Continued Radicals

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Abstract

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers and $f_j(x) = \sqrt{a_j + x}$, then $f_1 \circ f_2 \circ \cdots \circ f_n(0)$ is a nested radical with terms $a_1, \ldots, a_n$. If it exists, the limit as $n \to \infty$ of such an expression is a continued radical. We consider the set of real numbers $S(M)$ representable as a continued radical whose terms $a_1, a_2, \ldots$ are all from a finite set M. We give conditions on the set M for $S(M)$ to be (a) an interval, and (b) homeomorphic to the Cantor set.

1 Introduction

We will consider nested radicals of form

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots + \sqrt{a_n}}}}$$

which we will denote by $S_n = \sqrt{a_1, a_2, a_3, \ldots, a_n}$. The limit

$$\lim_{n\to\infty} S_n = \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}}, \qquad \text{denoted by} \quad \sqrt{a_1, a_2, a_3, \ldots},$$

is called a *continued radical*.

Continued radicals of this form are studied in [5], [7], and [2], and briefly in [9] and [3]. Ramanujan considered several continued radicals (see [2]) and showed that

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}}.$$

Continued radicals can be related to solutions of certain polynomial or radical equations (see [2] and [1]). Laugwitz [5] studies chain operations or iterated function systems of form $\lim_{n\to\infty}(f_1 \circ f_2 \circ \cdots f_n)(x)$. Observe that infinite series, infinite products, continued fractions, and continued radicals may all be so represented. Much work has been done on convergence criteria for continued radicals (see [1], [7], and [5]). Our emphasis is on the forms of the sets $S(M)$ of real numbers which are representable as a continued radical whose terms $a_1, a_2, \ldots$ are all from a finite set M. The analogous problem for continued fractions has been considered in [6] (see also [4]).

Perhaps the most familiar continued radical is $\sqrt{1, 1, 1, \ldots}$, whose value is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803$. The popular verification of this relies on the self-similarity of the continued radical: If $S = \sqrt{1, 1, 1, \ldots}$, then $S^2 = 1 + S$, and S must be the positive root of this quadratic equation. This argument has overlooked the serious issue of convergence. With similar careless regard for convergence, one might incorrectly conclude that

$$T = \sqrt{1 - \sqrt{1 - \sqrt{1 - \sqrt{1 - \cdots}}}}$$

is the positive solution of $T^2 = 1 - T$ and thus $T = \frac{-1+\sqrt{5}}{2} \approx .61803$. However, the partial expressions

$$\sqrt{1},\ \sqrt{1 - \sqrt{1}},\ \sqrt{1 - \sqrt{1 - \sqrt{1}}},\ \sqrt{1 - \sqrt{1 - \sqrt{1 - \sqrt{1}}}},\ \ldots$$

alternate $1, 0, 1, 0, \ldots$, and thus do not converge.

To guarantee that each partial expression S_n of a continued radical $\sqrt{a_1, a_2, a_3, \ldots}$ is defined, each *term* a_i must be nonnegative. We will restrict our attention to continued radicals $\sqrt{a_1, a_2, a_3, \ldots}$ whose terms a_i are whole numbers, though we note that the fundamental result below holds for any nonnegative terms.

Proposition 1.1 *(a) If $a_i \geq b_i \geq 0$ for all $i \in \mathbb{N}$, then $\sqrt{a_1, a_2, \ldots, a_k} \geq \sqrt{b_1, b_2, \ldots, b_k}$ for all $k \in \mathbb{N}$.*

(b) *If $a_i \geq 0$ for $i \in \mathbb{N}$, then the sequence $\sqrt{a_1}, \sqrt{a_1, a_2}, \sqrt{a_1, a_2, a_3}, \ldots$ of partial expressions of the continued radical $\sqrt{a_1, a_2, a_3, \ldots}$ is an increasing sequence.*

Part (a) follows from a direct calculation, using the fact that $f(x) = \sqrt{x}$ is an increasing function. Part (b) follows from part (a).

Observe that the converse of Part (a) does not hold. If $(a_i)_{i=1}^{\infty} = (8, 4, 0, 0, 0, \ldots)$ and $(b_i)_{i=1}^{\infty} = (6, 9, 0, 0, 0, \ldots)$, then $\sqrt{a_1, a_2, \ldots, a_k} \geq \sqrt{b_1, b_2, \ldots, b_k}$ for all $k \in \mathbb{N}$, but $a_2 \ngeq b_2$.

Proposition 1.2 *If the sequence $(a_i)_{i=1}^{\infty}$ of nonnegative numbers is bounded above, then $\sqrt{a_1, a_2, a_3, \ldots}$ converges.*

Proof. Suppose $(a_i)_{i=1}^{\infty}$ is bounded above by $M \geq 2$. We will show that the increasing sequence $S_1 = \sqrt{a_1}$, $S_2 = \sqrt{a_1, a_2}$, $S_3 = \sqrt{a_1, a_2, a_3}$, $\ldots$ of partial expressions is bounded above by M^2. By Proposition 1.1 (a),

$$S_k = \sqrt{a_1, a_2, \ldots, a_k} \leq \sqrt{M, M, \ldots, M} = q_k$$

where the number of Ms in the latter nested radical is k. Now $S_1 < q_1 = \sqrt{M} < M^2$. Suppose $q_k < M^2$. Then $q_{k+1} = \sqrt{M + q_k} < \sqrt{M + M^2} < \sqrt{2M^2} = M\sqrt{2} < M^2$. Thus, $(q_k)_{k=1}^{\infty}$ and therefore $(S_k)_{k=1}^{\infty}$ is bounded above by M^2, as needed. ■

The converse of the proposition fails. One can show, for example, that $\sqrt{1, 2, 3, 4, \ldots}$ converges even though $(1, 2, 3, 4, \ldots)$ is not bounded. Sizer [7] gives necessary and sufficient conditions on the terms of a continued radical to guarantee convergence. He shows that $\sqrt{a_1, a_2, a_3, \ldots}$ converges if and only if the set $\{\sqrt[2^i]{a_i} : i \in \mathbb{N}\}$ is bounded (see also Laugwitz [5]).

Proposition 1.2 shows that for any nonnegative number n, $\sqrt{n, n, n, \ldots}$ converges. We will denote the value of $\sqrt{n, n, n, \ldots}$ by φ_n. Now $\varphi_n^2 = n + \varphi_n$ and if $n \in \mathbb{N}$, the quadratic formula shows that

$$\sqrt{n, n, n, \ldots} = \varphi_n = \frac{1 + \sqrt{4n + 1}}{2}.$$

It is easy to verify that φ_n is a nonzero integer if and only if n is twice a triangular number. Specifically, $\varphi_n = k \in \mathbb{N}$ if and only if $n = k(k-1) = 2T_{k-1}$ for some integer $k \geq 2$. Recall that a triangular number is an integer of form $T_m = 1 + 2 + \cdots + m = \frac{m(m+1)}{2} = \binom{m+1}{2}$ for $m \in \mathbb{N}$.

