

Connected Subsets of an $n \times 2$ Rectangle



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Suppose an $n \times m$ grid is printed on an $n \times m$ note card. Cutting only along grid lines, in how many ways may a connected piece be cut from the card? In how many ways may the card be cut into two connected pieces? Three connected pieces?

We address these questions for an $n \times 2$ card. For example, Figure 1 shows four (of, as we will see, 286) ways a 5×2 card can be cut into three connected pieces. The solutions involve recursively defined sequences which exhibit some interesting interrelated patterns and provide a rich topic for independent research and discovery by students.

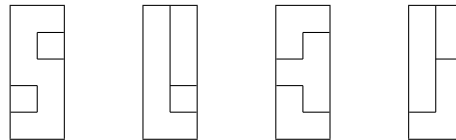


Figure 1. Four (of the 286) partitions of a 5×2 rectangle into three connected pieces.

Two interpretations of the grid-line restrictions are tilings and pixels. Problems on tilings have a long and rich history ([1], [3]), while applications in computer imaging have driven the more recent interpretation involving pixels ([2]). We will follow the latter interpretation, viewing an $n \times 2$ (pixelated) rectangle as a collection of $2n$ (1×1) pixels determined by the grid lines. By a subset of R , we mean a pixelated subset, that is, a subset of the $2n$ pixels of R . Two pixels are *adjacent* if they share a common edge. A subset S of R is *connected* if for any $x_0, x_k \in S$, there exists a sequence of pixels x_0, x_1, \dots, x_k in S with x_i adjacent to x_{i-1} for $i = 1, \dots, k$. Thus, S is

MSC: 11B37, 05A18

connected if the adjacency graph for S is a connected graph. Note that the empty set is connected. Recall that a partition of R is a collection of nonempty mutually disjoint subsets of R whose union is R . If $\{A, B\}$ is a partition of a pixelated rectangle, then A and B share no common pixels; this does not prohibit pixels from A and B from sharing a common edge in their geometric representations.

Before addressing $n \times 2$ rectangles, we note that a complete analysis of $n \times 1$ rectangles is easy. Any nonempty connected subset of an $n \times 1$ rectangle is determined by its top edge and bottom edge. Since there are $n + 1$ possible choices for edges, there are $\binom{n+1}{2}$ nonempty connected subsets. Or, one may observe that there is one $n \times 1$ connected subset, two $(n - 1) \times 1$ connected subsets, and so on, up to n (1×1) connected subsets, giving a total of $1 + 2 + \dots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$ nonempty connected subsets. A partition of an $n \times 1$ rectangle into k nonempty connected subsets is achieved by placing $k - 1$ internal dividers into the $n - 1$ internal gaps between the pixels. This can be done in $\binom{n-1}{k-1}$ ways.

Connected sets of height n in an $n \times 2$ rectangle

First, we will count the number of connected subsets which span the entire height of an $n \times 2$ rectangle. We will depict a subset of a rectangle by marking each included pixel with an X and each excluded pixel with an O . Suppose S is a connected subset of an $n \times 2$ rectangle with height n , and $1 \leq k < n$. If the k^{th} row of S is XX , then the $(k + 1)^{\text{st}}$ row could be XO , XX , or OX . If the k^{th} row is XO , then the $(k + 1)^{\text{st}}$ row could only be XO or XX . The case of the k^{th} row being OX is symmetric to the XO case, with the $(k + 1)^{\text{st}}$ row being either OX or XX . These observations give the recursion suggested shown in Figure 2.

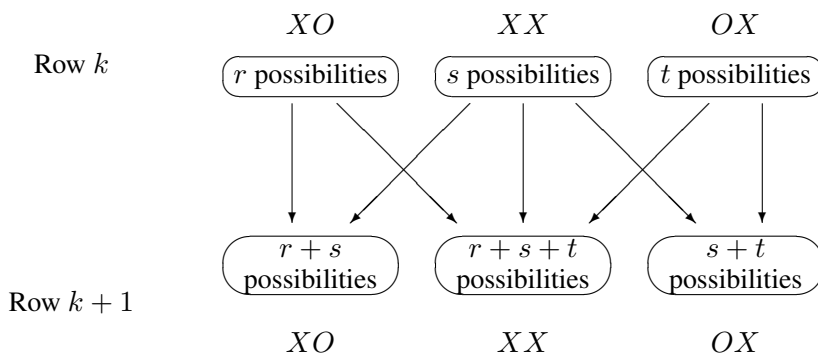


Figure 2. The recursion generating columns **A**, **B**, **C** and columns **E**, **F**, **G** of Table 1.

