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Complements of Convex Topologies on Products of Finite Totally Ordered Spaces

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Abstract Let X be a finite product of finite totally ordered topological spaces. We show that in the lattice of topologies on X, every convex topology τ on X has a convex complement τ'.

Keywords convex topology · ordered topological space · complement

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1 Introduction

A topology τ on a partially ordered set $(X, \leq)$ is said to be *locally convex* if every point has a neighborhood base of $\leq$-convex open sets, and *convex* if τ has a base of convex open sets. It is easy to show (see [8]) that these properties are equivalent. Let $Top(X)$ be the set of topologies on X ordered by inclusion. $Top(X)$ is a complete lattice whose greatest element is the discrete topology τ_d and whose least element is the indiscrete topology τ_i. If X is partially ordered, the collection $CvxTop(X)$ of convex topologies on X is also a complete lattice.

For two topologies τ, τ' in $CvxTop(X)$, their supremum in $CvxTop(X)$ is the same as their supremum in $Top(X)$, namely, $\tau \vee \tau' = [\tau \cup \tau']$, where the brackets indicate the topology generated by the enclosed subbasis. For

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$\tau, \tau' \in CvxTop(X)$, their infimum in $Top(X)$ may not be convex. For example, if $X = \{1,2\} \times \{1,2\}$ with the *product order* $(a,b) \leq (x,y)$ if and only if $a \leq x$ and $b \leq y$, let $\tau_1 = [\{\{(1,1)\}, \{(1,2),(2,2)\}, X\}]$, and $\tau_2 = [\{\{(1,1),(1,2)\}, \{(2,2)\}, X\}]$. In $Top(X)$ we have $\tau_1 \wedge \tau_2$ is the non-convex topology $\{\emptyset, \{(1,1),(1,2),(2,2)\}, X\}$. In $CvxTop(X)$, $\tau_1 \wedge \tau_2 = \tau_i$. In totally ordered spaces finite infima agree in $Top(X)$ and $CvxTop(X)$. This corrects a statement on p. 488 of [10], where the hypothesis of totally ordered was omitted. The authors are indebted to Hans-Peter Künzi for pointing this out.

A complement of a topology τ in the lattice $Top(X)$ is a topology $\tau' \in Top(X)$ with $\tau \vee \tau' = \tau_d$ and $\tau \wedge \tau' = \tau_i$. Complementation in the lattice of topologies has been extensively studied since the 1960s ([11], [13], [17], [18]), including an interest in topologies from a certain class having complements from a certain class ([1], [2], [7], [12], [14], [15], [16]). The number of complements of a given topology has received much attention; for a very brief summary and references, see [3], [10].

We will be concerned with topologies τ on finite sets X, so each point $x \in X$ has a smallest neighborhood $N(x) = \bigcap\{U : x \in U, U \in \tau\}$ called the *neighborhood core* of x (see [5]). Throughout, we will determine topologies τ and τ' on a finite set by their bases $\mathcal{B} = \{N(x) : x \in X\}$ and $\mathcal{B}' = \{N'(x) : x \in X\}$, respectively, of neighborhood cores.

There is a widely known close connection between topologies on finite sets and quasiorders (see [5], [6] or the survey [9]). If X is finite, a topology τ on X gives rise to the specialization quasiorder $\preceq$, and a quasiorder $\preceq$ on X gives rise to the specialization topology under the correspondence $x \preceq y$ if and only if $x \in cl(\{y\})$ if and only if $y \in N(x)$. Thus, on finite sets, the topological closure operator cl agrees with the order theoretic decreasing hull operator $d(A) = \{x \in X : \exists a \in A, x \preceq a\}$. The specialization quasiorder provides a directed graph on X, with a directed edge from x to y if and only if $x \preceq y$. If $x \preceq y$ (or equivalently, $y \in N(x)$) we say x *links to* y by τ, and we say $A \subseteq X$ *links to set* B *by* τ if there exist $a \in A$ and $b \in B$ such that a links to b by τ. We may also apply this terminiology to the topologies τ' and $\tau \wedge \tau'$. We emphasize that "links to" is a directed concept, just as it is when applied to webpages.

Recall that a directed graph G is strongly connected if for any two vertices x and y, there is a directed path from x to y. For a digraph G, $E(G)$ represents the set of edges of G.

The following observation will be useful in showing that X is the only nonempty set in X which is simultaneously τ-open and τ'-open, that is, in showing that $\tau \wedge \tau' = \tau_i$.

Proposition 1 *Suppose τ and τ' are two topologies on a finite set X with corresponding digraphs G and G' of their specialization quasiorders. If the union digraph $G \cup G'$ is strongly connected, then $\tau \wedge \tau' = \tau_i$.*

Proof: Suppose $G \cup G'$ is strongly connected and U is a nonempty $\tau \wedge \tau'$-open set, with $x \in U$. Now for $y \in X$, there exists a sequence of edges in $E(G)$ or $E(G')$ from $x = x_1$ to x_2 to x_3, and so on, to $x_n = y$. Now $(x_i, x_{i+1}) \in$

$E(G \cup G')$ implies $x_{i+1} \in N(x_i)$ or $x_{i+1} \in N'(x_i)$ so that $x_{i+1} \in N_{\tau \wedge \tau'}(x_i)$. Thus, every $\tau \wedge \tau'$-open set containing x_i must contain x_{i+1}, and in particular, U must contain y. □

The totally ordered set $X = \{1, 2, \ldots, n\}$ with the natural ordering $1 < 2 < \cdots < n$ will be denoted by $\mathbf{n}$.