We will consider continued radicals whose terms all come from a finite set $M = \{m_1, m_2, \dots, m_p\}$ of integral values. We will determine which such sets M permit us to represent all real numbers from an interval and which such sets M leave gaps in the set of numbers representable. We will consider uniqueness of representation. First we consider sets M of nonnegative terms in Section 2. Allowing zero as a term complicates matters, and this case is considered in Section 3. In Section 4 we consider the interesting pattern of gaps in the numbers representable by continued radicals whose terms all come from a two-element set $\{m_1, m_2\} \subseteq \mathbb{N}$.

2 Continued radicals with nonzero terms

Let us consider continued radicals $\sqrt{a_1, a_2, a_3, \dots}$ whose terms a_i come from a set $M = \{m_1, m_2, \dots, m_p\} \subseteq \mathbb{N}$ where $0 < m_1 < m_2 < \cdots < m_p$. We will be interested in sets M of "term values" which allow all points of a nondegenerate interval to be represented. To insure that no gaps occur in the set of values representable as a continued radical $\sqrt{a_1, a_2, a_3, \dots}$ with terms from M, it is necessary that the largest value representable with $a_1 = m_i$ equal or exceed the smallest value representable with $a_1 = m_{i+1}$ (for $i = 1, \dots, p-1$). That is, it is necessary that

$$\sqrt{m_i, m_p, m_p, m_p, \dots} \geq \sqrt{m_{i+1}, m_1, m_1, m_1, \dots} \quad \forall i \in \{1, \dots, p-1\},$$

or equivalently, that

$$\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + \varphi_{m_1}} \quad \forall i \in \{1, \dots, p-1\}.$$

In fact, this condition will be necessary and sufficient, as we will see in Theorem 2.2. First, we need a lemma.

Lemma 2.1 *Suppose $(a_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ are sequences of real numbers with $a_n \geq 1$ $\forall n \in \mathbb{N}$ and $r_n \geq 2$ $\forall n \in \mathbb{N}$, and for $k \in \mathbb{N}$, define*

$$\begin{aligned} h_k(x) &= \sqrt[r_1]{a_1 + \sqrt[r_2]{a_2 + \sqrt[r_3]{a_3 + \cdots + \sqrt[r_k]{a_k + x}}}} \\ &= \left(a_1 + \left(a_2 + \left(a_3 + \cdots + (a_k + x)^{\frac{1}{r_k}} \cdots\right)^{\frac{1}{r_3}}\right)^{\frac{1}{r_2}}\right)^{\frac{1}{r_1}}. \end{aligned}$$

Then the derivative of h_k over $[0, \infty)$ is positive and bounded above by 2^{-k}. That is, $0 < h_k'(c) \leq 2^{-k}$ for all $c \geq 0$.

Proof. Applying the chain rule for differentiation and recalling that $a_n \geq 1$ for all $n \in \mathbb{N}$, we find that

$$h_k'(c) = \frac{1}{r_1}(w_1)^{\frac{1}{r_1}-1} \cdot \frac{1}{r_2}(w_2)^{\frac{1}{r_2}-1} \cdot \frac{1}{r_3}(w_3)^{\frac{1}{r_3}-1} \cdots \frac{1}{r_k}(w_k)^{\frac{1}{r_k}-1}$$

where $w_n \geq 1$ for $n = 1, \ldots, k$. Observing that $r_n \geq 2$ for $n = 1, \ldots, k$, we see that each $\frac{1}{r_n} - 1$ is negative, so with $p_n = -(\frac{1}{r_n} - 1) > 0$, we have

$$0 < h_k'(c) = \frac{1}{r_1 r_2 r_3 \cdots r_k (w_1)^{p_1} (w_2)^{p_2} \cdots (w_k)^{p_k}} \leq \frac{1}{r_1 r_2 r_3 \cdots r_k} \leq \frac{1}{2^k} = 2^{-k}.$$

■

Theorem 2.2 *Suppose $M = \{m_1, m_2, \ldots, m_p\} \subseteq \mathbb{N}$ where $0 < m_1 < m_2 < \cdots < m_p$ and*

$$\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + \varphi_{m_1}} \quad \forall i \in \{1, \ldots, p-1\}.$$

Then the set of numbers representable as a continued radical $\sqrt{a_1, a_2, a_3, \ldots}$ with terms $a_i \in M$ is the interval $[\varphi_{m_1}, \varphi_{m_p}]$.

Proof. We present an algorithm to construct a representation $\sqrt{a_1, a_2, a_3, \ldots}$ ($a_i \in M \ \forall i \in \mathbb{N}$) for any given $b \in [\varphi_{m_1}, \varphi_{m_p}]$. Suppose $b \in [\varphi_{m_1}, \varphi_{m_p}]$ is given. Set $a_1 = m_p$ if $b \geq \sqrt{m_p + \varphi_{m_1}}$ and otherwise take $a_1 = m_i$ where m_i is the largest element of M for which $b \in [\sqrt{m_i + \varphi_{m_1}}, \sqrt{m_{i+1} + \varphi_{m_1}})$. Having found $a_1, a_2, \ldots, a_{n-1}$, we take a_n to be the largest element m_i of M for which

$$\sqrt{a_1, \ldots, a_{n-1}, m_i, m_1, m_1, m_1, \ldots} = \sqrt{a_1, \ldots, a_{n-1}, m_i + \varphi_{m_1}} \leq b.$$

The sequence of partial expressions $(S_n)_{n=1}^{\infty}$ determined by the terms $(a_1, a_2, a_3, \ldots)$ is bounded above by b, and therefore must converge. We will show that it converges to b by considering the auxiliary sequence $(b_n)_{n=1}^{\infty}$ defined by

$$\begin{aligned} b_{2n} &= \sqrt{a_1, \ldots, a_n, m_p, m_p, m_p, \ldots} = \sqrt{a_1, \ldots, a_n + \varphi_{m_p}} \quad \forall n \in \mathbb{N} \\ b_{2n-1} &= \sqrt{a_1, \ldots, a_n, 0, 0, 0, \ldots} = \sqrt{a_1, \ldots, a_n} = S_n \quad \forall n \in \mathbb{N}. \end{aligned}$$

Clearly $S_n = b_{2n-1} \leq b$ for all $n \in \mathbb{N}$. We will now show by induction that $b \leq b_{2n}$ for all $n \in \mathbb{N}$. For $n = 1$, suppose $a_1 = m_i < m_p$. Then

$$b_{2n} = \sqrt{a_1 + \varphi_{m_p}} = \sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + \varphi_{m_1}} > b$$

by the assignment of $a_1 = m_i$. If $a_1 = m_p$, then $b_{2n} = \sqrt{m_p + \varphi_{m_p}} = \varphi_{m_p} \geq b \in [\varphi_{m_1}, \varphi_{m_p}]$. Now suppose we have shown that $b_{2n-2} \geq b$. Consider $b_{2n} = \sqrt{a_1, \ldots, a_{n-1}, a_n + \varphi_{m_p}}$. If $a_n = m_p$, then

$$b_{2n} = \sqrt{a_1, \ldots, a_{n-1}, m_p + \varphi_{m_p}} = \sqrt{a_1, \ldots, a_{n-1} + \varphi_{m_p}} \geq b$$

by the induction hypothesis. If $a_n = m_i < m_p$, then

$$\begin{aligned} b_{2n} &= \sqrt{a_1, \ldots, a_{n-1}, m_i + \varphi_{m_p}} \\ &= \sqrt{a_1, \ldots, a_{n-1} + \sqrt{m_i + \varphi_{m_p}}} \\ &\geq \sqrt{a_1, \ldots, a_{n-1} + \sqrt{m_{i+1} + \varphi_{m_1}}} \\ &\geq b \end{aligned}$$

where the last inequality follows from the choice of $a_n = m_i$ to be the largest m_k for which $\sqrt{a_1, \ldots, a_{n-1}, m_k + \varphi_{m_1}} \leq b$. This completes the proof that $b_{2n-1} \leq b \leq b_{2n}\ \forall n \in \mathbb{N}$.

