &= \frac{t-1-r}{t} \binom{r}{s} \binom{t+s}{s+1} \\ &\quad + \frac{(s+1)t - rs + 1}{(t+1)(t+s+1)} \binom{r}{s} \binom{t+s+1}{s+1}, \end{aligned}$$

which simplifies to the required

$$\frac{t-r}{t+1} \binom{r}{s} \binom{t+s+1}{s+1}.$$

Thus, by induction on $t+r+s$, g_{rs}^t and h_{rs}^t are given by the respective formulae for all $(t, r, s) \in D$. □

It is now a simple matter to obtain our main result.

Theorem 4.4. *For $m, n \geq 1$, the number of nonempty convex subsets of $[n] \times [m]$ is*

$$Cvx(n, m) = \frac{1}{n+1} \sum_{r=0}^{n-1} \sum_{s=0}^r (n-r) \binom{r}{s} \binom{n+s+1}{s+1} \binom{m+r+1}{r+s+2}.$$

Proof. From Proposition 4.1 (c), Theorem 4.2 and Proposition 4.3, the number of convex subsets of $[n] \times [m]$ is

$$\begin{aligned}
g(m+1, n, 0) &= \binom{m}{0} + \sum_{r=0}^{n-1} \sum_{s=0}^r g_{rs}^n \binom{m+r+1}{r+s+2} \\
&= 1 + \frac{1}{n+1} \sum_{r=0}^{n-1} \sum_{s=0}^r (n-r) \binom{r}{s} \binom{n+s+1}{s+1} \binom{m+r+1}{r+s+2}.
\end{aligned}$$

Since the above enumeration includes the empty set, our result immediately follows. \square

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