We now describe the algorithm given in [10] to produce a convex complement for any convex topology τ on a finite totally ordered set $X = \mathbf{n}$. We say that the topology τ on $\mathbf{n}$ *breaks to the right of* j if there is some neighborhood core $N(k) \in \tau$ of form $N(k) = [a, j] = \{i : a \leq i \leq j\}$. Similarly, τ *breaks to the left of* j if j is the least element of some neighborhood core $N(k)$. A convex complement τ' of τ can be constructed by taking $N'(j) = [j_*, j^*]$ where j_* (j^*) is the largest number less (smallest number greater) than or equal to j at which τ does not break to the left (right), or the endpoint 1 (n) if no such number exists. It is shown in [10] that if j is not the left endpoint 1, then τ' breaks to the left of j if and only if τ does not, and dually. In particular, at each j which is not an endpoint 1 or n, exactly one of $N(x)$ or $N'(x)$ extends to the right of j and exactly one extends to the left of j. This construction guarantees that from any non-endpoint j, we have $\{j-1, j, j+1\} \subseteq N(j) \cup N'(j)$, so any $\tau \wedge \tau'$ open set containing j contains $j-1$ and $j+1$. That is, the τ- and τ'-neighborhood cores of j provide links to the left and to the right of j. Now repeating this argument for $j-1$ and $j+1$, we find that any $\tau \wedge \tau'$ neighborhood of j contains $\{j-2, j-1, j, j+1, j+2\}$. Iterating until we reach the endpoints, we see that the only nonempty $\tau \wedge \tau'$ open set is $\mathbf{n}$.

In the totally ordered case, from a point j, it was enough to consider links and breaks to $j-1$ and $j+1$ since these two points provided the only directions in which we could expand convexly from j. In a product of totally ordered sets, $\{j, x\}$ is convex for any x such that x is noncomparable to j, as well as any adjacent comparable x, making it more difficult to determine when links are needed (or prohibited). Example 7 explores this direction further.

2 Products of Two Totally Ordered Spaces

We start by considering the product of two totally ordered spaces. All products will carry the product (i.e., componentwise) order. To find a convex complement τ' of a convex topology τ on $\mathbf{m} \times \mathbf{n}$, we will string together the convex complements of the totally ordered rows $R_j = \mathbf{m} \times \{j\}$ viewed as subspaces, with appropriate connections at the endpoints providing enough links between rows to insure that the link graph is strongly connected.

Theorem 1 *Any convex topology τ on a product $X = \mathbf{m} \times \mathbf{n}$ of two totally ordered sets admits a convex complement τ'.*

The remainder of this section is devoted to the proof of this, which is by way of a constructive algorithm for the complement τ'. We assume τ is a convex topology on $\mathbf{m} \times \mathbf{n}$.

2.1 Linear Work

Each row R_j $(j = 1, \ldots n)$, with the subspace topology $\tau_j \equiv \tau|_{R_j}$ is a convex totally ordered space isomorphic to $\mathbf{m}$. Let $l_j = (1, j)$ be the left endpoint of row R_j, and let $r_j = (m, j)$ be the right endpoint of R_j. Construct the convex complement τ_j^* of τ_j using the algorithm from [10], to obtain τ_j^*-neighborhoods $N^*(x) \subseteq R_j$ for each $x \in R_j$. Now $\mathcal{B}^* = \{N^*(x) : x \in X = R_1 \cup \cdots \cup R_n\}$ is a basis for a convex topology τ^* on X. Now in $\tau \wedge \tau^*$, each point $x_j \in R_j$ is connected to every other point of its row R_j by a series of links by τ or τ^*. Our algorithm will modify the topology τ^* by replacing some of the neighborhood cores containing endpoints l_j or r_j which are not realized as intersections of other open sets by a larger convex union of elements of $\mathcal{B}^*$. Note that this procedure will produce a (coarser) convex topology τ' on X. Our strategy will be to preserve the τ^* links and breaks within each row and add enough τ' links between rows to the existing τ links between rows to make $X = \mathbf{m} \times \mathbf{n}$ strongly connected in the directed graph whose vertices are the points of X and the edges are the pairs $\{x, y\}$ for which x is linked to y by τ or by τ'.

2.2 How to Link

If we wish to convexly extend $N^*(l_j)$ downward to contain r_{j-1} and other points, then we must also extend each $N^*(y)$ containing l_j, for otherwise, the intersection of an "extended" neighborhood of l_j with one that was not extended would show that the neighborhood core of l_j was not extended. This simple observation is an important step used in [4] to obtain a recursive formula for the number of convex topologies on an n-element totally ordered space. Rather than extending each $N^*(y)$ containing l_j downward, in order to leave the τ^*-breaks within row R_j undisturbed, we will extend only the largest one $G(l_j) = \max\{N^*(y) : l_j \in N^*(y)\}$. That is, if we wish to link row R_j to a lower row, we will only extend the τ'-neighborhood cores for those points $y \in R_j$ with $N^*(y) = G(l_j)$. Dually, we take $G(r_j) = \max\{N^*(y) : r_j \in N^*(y)\}$. For $x_j \in \{l_j, r_j\}$, note that $G(x_j) \subseteq R_j$, and by convexity, $G(x_j) = \bigcup\{N^*(y) : x_j \in N^*(y)\}$. Combining the concepts, we see that introducing a link downward from R_j will involve extending $G(l_j)$ to include $G(r_{j-1})$, and possibly more. The basic linking mechanism is illustrated in Example 1 below. To provide a τ'-link down from R_j to R_{j-1}, for all $x \in R_j$ with $N^*(x) = G(l_j)$ we take $N'(x)$ to be $G(l_j) \cup G(r_{j-1})$. To provide a link up from R_j to R_{j+1}, for all $x \in R_j$ with $N^*(x) = G(r_j)$ we take $N'(x)$ to be $G(r_j) \cup G(l_{j+1})$. We take $N'(x) = N^*(x)$ for all other points x whose τ^*-neighborhood core was not expanded by linking.