For a fixed $k \in \mathbb{N}$, let h_k be defined as in Lemma 2.1, with $r_n = 2\ \forall n \in \mathbb{N}$. Now given $i > j > 2k$, we have $b_i = h_k(x_1)$ for some $x_1 \in [0, \varphi_{m_p}]$ and $b_j = h_k(x_0)$ for some $x_0 \in [0, \varphi_{m_p}]$. The mean value theorem applies to h_k over the interval with endpoints x_0 and x_1, so

$$|b_i - b_j| = |h_k(x_1) - h_k(x_0)| = h_k'(c)|x_1 - x_0|$$

for some c between x_0 and x_1. Now $|x_1 - x_0| \leq \varphi_{m_p}$ and, by Lemma 2.1, $h_k'(c) \leq 2^{-k}$, so we have $|b_i - b_j| \leq \varphi_{m_p} 2^{-k}$ for any $i, j > k$. Given any $\epsilon > 0$, we may find $k \in \mathbb{N}$ such that $\varphi_{m_p} 2^{-k} < \epsilon$, and it follows that $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence and must converge. Now the two subsequences $(b_{2n-1})_{n=1}^{\infty}$ and $(b_{2n})_{n=1}^{\infty}$ of the convergent sequence $(b_n)_{n=1}^{\infty}$ must have the same limit L, and the inequality $S_n = b_{2n-1} \leq b \leq b_{2n}\ \forall n \in \mathbb{N}$ shows that $L \leq b \leq L$, so $L = b$. In particular, $(b_{2n-1})_{n=1}^{\infty} = (S_n)_{n=1}^{\infty}$ must converge to b. ■

The argument of the last paragraph above can be used to show that if $M = \{m_1, \ldots, m_p\}$ where $0 < m_1 < \ldots < m_p$ and $a_n \in M$ for $n =$

$1, \ldots, k, \ldots, j$, then the nested radicals $\sqrt{a_1, \ldots, a_k}$ and $\sqrt{a_1, \ldots, a_j}$ differ by no more than $2^{-k}\varphi_{m_p}$.

In choosing sets of values $M = \{m_1, \ldots, m_p\}$ to serve as terms of continued radical representations of the elements of an interval $[\varphi_{m_1}, \varphi_{m_p}]$, we might want the most efficient selection of terms. Suppose there exists $i \in \{1, 2, \ldots, p-1\}$ such that $\sqrt{m_i + \varphi_{m_p}} > \sqrt{m_{i+1} + \varphi_{m_1}}$. Then the algorithm of Theorem 2.2 applied to $b = \sqrt{m_i + \varphi_{m_p}} = \sqrt{m_i, m_p, m_p, m_p, \ldots}$ greedily chooses the initial term as large as possible and produces a representation of form $b = \sqrt{m_{i+1}, a_2, a_3, \ldots}$. Thus, we do not have uniqueness of representation if $\sqrt{m_i + \varphi_{m_p}} > \sqrt{m_{i+1} + \varphi_{m_1}}$ for some $i \in \{1, \ldots, p-1\}$. Consequently, in choosing the values $\{m_1, \ldots, m_p\}$ most efficiently, the inequalities

$$\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + \varphi_{m_1}} \quad \forall i \in \{1, \ldots, p-1\}$$

to insure that every value in $[\varphi_{m_1}, \varphi_{m_p}]$ is representable must actually be equalities

$$\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}} \quad \forall i \in \{1, \ldots, p-1\}$$

to prevent unnecessary duplication of representation. These equalities will not eliminate all duplication of representation, but will limit it to situations involving overlapping endpoints of the intervals $(I_i)_{i=1}^p$ where $I_i = [\sqrt{m_i + \varphi_{m_1}}, \sqrt{m_i + \varphi_{m_p}}]$ contains all the points representable as $\sqrt{m_i, a_2, a_3, \ldots}$. If there exists $b \in [\varphi_{m_1}, \varphi_{m_p}]$ having distinct representations $b = \sqrt{a_1, a_2, \ldots} = \sqrt{b_1, b_2, \ldots}$ with $a_1 \neq b_1$, then $a_1 = m_i$ implies $b \in I_i$ and $b_1 = m_j$ implies $b \in I_j$. Since $i \neq j$ and $I_i \cap I_j \neq \emptyset$, we must have (assuming, without loss of generality, that $i < j$) $b = \sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}}$, the shared endpoint of adjacent intervals I_i and I_j. It follows that the only duplication of representations must be of form $\sqrt{c_1, \ldots, c_z, a_1, a_2, \ldots} = \sqrt{c_1, \ldots, c_z, b_1, b_2, \ldots}$, where $\sqrt{a_1, a_2, \ldots} = \sqrt{b_1, b_2, \ldots}$ are as above. That is, the only possible duplication of representation must be of form

$$\begin{aligned} \sqrt{c_1, \ldots, c_z, m_i + \varphi_{m_p}} &= \sqrt{c_1, \ldots, c_z, m_{i+1} + \varphi_{m_1}} \\ \sqrt{c_1, \ldots, c_z, m_i, m_p, m_p, m_p, \ldots} &= \sqrt{c_1, \ldots, c_z, m_{i+1}, m_1, m_1, m_1, \ldots}, \end{aligned}$$

where repeating the largest value m_p is equal to raising the preceding term from m_i to m_{i+1} and repeating the smallest value m_1. (Compare to the decimal equation $1.3\overline{999} = 1.4\overline{000}$.)

To investigate when $\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}}$ may occur, we start with a lemma.

Lemma 2.3 *Suppose z and w are distinct natural numbers. Then $\sqrt{z} - \sqrt{w} \in \mathbb{Z}$ if and only if $\sqrt{z}$ and $\sqrt{w}$ are natural numbers.*

Proof. Suppose $z, w \in \mathbb{N}$ and $\sqrt{z} - \sqrt{w} = r \in \mathbb{Z}\setminus\{0\}$. Then $r^2 = z - 2\sqrt{zw} + w \in \mathbb{N} \Rightarrow \sqrt{zw} \in \mathbb{Q} \Rightarrow \sqrt{zw} \in \mathbb{N}$. Now $r\sqrt{z} = (\sqrt{z} - \sqrt{w})\sqrt{z} = z - \sqrt{zw} = s \in \mathbb{Z}$ (recalling that $\sqrt{zw} \in \mathbb{N}$). Dividing $r\sqrt{z} = s$ by $r \neq 0$ shows $\sqrt{z} \in \mathbb{Q}$ and thus $\sqrt{z} \in \mathbb{N}$. Similarly, $r\sqrt{w} \in \mathbb{Z}$ implies $\sqrt{w} \in \mathbb{N}$. The converse is immediate. ■

Now suppose $\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}}$. Squaring both sides of this equation leads to $\varphi_{m_p} - \varphi_{m_1} = m_{i+1} - m_i \in \mathbb{N}$. Since $\varphi_x = \frac{1+\sqrt{4x+1}}{2}$, we find that $\varphi_{m_p} - \varphi_{m_1} \in \mathbb{N}$ if and only if $\sqrt{4m_p + 1} - \sqrt{4m_1 + 1}$ is an even integer. By Lemma 2.3, $\sqrt{4m_p + 1}$ and $\sqrt{4m_1 + 1}$ must both be integers, and as square roots of odd numbers, they must both be odd integers (so their difference is even). Adding 1 to each and dividing by 2 shows that φ_{m_p} and φ_{m_1} are integers.

Thus, to avoid unnecessary duplication of representation, we must have $\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}}$, or $m_{i+1} = m_i + \varphi_{m_p} - \varphi_{m_1}$, and φ_{m_1} and φ_{m_p} must both be integers, say $n + 1$ and $j + 1$. Recall that this occurs if and only if $m_1 = n(n+1)$ and $m_p = j(j+1)$ for some values $n, j \in \mathbb{N}$ with $j > n$, and it follows that

$$M = \{n(n+1), n(n+1) + 1(j-n), n(n+1) + 2(j-n), \dots, j(j+1)\}$$

has $j + n + 2$ equally spaced terms.

Below, we summarize our results on "efficient" sets M for which $S(M)$ is an interval and the terms of M are spaced as widely as possible to eliminate unnecessary duplication of representations.

Theorem 2.4 *Suppose $M = \{m_1, m_2, \dots, m_p\} \subseteq \mathbb{N}$ where $0 < m_1 < m_2 < \dots < m_p$ and*

$$\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}} \quad \forall i \in \{1, \dots, p-1\}.$$

Then $M = \{n(n+1), n(n+1) + 1(j-n), n(n+1) + 2(j-n), \dots, j(j+1)\}$, $\varphi_1 = n + 1$ and $\varphi_p = j + 1$ are natural numbers, $m_{i+1} = m_i + \varphi_{m_p} - \varphi_{m_1} = m_i + j - n$ for each $i \in \{1, \dots, p-1\}$, each number in the interval $[\varphi_{m_1}, \varphi_{m_p}]$ is representable as a continued radical $\sqrt{a_1, a_2, a_3, \dots}$ with terms $a_i \in M$, and the representation is unique except for expressions of form

$$\sqrt{c_1, \dots, c_z, m_i, m_p, m_p, m_p, \dots} = \sqrt{c_1, \dots, c_z, m_{i+1}, m_1, m_1, m_1, \dots}.$$

Example 2.5 *Let* $M = \{2, 4, 6, 8, 10, 12\}$. *Now* $m_1 = 2 = 1(2) = 2T_1$, $m_p = 12 = 3(4) = 2T_3$, $\varphi_{m_1} = \varphi_2 = 2$, *and* $\varphi_{m_p} = \varphi_{12} = 4$. *Observe that* $\varphi_{m_p} - \varphi_{m_1} = 4 - 2 = 2$, *and our set* $M = \{m_1, \ldots, m_p\}$ *does satisfy* $m_{i+1} = m_i + \varphi_{m_p} - \varphi_{m_1}$ *for all* $i = 1, \ldots, p-1$, *that is* $\sqrt{m_i + \varphi_{m_p}} = \sqrt{m_{i+1} + \varphi_{m_1}}$. *Thus, every real number* $b \in [2, 4] = [\varphi_{m_1}, \varphi_{m_p}]$ *has a representation* $\sqrt{a_1, a_2, \ldots}$ *where* $a_i \in \{2, 4, 6, 8, 10, 12\}$ $\forall i \in \mathbb{N}$. *Duplication of representations occur, for example, in*

$$\begin{aligned}\sqrt{8, 4, 12, 12, 12, \ldots} &= \sqrt{8, 4 + \varphi_{12}} = \sqrt{8, 4 + 4} \\ &= \sqrt{8, 6 + 2} = \sqrt{8, 6 + \varphi_2} = \sqrt{8, 6, 2, 2, 2, \ldots}.\end{aligned}$$

To find the representation of $\pi \in [2, 4]$, *note that*

$$\sqrt{8} \approx 2.82842712 \leq \pi < \sqrt{10} \approx 3.16227766,$$

so our algorithm assigns $a_1 = 6$, *the largest value of* $m_i \in M$ *for which* $\sqrt{m_i + \varphi_2} = \sqrt{m_i + 2} \leq \pi$. *Having found* a_1, *we note that*

$$\sqrt{6, 12 + \varphi_2} = \sqrt{6, 14} \approx 3.12116282 \leq \pi,$$

so we have $a_2 = 12$. *Next,*

$$\begin{aligned}\sqrt{6, 12, 6 + \varphi_2} &= \sqrt{6, 12, 8} \approx 3.13859358 \\ &\leq \pi < \sqrt{6, 12, 10} = \sqrt{6, 12, 8 + \varphi_2} \approx 3.14545337,\end{aligned}$$

so $a_3 = 6$ *and* $\sqrt{a_1, a_2, a_3} = \sqrt{6, 12, 6}$. *Next,*

$$\begin{aligned}\sqrt{6, 12, 6, 6 + \varphi_2} &= \sqrt{6, 12, 6, 8} \approx 3.14153977 \\ &\leq \pi < \sqrt{6, 12, 6, 10} \approx 3.14268322,\end{aligned}$$

so $a_4 = 6$ *and* $\sqrt{a_1, a_2, a_3, a_4} = \sqrt{6, 12, 6, 6}$. *Continuing, we find that*

$$\pi = \sqrt{6, 12, 6, 6, 2, 2, 10, 4, 4, 2, 8, 10, 12, 6, \ldots}.$$

The theorem below shows that when using a set of terms M as described in Theorem 2.4, any number having a periodic sequence of terms is either irrational or an integer.