Example 1 (Basic linking.) A convex topology τ on $\mathbf{5} \times \mathbf{3}$ is shown in Figure 1. Note that τ links R_2 up to R_3 since there are points of R_2 whose τ-neighborhood cores extends to R_3, and τ provides no other links from one row to another. The topology τ^* is shown, with $G(l_3)$, $G(l_2)$, $G(r_1)$, and $G(r_2)$ shown in bold. To provide a sequence of links between all rows, it suffices to add

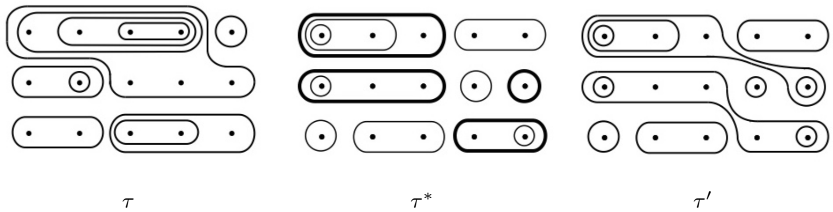


Fig. 1 Linking Example

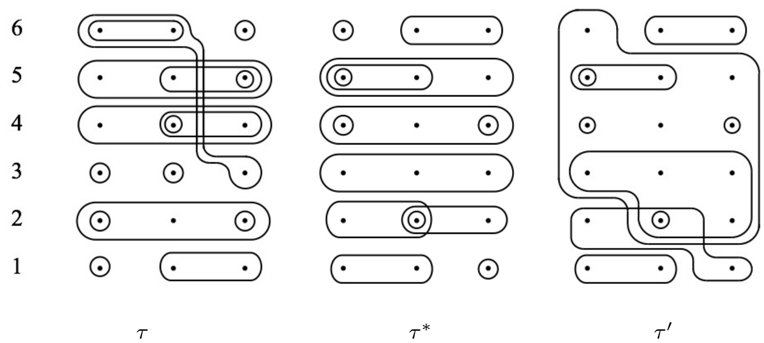


Fig. 2 Multiple Row Linking

τ'-links from R_3 down to R_2, from R_2 down to R_1, and from R_1 up to R_2. To link R_3 down to R_2, we expand $G(l_3)$ to include $G(r_2)$. Specifically, $x = (3,3)$ is the only point in R_3 with $N^*(x) = G(l_3)$, so we take $N'((3,3)) = G(l_3) \cup G(r_2)$ and $N'(x) = N^*(x)$ for all $x \in R_3 \setminus \{(3,3)\}$. To link R_2 down to R_1 and R_1 up to R_2, in transforming from τ^* to τ' we expand $G(l_2)$ to include $G(r_1)$ and expand $G(r_1)$ to include $G(l_2)$; that is, we replace both $G(l_2)$ and $G(r_1)$ by their union. Since $G(l_2) = N^*((2,2)) = N^*((3,2))$ and $G(r_1) = N^*((4,1))$, we take $N'((2,2)) = N'((3,2)) = N'((4,1)) = G(l_2) \cup G(r_1)$, and $N'(x) = N^*(x)$ for all other points $x \in R_2 \cup R_1$.

Example 2 (Multiple-row linking.) The topology τ shown in Figure 2 links up from R_3 to R_6. The techniques below will show that we will want τ' to link every row down to the bottom, rows R_1 and R_2 up to R_3, and rows R_4 and R_5 up to R_6. Note that for rows R_3, R_4, and R_5, the largest τ^* neighborhoods used for linking are entire rows. That is, these rows satisfy the following equivalent conditions: (a) $l_j \in G(r_j)$, (b) $r_j \in G(l_j)$, (c) $G(r_j) = R_j$, (d) $G(l_j) = R_j$, (e) $G(r_j) = G(l_j)$, (f) R_j is a neighborhood core in τ_j^*.

Now linking R_6 down to R_5 would replace $G(l_6)$ by $G(l_6) \cup G(r_5) = G(l_6) \cup R_5$. Since $G(l_6) = \{(1,6)\}$ is the τ^*-neighborhood core of only $(1,6)$, we take $N'((1,6)) = G(l_6) \cup R_5$. But we wish to continue to link down to R_4, so $N'((1,6))$ should be $G(l_6) \cup R_5 \cup G(r_4) = G(l_6) \cup R_5 \cup R_4$. Similarly, linking down to R_3 requires adjoining $G(r_3) = R_3$. Now linking down to R_2 adjoins

$G(r_2) = \{(2,2),(3,2)\}$, which is not the entire row. Thus, $N'((1,6)) = G(l_6) \cup R_5 \cup R_4 \cup R_3 \cup G(r_2)$, and R_6 links down to R_2 with a single τ'-neighborhood core. To link R_6 on down to R_1, we require one more link from R_2 down to R_1, accomplished by taking $G(l_2) \cup G(r_1)$ as the τ'-neighborhood core of every $y \in R_2$ with $N^*(y) = G(l_2)$. That is, we take $N'((1,2)) = G(l_2) \cup G(r_1)$. This illustrates multi-row linking.