Theorem 2.6 *Suppose* M *satisfies the conditions of Theorem 2.4,* x *is a rational number in* $S(M) = [\varphi_{m_1}, \varphi_{m_p}]$, *and* $x = \sqrt{a_1, a_2, a_3, \ldots}$ *where* $a_i \in M$ *for all* $i \in \mathbb{N}$. *Then the sequence* $a_1, a_2, a_3, \ldots$ *of terms of* x *is eventually periodic if and only if* x *is an integer.*

Proof. Suppose $x \in S(M) \cap \mathbb{Z}$. If $\sqrt{a_1, a_2, a_3, \ldots}$ is not the unique representation of x, then by Theorem 2.4 the sequence of terms is either eventually constantly m_1 or eventually constantly m_p, and we are done. Thus, suppose $\sqrt{a_1, a_2, a_3, \ldots}$ is the unique representation of x. The algorithm to find the terms $a_1, a_2, \ldots$ chooses a_1 as large as possible so that $x = \sqrt{a_1 + b_1}$ where $b_1 = \sqrt{a_2, a_3, \ldots} \in S(M)$. Furthermore, $b_1 = x^2 - a_1$ is an integer in $S(M)$. Now $b_1 = \sqrt{a_2 + b_2}$ where $b_2 = \sqrt{a_3, a_4, \ldots} \in S(M)$, and $b_2 = b_1^2 - a_2$ is an integer in $S(M)$. Continuing in this manner, we find that $b_j = \sqrt{a_{j+1}, a_{j+2}, \ldots}$ is an integer in $S(M)$ for each $j \in \mathbb{N}$. Since $S(M) \cap \mathbb{Z}$ is finite, there must exist a smallest pair of indices j and $j + k$ such that $b_j = b_{j+k}$. Thus, $\sqrt{a_{j+1}, a_{j+2}, \ldots} = \sqrt{a_{j+k+1}, a_{j+k+2}, \ldots}$, and uniqueness of representation implies $a_r = a_{r+k}$ for all integers $r > j$.

For the converse, first suppose $x = \sqrt{a_1, \ldots, a_k, a_1, \ldots, a_k, \ldots}$ has a purely periodic representation and $x \in \mathbb{Q}$. Then x is a solution to $p(x) = x$ or $p(x) - x = 0$ where $p(x) = (\cdots((x^2 - a_1)^2 - a_2)^2 - \cdots - a_{k-1})^2 - a_k$ is a monic polynomial with integer coefficients. By the rational root theorem, x must be an integer. Next, if $x = \sqrt{b_1, \ldots, b_j, a_1, \ldots, a_k, a_1, \ldots, a_k, \ldots} \in \mathbb{Q}$ has an eventually periodic representation, then x is a root of the monic polynomial with integer coefficients $p(x) - a$ where $p(x) = (\cdots((x^2 - b_1)^2 - b_2)^2 - \cdots - b_{j-1})^2 - b_j$ and $a = \sqrt{a_1, \ldots, a_k, a_1, \ldots, a_k, \ldots} \in \mathbb{Z}$. Again, the rational root theorem implies $x \in \mathbb{Z}$. ∎

3 Representation allowing 0 as a term

Allowing zero as a term in our continued radicals introduces some minor complications. We will assume our terms a_n all come from a set $M = \{m_1, m_2, \ldots, m_p\} \subseteq \mathbb{N} \cup \{0\}$ where $0 = m_1 < m_2 < \cdots < m_p$. To prevent gaps in the set $S(M)$ of numbers representable with these terms, the largest value of form $\sqrt{m_i, a_2, a_3, \ldots}$ must equal or exceed the smallest value of form $\sqrt{m_{i+1}, b_2, b_3, \ldots}$. That is, we must have

$$\begin{aligned} \sqrt{m_i, m_p, m_p, m_p, \ldots} &\geq \sqrt{m_{i+1}, 0, 0, 0, \ldots} \\ \sqrt{m_i + \varphi_{m_p}} &\geq \sqrt{m_{i+1}}. \end{aligned}$$

However, note that besides the single value $\sqrt{m_{i+1}} = \sqrt{m_{i+1}, 0, 0, 0, \ldots}$, every other value representable as $\sqrt{m_{i+1}, b_2, b_3, \ldots}$ must be greater than

$$\sqrt{m_{i+1}, 1} = \sqrt{m_{i+1} + 1} = \lim_{k \to \infty} \sqrt{m_{i+1}, 0, 0, \ldots, 0, 0, b_k, 0, 0, \ldots} \text{ where } b_k \neq 0.$$

We saw in the previous section that if all term values $m_i \in M$ are positive, then the numbers representable as $\sqrt{m_i, a_2, \dots}$ where $a_n \in M \ \ \forall n \in \mathbb{N}$ formed a closed interval $I_i = [\sqrt{m_i + \varphi_{m_1}}, \sqrt{m_i + \varphi_{m_p}}\,]$ (assuming the m_i's were chosen so that $I_i \cap I_{i+1} \neq \emptyset$ for $i = 1, \dots, p-1$). We now see that if $m_1 = 0$, then the numbers representable as $\sqrt{m_i, a_2, \dots}$ where $a_n \in M \ \ \forall n \in \mathbb{N}$ will be a subset of $\{\sqrt{m_i}\,\} \cup (\sqrt{m_i + 1}, \sqrt{m_i + \varphi_{m_p}}\,] = J_i$. To prevent any gaps in $S(M)$, it is necessary that $\bigcup\{J_i : i = 1, 2, \dots, p\}$ forms a solid interval. Consequently, it is necessary that

$$\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + 1} \ \ \forall i = 1, 2, \dots, p-1.$$

Squaring both sides of this equation gives the necessary condition that

$$m_{i+1} \leq m_i + (\varphi_{m_p} - 1) \ \ \forall i = 1, 2, \dots, p-1.$$

One would again expect that choosing the values of $m_1, \dots, m_p$ so that the above inequalities are equalities will result in the most efficient representation of the largest possible interval using the smallest number of terms. For equality to hold, we must have that $\varphi_{m_p} \in \mathbb{N}$, and thus $m_p = (q+1)q$ for some $q \in \mathbb{N}$. Then since $\varphi_{(q+1)q} = q + 1$, the equation $m_{i+1} = m_i + (\varphi_{m_p} - 1)$ becomes $m_{i+1} = m_i + q$, and thus $M = \{0, q, 2q, 3q, \dots, (q+1)q\}$. The theorem below confirms these expectations.

Theorem 3.1 *Suppose $q \in \mathbb{N}$. Any $b \in (1, q+1]$ can be represented as a continued radical $\sqrt{a_1, a_2, a_3, \dots}$ where $a_i \in \{0, q, 2q, 3q, \dots, (q+1)q\} = M$ for every $i \in \mathbb{N}$. Furthermore, if any $b \in (1, q+1]$ can be represented as a continued radical $\sqrt{a_1, a_2, a_3, \dots}$ where $a_i \in J$ for every $i \in \mathbb{N}$ and $|J| \leq |M|$, then $J = M$. That is, M is the unique set of $q+2$ nonnegative integer terms allowing every $b \in (1, q+1]$ to be represented, and there is no set of $q+1$ or fewer nonnegative integer terms which allow every $b \in (1, q+1]$ to be represented.*

Proof: Given $q \in \mathbb{N}$ and $b \in (1, q+1]$, we present an algorithm to generate a sequence $(a_i)_{i=1}^{\infty}$ such that $b = \sqrt{a_1, a_2, a_3, \dots}$ where each $a_i \in M = \{m_i : i = 1, \dots, q+2\}$ and $m_i = (i-1)q$. The algorithm is greedy, taking each a_i as large as possible subject to the restriction that $\sqrt{a_1, \dots, a_k} \leq b \ \ \forall k \in \mathbb{N}$. Set $a_1 = m_i$ if $b = \sqrt{m_i}$ and otherwise take a_1 to be the largest m_i for which $\sqrt{m_i + 1} < b$ (or $\sqrt{m_i + 1} \leq b$ in case $q = 1$). Having found $a_1, a_2, \dots, a_{n-1}$, take $a_n = m_i$ if $\sqrt{a_1, \dots, a_{n-1}, m_i} = b$ and otherwise take a_n to be the largest

m_i for which $\sqrt{a_1, \ldots, a_{n-1}, m_i + 1} < b$ (or $\sqrt{a_1, \ldots, a_{n-1}, m_i + 1} \leq b$ in case $q = 1$).

The sequence $(S_n)_{n=1}^{\infty}$ of partial expressions determined by the terms $(a_1, a_2, a_3, \ldots)$ generated by the algorithm is bounded above by b and therefore converges. Now the algorithm assigns $a_i = 0$ for all $i > n$ if and only if $b = S_n = \sqrt{a_1, \ldots, a_n}$ for some $n \in \mathbb{N}$, and in this case $(S_n)_{n=1}^{\infty}$ is eventually constantly b and thus clearly converges to b. Thus, we will assume that $(a_n)_{n=1}^{\infty}$ is not eventually constantly 0. Letting $(b_n)_{n=1}^{\infty}$ be the auxiliary sequence as defined in Theorem 2.2, we have, as before, $b_{2n-1} \leq b \leq b_{2n}$ for all $n \in \mathbb{N}$. Let $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ be the subsequence of nonzero terms of $(a_n)_{n=1}^{\infty}$. Considering the effect of the zeros in the sequence $(a_n)_{n=1}^{\infty}$, for $i, j \geq 2n_k$, we have $b_i = h_k(x_0)$ and $b_i = h_k(x_1)$ for some $x_0, x_1 \in [0, q+1]$ where

$$h_k(x) = \sqrt[r_1]{a_{n_1} + \sqrt[r_2]{a_{n_2} + \sqrt[r_3]{a_{n_3} + \cdots + \sqrt[r_k]{a_{n_k} + x}}}},$$

where $r_1 = 2^{n_1}$, and $r_i = 2^{(n_i - n_{i-1})}$ for $i = 2, \ldots, k$. By the mean value theorem, there exists c between x_0 and x_1 with

$$|b_i - b_j| = |h_k(x_1) - h_k(x_0)| = |x_1 - x_0| h_k'(c) \leq (q+1)2^{-k},$$

where the last inequality follows from Lemma 2.1. As before, $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence and must converge to b, and consequently, the subsequence $(b_{2n-1})_{n=1}^{\infty} = (S_n)_{n=1}^{\infty}$ converges to b.

Now the elements of M were chosen so that the necessary inequalities $m_{i+1} \leq m_i + (\varphi_{m_p} - 1)\ \ \forall i = 1, 2, \ldots, p-1$ were actually equalities, and thus elements of M are spaced as widely as possible without introducing gaps in the set of numbers representable using terms from M. Consequently, if the continued radicals whose terms come from $J = \{0, n_1, \ldots, n_s\}$ where $|J| \leq |M|$ also cover the interval $(1, q+1]$, then $\varphi_{n_s} \geq q+1$ so $n_s \geq (q+1)q$. Now if $J \neq M$, it follows that at least one pair of consecutive entries of J differ by more than the uniform distance j between consecutive entries of M, contrary to the fact that the entries of M were already chosen as widely spaced as possible. ■

We now turn our attention to the question of uniqueness of representation. Uniqueness of representation fails if the set M of possible values for the terms contains $0, n$ and φ_n for some integer $n \in \mathbb{N}$, for then

$$\sqrt{\varphi_n, 0, 0, 0, \ldots} = \sqrt{0, n, n, n, \ldots}.$$

In case $M = \{0, q, 2q, 3q, \dots, (q+1)q\} = \{m_1, \dots, m_{q+2}\}$, we will see that the converse of the implication is also true. Let us first assume $q \geq 2$.

Suppose $\sqrt{c_1, c_2, \dots, c_z, a_1, a_2, a_3, \dots}$ and $\sqrt{c_1, c_2, \dots, c_z, b_1, b_2, b_3, \dots}$ are distinct representations of $b \in (1, q+1]$ where $a_1 \neq b_1$. Squaring both sides of the equation

$$\sqrt{c_1, c_2, \dots, c_z, a_1, a_2, a_3, \dots} = \sqrt{c_1, c_2, \dots, c_z, b_1, b_2, b_3, \dots}$$

and subtracting c_i repeatedly as i ranges from 1 to z gives

$$\sqrt{a_1, a_2, a_3, \dots} = \sqrt{b_1, b_2, b_3, \dots} \text{ where } a_1 \neq b_2.$$

Now from the algorithm of Theorem 3.1, the numbers representable as $\sqrt{m_i, a_2, \dots}$ where $a_n \in M_q \ \forall n \in \mathbb{N}$ are the elements of $J_i = \{\sqrt{m_i}\} \cup I_i$ where $I_i = (\sqrt{m_i + 1}, \sqrt{m_i + \varphi_{m_p}}]$. The intervals I_i are mutually disjoint, so any duplication of representation can only occur for a number of form $\sqrt{m_i} \in I_{i-1}$. Now if we have distinct representations of a number

$$\sqrt{m_i} = \sqrt{m_{i-1}, b_2, b_3, b_4, \dots},$$

squaring both sides and recalling that $m_i - m_{i-1} = q$ gives

$$q = \sqrt{b_2, b_3, b_4, \dots} \quad (b_n \in \{0, q, 2q, \dots, (q+1)q\} \ \forall n \geq 2).$$

Now we show that the only such representations of q are

$$\begin{aligned} q &= \sqrt{q^2, 0, 0, 0, \dots} \\ &= \sqrt{(q-1)q, (q-1)q, \dots, (q-1)q, q^2, 0, 0, 0, \dots} \\ &= \sqrt{(q-1)q, (q-1)q, (q-1)q, \dots} \end{aligned}$$

Suppose $q = \sqrt{b_2, b_3, b_4, \dots}$ where $b_n \in \{0, q, 2q, \dots, (q+1)q\}$ for $n = 2, 3, 4, \dots$ and $(b_i)_{i=2}^{\infty} \neq (q^2, 0, 0, 0, \dots)$. Clearly $b_2 \neq (q+1)q = q^2 + q$, for then $\sqrt{b_2, b_3, b_4, \dots} > q$ and $b_2 \neq q^2$ for then $(b_i)_{i=2}^{\infty} = (q^2, 0, 0, 0, \dots)$. Furthermore, $b_2 \not\leq (q-2)q$, for if $b_2 \leq (q-2)q$, then

$$\begin{aligned} \sqrt{b_2, b_3, b_4, \dots} &\leq \sqrt{(q-2)q, (q+1)q, (q+1)q, (q+1)q, \dots} \\ &= \sqrt{q^2 - 2q + \varphi_{(q+1)q}} \\ &= \sqrt{q^2 - 2q + q + 1} \\ &< q \quad \text{since } q \geq 2. \end{aligned}$$