To complete the example, since R_1 should link up to R_2, we take $N'(3,1) = G(l_2) \cup G(r_1) = N'((1,2))$. Now R_2 and R_3 need to be linked in both directions, and since R_3 already τ-links up, we do not want R_2 or R_3 to τ'-link up, so we take $N'((3,2)) = N'((1,3)) = N'((2,3)) = N'((3,3)) = G(l_3) \cup G(r_2) = \{(2,2),(3,2),(1,3),(2,3),(3,3)\}$. Also, R_4 and R_5 need to be linked in both directions. This is accomplished by taking $N'(2,4) = N'(3,5) = N'(1,6) = G(l_6) \cup R_5 \cup R_4 \cup R_3 \cup G(r_2)$.

2.3 When to Link and Break

If $x_j \in R_j$, then the breaks within that row provided by $N^*(x_j) \subseteq R_j$ guarantee that $N(x_j) \cap N^*(x_j) \cap R_j = \{x_j\}$. We will define τ' to maintain the τ^*-breaks between elements on a row. So, if $N(x_j) \cap N'(x_j) \neq \{x_j\}$, the points of intersection would have to occur outside R_j. Our construction will prohibit that by insuring that $N'(x_j)$ "breaks" before the first rows—above and below—that $N(x_j)$ links out to. If $N(x_j)$ links row R_j down, we define the *downward link/break interval associated with* $x_j \in R_j$ to be $(k, j]$, where k is the largest integer less than j for which $N(x_j)$ links down to R_k. The *upward link/break interval* $[j, i)$ associated with $x_j \in R_j$ is defined dually. Link/break intervals are directional, half-open intervals. If $(k, j]$ and $[j, i)$ are the downward and upward link/break intervals associated with $x_j \in R_j$, then requiring that $N'(x_j) \subseteq \bigcup\{R_z : z \in (k,i)\} = R_{k+1} \cup \cdots \cup R_{i-1}$ will insure that $N(x_j) \cap N'(x_j) = \{x_j\}$ (and furthermore, that $N(x_j)$ is not τ'-open unless it is X and that $N'(x_j)$ is not τ-open unless it is X).

For each $j \in \{1, \ldots, n\}$, we construct the link/break intervals for each $x_j \in R_j$ for which $N(x_j)$ links out of R_j. Note that if $x_j \in R_j, x_k \in R_k$, $j \neq k$ and $N = N(x_j) = N(x_k)$, then N may generate up to two link/break intervals as the neighborhood core of x_j and up to two more as the neighborhood core of x_k (and up to two for each other point x_i from a different row with $N(x_i) = N$).

Some link/break intervals associated with τ-neighborhood cores are shown in Figure 3. If $(k, j]$ is a downward link/break interval, we will say "k is a ⫰" and "j is a ╿", and will interpret the symbols ⫰ and ╿ as indices k and j, respectively, of rows containing the corresponding endpoints of a downward link/break interval $(k, j]$. Dually, we interpret ⫯ and ╽ as row indices corresponding to endpoints of upward link/break intervals. This allows us to use phrases such as "╽ $\geq j$ ". Furthermore, by $x \in$ ╿ we mean $x \in (k, j]$ for some downward link/break interval, and if the exact link/break interval is needed, we may write $x \in$ ╿⫰ $= (k, j]$.

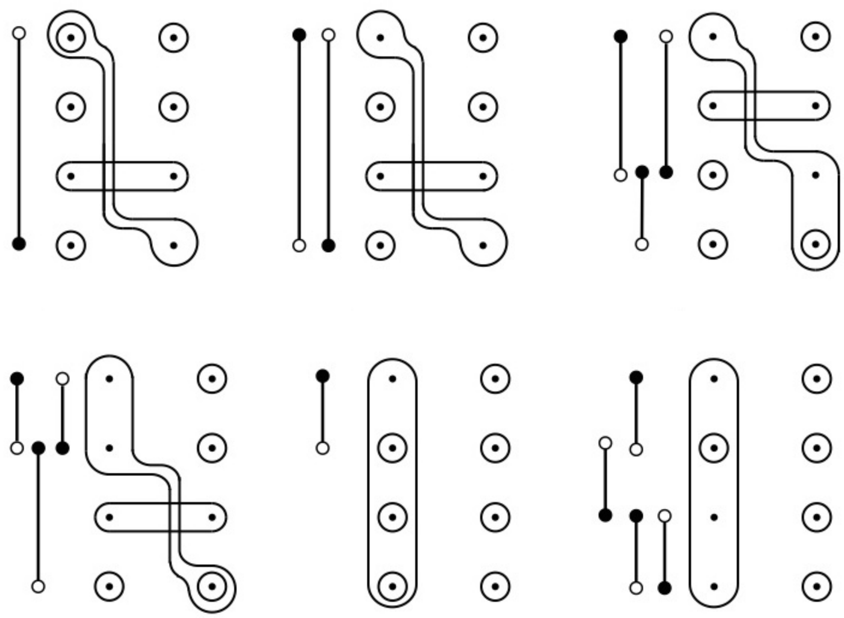


Fig. 3 Link/Break intervals

Given a convex topology τ on a product $X = \mathbf{m} \times \mathbf{n}$ of totally ordered sets, record the link/break intervals. Now for each j with $1 \leq j \leq n$ we define an interval (j_-, j^+) of rows to be τ'-linked.

Given j, take

$$j^+ = \begin{cases} \text{the smallest integer above the first } \overset{|}{\bullet} \ \geq j & \text{if } \exists \ \overset{|}{\bullet} \ \geq j \\ n+1 & \text{otherwise} \end{cases}$$

$$j_- = \begin{cases} \text{the greatest } \overset{|}{\circ} \ < j & \text{if } j \in \text{any } \underset{|}{\bullet} \\ \text{the greatest } \overset{|}{\circ} \text{ below the first } \underset{|}{\bullet} < j & \text{if } j \notin \text{any } \underset{|}{\bullet} \\ 0 & \text{otherwise.} \end{cases}$$

Now we τ' link each row R_j to all the rows in (j_-, j^+) as described in Section 2.2. To show that τ' is indeed a complement of τ, there remain three things to check:

(a) Intersections of τ' sets do not introduce new τ'-sets or τ'-breaks.
(b) Enough τ' breaks are provided to guarantee $N(x) \cap N'(x) = \{x\}$ for every $x \in X$.
(c) Enough τ' links are provided to make X strongly $\tau \wedge \tau'$-connected (see Proposition 1) to insure X is the only nonempty set open in both τ and τ'.

(a) By the construction, each $N'(x)$ preserves the τ^* links and breaks within each row it intersects, so we need only check intersections which intersect more than one row. Thus, to show that intersections of τ' neighborhoods defined by the algorithm do not produce new unwanted τ' open sets which might be τ-open, we will show that if $(j_-, j^+) \cap (k_-, k^+)$ is nonempty, then $(j_-, j^+) \cap (k_-, k^+) = (r_-, r^+)$ for some r. If one of (j_-, j^+) is contained in the other, this is clear, so without loss of generality we may assume $k_- < j_- < k^+ < j^+$ so $(j_-, j^+) \cap (k_-, k^+) = (j_-, k^+) \neq \emptyset$. Let $r = k^+ - 1$. From the definition of k^+, we have r is a $\overset{|}{\bullet}$, $r^+ = k^+$, and $j_- < r < j$ (or else $j^+ = k^+$). Consider the three ways j_- may have been defined. If $j_- = 0$, then we cannot have $k_- < j_-$. If $j \in$ some $\overset{\bullet}{\underset{\circ}{|}} = (y, x]$ and j_- was the first $\overset{|}{\circ}$ below j, then $r \in (y, x]$ and $r_- = j_-$. If j is not in any $\underset{|}{\bullet}$, then j_- is the first $\overset{|}{\circ}$ below the first $\underset{|}{\bullet}$ below j. But $j_- < r < j$ implies $r_- = j_-$, whether r is in some $\underset{|}{\bullet}$ or not. In both allowed cases, we have $r^+ = k^+$ and $r_- = j_-$, so the intersection (j_-, k^+) is already an interval of form (r_-, r^+) generated by the algorithm.

(b) For each $\overset{|}{\bullet}$, the construction provides a τ' break immediately above, and for each $\underset{|}{\bullet}$, the construction provides a τ' break before reaching the $\overset{|}{\circ}$ at the other end of that link/break interval. This guarantees that enough breaks are provided.

Before proving (c), we mention the simplest case of (c): No τ'-neighborhood core $N' \neq X$ is τ-open. We have seen one justification for this in the first paragraph of Section 2.3. We give a second justification for this. If N' has a mid-row break, then this break was a τ^* break, and thus was not a τ break. Thus, τ reaches across the mid-row break and thus out of N', so N' was not τ-open. If N' has no mid-row breaks, then the top (bottom) break occurs above (below) a row R_j which had to satisfy the equivalent conditions: (a) $l_j \in G(r_j)$, (b) $r_j \in G(l_j)$, (c) $G(r_j) = R_j$, (d) $G(l_j) = R_j$, (e) $G(r_j) = G(l_j)$, (f) R_j is a neighborhood core in τ_j^*. Now if the top row R_j of N' is not R_n, then R_j had a $\overset{|}{\bullet}$, so τ linked up from R_j out of N', so N' is not τ-open. If the top row of N' is R_n and the bottom row of N' is R_k, so that $N' = R_n \cup R_{n-1} \cup \cdots \cup R_k$, then there had to be a $\overset{|}{\circ}$ at $k-1$, so τ linked from somewhere above R_k down to R_{k-1}, again showing that τ linked out of N', so N' is not τ-open. Thus, no single $N'(x) \neq X$ is τ-open.

Item (c) shows more: not only is no proper $N'(x)$ τ-open, no union $\bigcup_{x \in I} N'(x)$ is τ-open unless it is X. We need the following two Lemmas to show (c).

Lemma 1 *Every row R_j $(1 \leq j < n)$ links upward and every row R_j is linked to the top row by a sequence of τ- or τ'-links.*

Proof: If the first $\overset{|}{\bullet} \geq j$ is $> j$, then the definition of j^+ shows that R_j has a τ'-link up. If the first $\overset{|}{\bullet} \geq j$ is $= j$, then R_j has a τ-link up. Iterating the upward links, we will eventually reach the top. □

Lemma 2 *Every row links down and is linked to from the top by a sequence of τ- or τ'-links.*

Proof: We proceed by induction from the top row downward. Clearly the top row is linked to from the top. If the top row R_n does not τ-link down then n_- would be the first ⫰ below the first ⫯ below n—say this ⫯ occurs at row x—which insures that $n_- < x \leq n-1$ so R_n τ'-links down (to R_{n-1}) and R_{n-1} is linked to from the top.