Thus, we must have $b_2 = (q-1)q$ and we now have

$$q = \sqrt{(q-1)q, b_3, b_4, \ldots}.$$

Squaring this equation yields

$$q = \sqrt{b_3, b_4, \ldots}.$$

Repeating the argument above, we find that $(b_3, b_4, b_5, \ldots) = (q^2, 0, 0, \ldots)$ or $b_3 = (q-1)q$. Iterating, we find that $(b_2, b_3, b_4, \ldots)$ is either constantly $(q-1)q$ or has a finite number of initial terms equal to $(q-1)q$ followed by $q^2, 0, 0, 0, \ldots$. This completes our claim about the possible representations of $q = \sqrt{b_2, b_3, b_4, \ldots}$.

Now it follows that the only possible duplicate representations of $\sqrt{a_1, a_2, \ldots} = \sqrt{b_1, b_2, \ldots}$ in which $a_1 \neq b_1$ are of form

$$\begin{aligned}
\sqrt{m_i} &= \sqrt{m_{i-1}, q^2, 0, 0, 0, \ldots} \\
&= \sqrt{m_{i-1}, (q-1)q, (q-1)q, \ldots, (q-1)q, q^2, 0, 0, 0, \ldots} \\
&= \sqrt{m_{i-1}, (q-1)q, (q-1)q, (q-1)q, \ldots}.
\end{aligned}$$

Furthermore, all possible duplicate representations are of form

$$\sqrt{c_1, \ldots, c_z, a_1, a_2, \ldots} = \sqrt{c_1, \ldots, c_z, b_1, b_2, \ldots}$$

where $\sqrt{a_1, a_2, a_3, \ldots}$ and $\sqrt{b_1, b_2, b_3, \ldots}$ are as above.

We summarize our results, stated contrapositively, below.

Theorem 3.2 *A real number $b \in (\sqrt{q}, q+1)$ has a unique representation as $\sqrt{a_1, a_2, a_3, \ldots}$ where $a_i \in M_q = \{0, q, 2q, \ldots, (q+1)q\}$ with $q \geq 2$ if and only if it cannot be represented as a terminating continued radical $\sqrt{a_1, a_2, \ldots, a_z, 0, 0, 0, \ldots}$. A number $b \in (\sqrt{q}, q+1)$ has a terminating continued radical representation $\sqrt{a_1, a_2, \ldots, a_z, 0, 0, 0, \ldots}$ if and only if it has a continued radical representation ending in repeating $(q-1)q$'s. Observe that*

$$\sqrt{q, 0, 0, 0, \ldots} \quad \textit{and} \quad \sqrt{(q+1)q, (q+1)q, (q+1)q, \ldots}$$

respectively are the unique representations of $\sqrt{q}$ and $\varphi_{(q+1)q} = q+1$.

Proof. The first statement was proved above. The second statement follows from the equation

$$\sqrt{c_1,\ldots,c_z,m_i,0,0,\ldots} = \sqrt{c_1,\ldots,c_z,m_{i-1},(q-1)q,(q-1)q,\ldots}$$

for $q \geq 2$ and $i \in \{2,\ldots,q+2\}$. ■

To complete the uniqueness discussion, we now consider the special case $q = 1$ in which our terms are selected from $M_1 = \{0,1,2\}$. The arguments required for this case are similar to those given above, but the illustrative nature of a specific case may be helpful. This result appeared in Sizer [7].

Theorem 3.3 *Any number $b \in (1,2)$ can be represented as a continued radical $\sqrt{a_1,a_2,\ldots}$ where $a_i \in \{0,1,2\}$. This representation is unique unless b has such a representation ending in repeating 0s. A number $b \in (1,2)$ has such a representation ending in repeating 0s if and only if it has such a representation ending in repeating 2s.*

Note that $\sqrt{1,0,0,0,0,\ldots} = 1$ and $\sqrt{2,2,2,2,2,\ldots} = 2$ are the unique representations of 1 and 2.

Proof: The existence of such a representation of $b \in (1,2)$ follows from Theorem 3.1.

Suppose $\sqrt{c_1,\ldots,c_z,a_1,a_2,a_3,\ldots}$ and $\sqrt{c_1,\ldots,c_z,b_1,b_2,b_3,\ldots}$ are distinct representations of $b \in (1,2)$ and $a_1 \neq b_1$. Now if $\sqrt{a_1,a_2,a_3,\ldots} = b' = \sqrt{b_1,b_2,b_3,\ldots}$, the observation that

$$\begin{aligned}
&\sqrt{0,x_2,x_3,\ldots} \in \{0\} \cup \in [1,\sqrt{2}]\\
&\sqrt{1,x_2,x_3,\ldots} \in \{1\} \cup [\sqrt{2},\sqrt{3}]\\
&\sqrt{2,x_2,x_3,\ldots} \in \{\sqrt{2}\} \cup [\sqrt{3},2]
\end{aligned}$$

implies that either

$$b' = \sqrt{2} = \sqrt{2,0,0,0,\ldots} = \sqrt{1,1,0,0,0,\ldots} = \sqrt{0,2,2,2,\ldots}$$

or

$$b' = \sqrt{3} = \sqrt{2,1,0,0,0,\ldots} = \sqrt{1,2,2,2,\ldots}\,.$$

Inserting the initial terms $c_1,\ldots,c_z$, we find that any distinct representations of b by continued radicals end in repeating zeros or repeating twos.

Suppose $b \in (1,2)$ and $b = \sqrt{a_1,\ldots,a_n,2,2,2,\ldots}$, where $a_n \neq 2$. (Since $b \neq 2$, $n \geq 1$.) Now either $a_n = 0$ and

$$b = \sqrt{a_1,\ldots,a_{n-1},0,2,2,2,\ldots} = \sqrt{a_1,\ldots,a_{n-1},2,0,0,0,\ldots},$$

or $a_n = 1$ and

$$b = \sqrt{a_1, \dots, a_{n-1}, 1, 2, 2, 2, \dots} = \sqrt{a_1, \dots, a_{n-1}, 2, 1, 0, 0, \dots}.$$

Similarly, suppose $b \in (1,2)$ and $b = \sqrt{a_1, \dots, a_n, 0, 0, 0, \dots}$, where $a_n \neq 0$. If $a_n = 2 = \sqrt{2, 2, 2, \dots}$, then $b = \sqrt{a_1, \dots, a_{n-1}, 0, 2, 2, 2, \dots}$ gives the desired representation. If $a_n = 1$, then $b = \sqrt{a_1, \dots, a_{n-1}, 1}$ and without loss of generality, we may assume $a_{n-1} \neq 0$. If $a_{n-1} = 1$, then $\sqrt{a_{n-1}, a_n} = \sqrt{1, 1} = \sqrt{2} = \sqrt{0, 2, 2, 2, \dots}$, so $b = \sqrt{a_1, \dots, a_{n-2}, 0, 2, 2, 2, \dots}$ is the desired representation. If $a_{n-1} = 2$, then $\sqrt{a_{n-1}, a_n} = \sqrt{2, 1} = \sqrt{3} = \sqrt{1, 2, 2, 2, \dots}$, so $b = \sqrt{a_1, \dots, a_{n-2}, 1, 2, 2, 2, \dots}$ is the desired representation. ■

We note that the analog to Theorem 2.6 holds for continued radicals in $S(M)$ where M is as in Theorem 3.1.

Finally, we observe that taking the terms of $\sqrt{a_1, a_2, a_3, \dots}$ to be elements of $\{0, q, 2q, 3q, \dots, (q+1)q\}$ where $q \in \mathbb{N}$, we can represent every element of $(1, q+1]$, and since this interval has length $q \geq 1$, it follows that every real number can be represented as $a_0 + \sqrt{a_1, a_2, a_3, \dots}$ where $a_0 \in \mathbb{Z}$ and $a_i \in M \;\; \forall i \in \mathbb{N}$.

4 Continued radicals whose terms assume only two values

We note that one cannot represent all points of an interval using continued radicals whose terms come from a set of two values $M = \{m_1, m_2\} \subseteq \mathbb{N} \cup \{0\}$. If both values are nonnegative, then the requirement from Theorem 2.2 that $\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + \varphi_{m_1}}$ implies that

$$\sqrt{m_1 + \frac{1 + \sqrt{4m_2 + 1}}{2}} \geq \sqrt{m_2 + \frac{1 + \sqrt{4m_1 + 1}}{2}},$$

and it follows that $\sqrt{4m_2 + 1} - 2m_2 \geq \sqrt{4m_1 + 1} - 2m_1$. This last inequality must fail since $m_2 > m_1$ but $f(x) = \sqrt{4x + 1} - 2x$ is strictly decreasing on $[0, \infty)$. Similarly, if $M = \{0, m_2\}$ where $m_2 \in \mathbb{N}$, the requirement $\sqrt{m_i + \varphi_{m_p}} \geq \sqrt{m_{i+1} + 1}$ to insure no gaps becomes

$$\sqrt{0 + \frac{1 + \sqrt{4m_2 + 1}}{2}} \geq \sqrt{m_2 + 1},$$

which is easily seen to have no positive solutions.

Furthermore, the set D of real numbers representable using terms from a two element set $M = \{m_1, m_2\} \subseteq \mathbb{N}$ will have a familiar pattern of gaps.

Theorem 4.1 *If m_1 and m_2 are natural numbers with $m_1 < m_2$, then the set $D = \{\sqrt{a_1, a_2, \dots} : a_i \in \{m_1, m_2\}\, \forall i \in \mathbb{N}\}$ is homeomorphic to the Cantor ternary set $\mathcal{C}$.*

Proof. The Cantor set $\mathcal{C}$ is the set of real numbers in $[0,1]$ which have ternary representations of form $0.c_1c_2c_3\dots$ where each digit c_i is either 0 or 2. Each element of $\mathcal{C}$ has a unique such representation (though some have other representations using 1's.) Note that the arguments of the previous sections insure that each element of D is determined by a unique sequence of terms in M. Letting $g(0) = m_1$ and $g(2) = m_2$, the function $h : \mathcal{C} \to D$ which maps $0.c_1c_2c_3\dots$ to $\sqrt{g(c_1), g(c_2), g(c_3), \dots}$ is a bijection. The continuity of h and h^{-1} follows from the fact that these functions both preserve limits: Suppose $(\overline{c_i})_{i=1}^{\infty}$ is a sequence in $\mathcal{C}$ converging to $\overline{c_0} \in \mathcal{C}$ where for $i \in \mathbb{N} \cup \{0\}$, $\overline{c_i}$ has ternary representation $0.c_{i,1}c_{i,2}c_{i,3}\dots$ with $c_{i,n} \in \{0,2\}$ for each $n \in \mathbb{N}$. Now the convergence of $\overline{c_i}$ to $\overline{c_0}$ implies that the sequences $(c_{i,1}, c_{i,2}, c_{i,3}, \dots)$ of digits of $\overline{c_i}$ must "converge" to the sequence of digits of $\overline{c_0}$ in the sense that the sequence $(k_i)_{i=1}^{\infty}$ converges to ∞ where $k_i \in \mathbb{N} \cup \{\infty\}$ is the smallest value of k for which $c_{i,k} \neq c_{0,k}$. It follows that the corresponding sequence $(g(c_{i,1}), g(c_{i,2}), g(c_{i,3}), \dots)$ of terms of $h(\overline{c_i})$ must "converge" to the sequence of terms of $h(\overline{c_0})$ in the same sense, so h preserves limits. A similar argument shows that h^{-1} also preserves limits. ∎

The theorem above only used the assumption that the integers m_1, m_2 be nonzero to insure uniqueness of representation of continued radicals of form $\sqrt{a_1, a_2, \dots}$. If $M = \{0, m_2\}$ where $m_2 \in \mathbb{N}$, and b has two representations as continued radicals having terms from M, removing any initial identical terms gives a number having two representations which differ in the first term. But $\sqrt{0, a_2, a_3, \dots} \in \{0\} \cup (1, \sqrt{\varphi_{m_2}}\,]$ and $\sqrt{m_2, b_2, b_3, \dots} \in \{\sqrt{m_2}\} \cup (\sqrt{m_2 + 1}, \varphi_{m_2}]$, so considering the remarks of the first paragraph of this section, the only possible duplication of representation could occur for $\sqrt{m_2}$ if it is less that or equal to $\sqrt{\varphi_{m_2}}$. This would lead to $-1 \leq \sqrt{4m_2+1} - 2m_2 = f(m_2)$ where $f(x)$ is as defined before Theorem 4.1. Since $f(x)$ is strictly decreasing and $f(2) = -1$, it follows that the only two-point sets $M = \{m_1, m_2\} \subseteq \mathbb{N} \cup \{0\}$ for which the proof of Theorem 4.1 fails to show $D = S(M) = \{\sqrt{a_1, a_2, \dots} : a_i \in \{m_1, m_2\}\, \forall i \in \mathbb{N}\}$ is homeomorphic to the Cantor ternary set, are the sets $M = \{0, 1\}$ and $M = \{0, 2\}$. With $M = \{0, 1\}$, for example, we have $\sqrt{a_1, \ldots, a_n, 1, 0, 0, 0, \ldots} = \sqrt{a_1, \ldots, a_n, 0, 1, 0, 0, \ldots}$, and thus h^{-1}, as defined in the proof of Theorem 4.1, is not well-defined at these points.

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