Now suppose rows $n, n-1, \ldots, j+1$ link down and rows $n, n-1, \ldots, j+1, j$ are linked to from the top by a sequence of τ- and τ'-links ($j < n$).

If $j_- < j-1$, then R_j τ'-links down to row $j-1$, and since R_j was linked to from the top, with this one extra τ'-link, R_{j-1} is linked to from the top.

If $j_- = j-1$, then there is a link/break interval with its ⫰ on row $j-1$ The other endpoint of the link/break interval had to occur at $y \geq j$. Now y is reachable from the top by a sequence of links by the induction hypothesis; adding the τ-link generating the link/break interval $(j-1, y]$ shows that $j-1$ is reachable from the top. Furthermore, from Lemma 1, recall that row j reaches to the top. Since $j-1$ is reachable from the top, there is a sequence of links from row j to the top row and on down to row $j-1$, and thus row j links down. Now the result follows by mathematical induction. □

Now we are ready to show that the digraph $G \cup G'$ (as described in Proposition 1) is strongly connected. Pick arbitrary $x \in R_j$ and $y \in R_k$ from X. Now the links and breaks from the linear work link x to every point on R_j and y to every point on R_k. If $j = k$, we are done. Otherwise, applying the lemmas, R_j links up to the top and the top links down to R_k by sequences of τ- or τ'-links providing a sequence of links from x to y.

This completes the proof.

3 Examples

Example 3 If τ and τ' are complementary topologies on X and $A \subseteq X$, in general it is not true that the subspace topologies $\tau|_A$ and $\tau'|_A$ are complementary topologies on A. The particular complements constructed by our algorithms are no exception. For $X = \mathbf{3} \times \mathbf{2}$, consider the topology τ shown in Figure 4. Also shown are τ^*, τ', and the subspaces of $\tau|_A$ and $\tau'|_A$ where $A = \{2,3\} \times \mathbf{2}$. Since R_1 (or R_2) is a proper nonempty set open in both $\tau|_A$ and $\tau'|_A$, these two topologies on A are not complements.

Example 4 If $X = \mathbf{m} \times \mathbf{n} = \bigcup\{R_j : 1 \leq j \leq n\}$ and the τ-neighborhood core $N(x_j)$ of each point $x_j \in R_j$ satisfies $N(x_j) \subseteq R_j$, then τ provides no links between rows. In this case, consider the totally ordered space $(X, \tau, \leq_{lex})$ where $\leq_{lex}$ is the (reverse) lexicographic order on $X = \mathbf{m} \times \mathbf{n}$ defined by $(i,j) \leq_{lex} (k,l)$ if and only if $j < l$ or $j = l$ and $i \leq k$. Thus, $\leq_{lex}$ simply strings the rows of X together linearly. In this case, the complement τ' of $X = \mathbf{m} \times \mathbf{n}$ produced by the algorithm is the complement of the totally ordered space $(X, \tau, \leq_{lex})$ produced by the algorithm of [10].

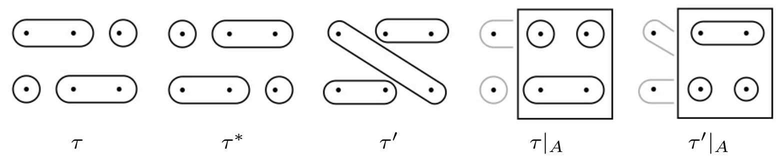


Fig. 4 The subspace of a complement is not a complement of the subspace.

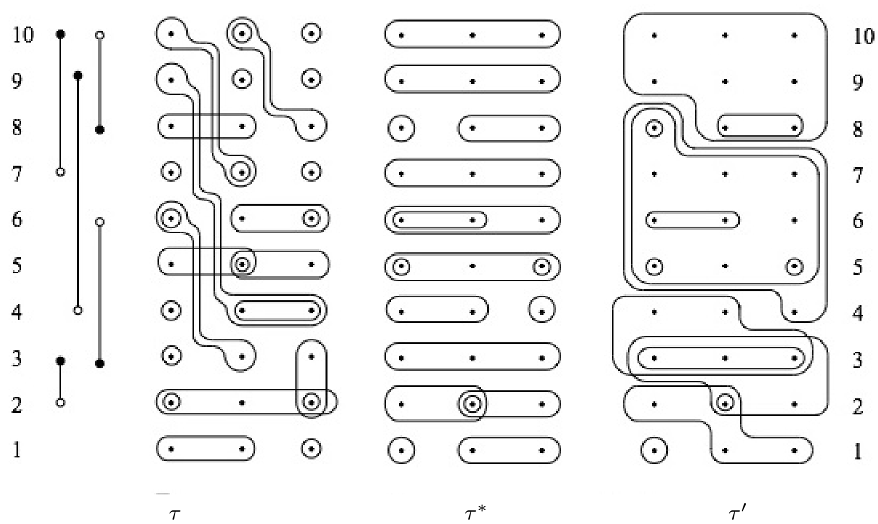


Fig. 5 An example

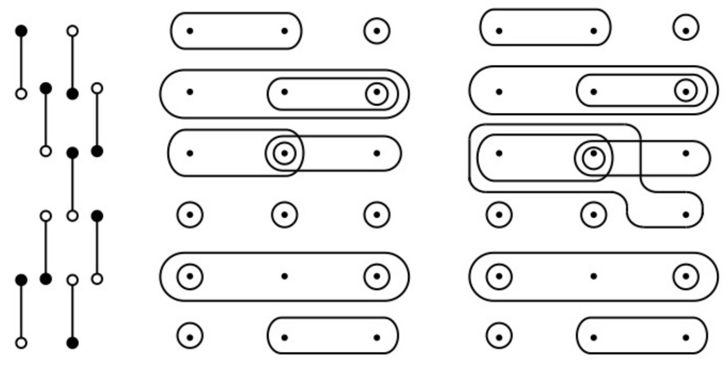
Example 5 Figure 5 below shows a convex topology τ on $\mathbf{3}\times\mathbf{10}$, the associated link/break intervals, the intermediate topology τ^*, and the convex complement τ' generated by the algorithm.

Example 6 Let τ be the convex topology on $\mathbf{3} \times \mathbf{6}$ shown in Figure 2. The topology τ' shown there is the convex complement of τ generated by our algorithm. We may apply the algorithm to τ' to find τ''. The link/break intervals from τ', the intermediate space τ'^*, and τ'' are shown in Figure 6.

Note that $\tau'' \neq \tau$. However, further analysis would show that for this example, $\tau' = \tau'''$.

In the case of linearly ordered spaces, it is shown in [10] that using the algorithm introduced there to generate the complements, it always holds that $\tau' = \tau'''$. It remains to be seen whether this always holds for the complements of topologies on products of totally ordered spaces produced by the current algorithm.

Of course, a convex topology may have many convex complements, so there may be other algorithms which generate (possibly different) convex complements. For example, the up/down dual of algorithm given above would also



τ'^{*} τ''

Fig. 6 Iterating the algorithm for τ' of Figure 2

generate a convex complement. Specifically, the dual algorithm would be as the original except that for each j we take

$$j_- = \begin{cases} \text{the greatest integer below the first } \overset{\bullet}{\vert} \ \leq j & \text{if } \exists \ \overset{\bullet}{\vert} \ \leq j \\ 0 & \text{otherwise} \end{cases}$$

$$j^+ = \begin{cases} \text{the least } \overset{\circ}{\vert} \ > j & \text{if } j \in \text{any } \underset{\bullet}{\vert} \\ \text{the least } \overset{\circ}{\vert} \text{ above the first } \underset{\bullet}{\vert} \ > j & \text{if } j \notin \text{any } \underset{\bullet}{\vert} \\ n+1 & \text{otherwise.} \end{cases}$$

If τ is the topology on $\mathbf{3} \times \mathbf{6}$ shown in Figure 2, the dual algorithm produces the convex complement $\tau^{\backprime}$ shown in Figure 7. Compare this with the complement τ' from Figure 2.

Example 7 Here we depart from our algorithm and illustrate some primitive techniques which may yield a convex complement for simple convex topologies on a product $\mathbf{m} \times \mathbf{n}$. For such a topology τ, the strategy is based on indicating the required breaks which the complement τ' must add and the permitted links in the directions of the "nearest neighbors" for each point. In the grid $\mathbf{m} \times \mathbf{n}$, a point x has eight "nearest neighbors" which are north, south, east, west, northeast, northwest, southeast, and southwest of it. However, if the point x_{ne} northeast of x is included in a convex set N containing x, then the points x_n and x_e north and east (respectively) of x must be included. But, then x_{ne} is north of x_e (and east of x_n), so linking and breaking in the northeast direction can always be obtained from linking and breaking in the north and east directions. Thus, we need not consider links and breaks in the northeast, and dually southwest, directions. This gives six nearest neighbors to consider. We draw a hexagon around each point of X with a solid edge indicating a

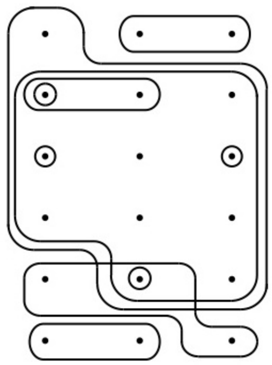


Fig. 7 The complement $\tau^{\backprime}$ of τ in Figure 2 from the dual algorithm

required τ'-break in that direction and a dotted edge indicating a permitted τ'-link in that direction, as seen in Figure 8. Then by inspection, we draw in minimal τ' neighborhoods which provide the indicated breaks. However, there is no unique way to accomplish this. Two possible ways, τ' and $\tau^{\sim}$ are shown in Figure 8. Such ambiguity in how to complete the algorithm indicates that

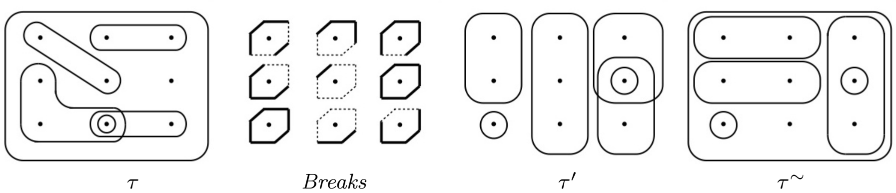


Fig. 8 A primitive approach

this technique has limitations, but there are other more serious limitations. In the topology τ shown, the only diagonal breaks needed were at 45° angles and we only considered neighbors at $45k^\circ$ angles ($k \in \mathbb{N}$). However, breaks in other directions may be needed, and links in other directions may be permitted. Had the neighborhood core of $(1,3)$ been $\{(1,3),(3,2)\}$ instead of $\{(1,3),(2,2)\}$, we would have to consider breaks (and possible links) in the direction from $(1,3)$ to $(3,2)$. In larger spaces, this problem could make this primitive strategy ineffective.

4 Higher Dimensional Products

The techniques presented here will apply to higher dimensional products of totally ordered spaces. The proof of the two-factor case only required linearity in the second factor, in order to define the link/break intervals and the intervals

(j_-, j^+). All that was required of the first factor was an algorithm to produce a convex complement with minimum and maximum elements r_i and l_i and good neighborhoods $G(r_i)$ and $G(l_i)$ to expand from.

Theorem 2 *Any convex topology τ on a finite product of finite totally ordered spaces $\mathbf{m_1}, \ldots, \mathbf{m_n}$ admits a convex complement.*

Proof: We proceed inductively on the number n of totally ordered spaces $\mathbf{m_i}$. The algorithms of [10] and Theorem 1 establish the result for $n = 1$ and $n = 2$. Having established the result for $n = k$, observe that $X = \mathbf{m_1} \times \cdots \times \mathbf{m_k} \times \mathbf{m_{k+1}}$ with topology τ may be partitioned into blocks (or "rows") $R_j = \mathbf{m_1} \times \cdots \times \mathbf{m_k} \times \{j\}$ for $j \in \mathbf{m_{k+1}}$. Each such block R_j, viewed as a subspace of X, is (homeomorphic and order isomorphic to) a product of k totally ordered spaces with a convex topology, and we may find a convex complement τ_j^* for each block R_j, and we take $\tau^* = [\bigcup\{\tau_j^* : j \in \mathbf{m_{k+1}}\}]$. As above, we take l_j and r_j to be the minimum and maximum elements of R_j and for $x_j \in \{l_j, r_j\}$ we take $G(x_j) = \bigcup\{N^*(y) \subseteq R_j : x_j \in N^*(y)\}$. Link/break intervals in the totally ordered factor $\mathbf{m_{k+1}}$ are defined as above. For each $j \in \mathbf{m_{k+1}}$, apply the linking techniques of the algorithms above to link the blocks $\{R_i : i \in (j_-, j^+)\}$ to each other through downward links from neighborhoods $G(l_i)$ of minimal points l_i in R_i and upward links from neighborhoods $G(r_i)$ of maximum points r_i of R_i. The arguments for the two-factor case above apply to show that this construction yields a convex complement of $X = \mathbf{m_1} \times \cdots \times \mathbf{m_{k+1}}$. □

Figure 9 suggests the required linking in the product of three totally ordered spaces.

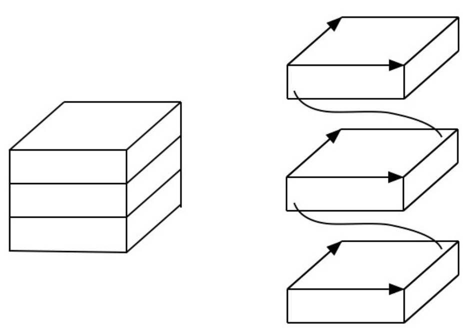


Fig. 9 Partition of a product into lower dimensional blocks, with linking between blocks.

References

1. B. A. Anderson, A class of topologies with T_1-complements. Fund. Math. **69** 267–277(1970)
2. B. A. Anderson and D. G. Stewart, T_1-complements of T_1 topologies. Proc. Amer. Math. Soc. **23(1)** 77-81 (1969)
3. J. I. Brown, and S. Watson, The number of complements of a topology on n points is at least 2^n (except for some special cases). Discrete Math. **154** 27–39 (1996)
4. T. Clark and T. Richmond, The number of convex topologies on a finite totally ordered set, Involve, **8(1)** 25–32 (2015)
5. M. Erné; The ABC of order and topology. In H. Herrlich and H.-E. Porst (Eds.) *Category Theory at Work*, Heldermann, Berlin, 57–93, 1991
6. J. Goubault-Larrecq, *Non-Hausdorff Topology and Domain Theory*, New Mathematical Monographs: 22, Cambridge University Press (2013)
7. M. Lamper, Complements in the lattice of all topologies of topological groups. Arch Math. (Brno) **10(4)** 221-230 (1974) (1975)
8. L. Nachbin, *Topology and Order*, Van Nostrand Math. Studies **4**, Princeton, NJ (1965)
9. T. Richmond, Quasiorders, principal topologies, and partially ordered partitions, Internat. J. Math. & Math. Sci. **21(2)** 221-234 (1998)
10. T. Richmond, Complementation in the lattice of locally convex topologies, Order **30(2)** 487–496 (2013)
11. P. S. Schnare, The topological complementation theorem a la Zorn. Proc. Amer. Math. Soc. **35(1)** 285–286 (1972)
12. D. Shakhmatov, D. and M. Tkachenko, A compact Hausdorff topology that is a T_1-complement of itself. Fund. Math. **175(2)** 163–173 (2002)
13. A. K. Steiner, The lattice of topologies: Structure and complementation. Trans. Amer. Math. Soc. **122** 379–398 (1966)
14. A. K. Steiner, Complementation in the lattice of T_1-topologies, Proc. Amer. Math. Soc. **17** 884–886 (1966)
15. A. K. Steiner and E. F. Steiner, Topologies with T_1-complements. Fund. Math. **61** 23–28 (1967)
16. A. K. Steiner and E. F. Steiner, A T_1-complement for the reals. Proc. Amer. Math. Soc. **19** 177-179 (1968)
17. C. Uzcátegui, Maximal complements in the lattice of pre-orders and topologies. Topology Appl. **132(2)** 147–157 (2003)
18. A. C. M. Van Rooij, The lattice of all topologies is complemented. Canad. J. Math. **20** 805–807 (1968)