Using this recurrence pattern, Table 1(a) shows the number of connected subsets of height n in an $n \times 2$ rectangle having top row XO , and Table 1(b) shows the number of connected subsets of height n in an $n \times 2$ rectangle having top row XX . The top row of a nonempty connected subset could also be OX , but clearly the associated table would be a reflection (around the XX column) of Table 1(a).

We will denote the n^{th} entry of column **A** by A_n , with similar notation applying to each of the other columns **B** through **H**. Thus, $B_8 = 169$ says that there are 169 nonempty connected subsets of height 8 in an 8×2 rectangle which have first row XO and eighth row XX . Since $D_8 = 408$, there are 408 nonempty connected subsets of height 8 in an 8×2 rectangle which have first row XO .

Top Row XO					Top Row XX				
	bottom row XO	bottom row XX	bottom row OX	Sum		bottom row XO	bottom row XX	bottom row OX	Sum
n	A	B	C	D	n	E	F	G	H
1	1	0	0	1	1	0	1	0	1
2	1	1	0	2	2	1	1	1	3
3	2	2	1	5	3	2	3	2	7
4	4	5	3	12	4	5	7	5	17
5	9	12	8	29	5	12	17	12	41
6	21	29	20	70	6	29	41	29	99
7	50	70	49	169	7	70	99	70	239
8	120	169	119	408	8	169	239	169	577
9	289	408	288	985	9	408	577	408	1393
10	697	985	696	2378	10	985	1393	985	3363
11	1682	2378	1681	5741	11	2378	3363	2378	8119
12	4060	5741	4059	13860	12	5741	8119	5741	19601

(a) (b)

Table 1. Number of connected sets of height n in an $n \times 2$ rectangle (a) with top row XO and (b) with top row XX .

There are some interesting relations between the columns of Table 1. Columns **B**, **E** and **G** are equal. Column **D** is column **B** shifted up by one place, and column **H** is column **F** shifted up by one place.

We state these and some other relations between the columns here.

Proposition 1. For every natural number $n > 1$,

- (a) $B_n = E_n = G_n$.
- (b) $B_n = D_{n-1}$.
- (c) $F_n = H_{n-1}$.
- (d) $A_n + C_{n-1} = B_n$.
- (e) $A_n + C_n = B_{n-1} + B_n$.
- (f) $A_n = 1 + C_n$.

Proof. (a) The equality of columns **E** and **G** follows from the one-to-one correspondence (realized by reflecting over a vertical line) between connected sets whose first row is XX and last row is XO and connected sets whose first row is XX and last row is OX . Also, $B_n = E_n$, since one counts the connected sets starting with OX and ending with XX , while the other counts the (horizontally reflected) connected sets starting with XX and ending with OX .

(b) Recall that B_n counts the connected sets of height n starting with XO and ending with XX . But if the n^{th} row is XX , the $(n - 1)^{\text{st}}$ row could have been XO , XX , or OX , and D_{n-1} gives the number of connected sets of height $n - 1$ starting with XO and ending with any of the three options XO , XX , and OX . Thus, $B_n = D_{n-1}$. A similar argument shows (c) $F_n = H_{n-1}$.

(d) Since $A_n = A_{n-1} + B_{n-1}$ from the recursion of Figure 2, we have $A_n +$

$C_{n-1} = A_{n-1} + B_{n-1} + C_{n-1}$ which, again by the recursion of Figure 2, equals B_n .

The proofs of (e) and (f) are left to the reader. ■

While these observations relate the entries of various columns, next we turn to the relation between the entries within a single column. We start with column **B**.

Proposition 2. For any $n \geq 3$, $B_n = 2B_{n-1} + B_{n-2}$.

Proof. From Proposition 1, $B_n = D_{n-1} = A_{n-1} + B_{n-1} + C_{n-1}$, and $A_{n-1} + C_{n-1} = B_{n-2} + B_{n-1}$. Substituting the latter into the former gives desired result. ■

A more visual proof of Proposition 2 is suggested in Figure 3. By tracing the ancestry of the B_n paths reaching XX on the n^{th} row, it is easy to see that there are $B_{n-2} + 2B_{n-1}$ of them, proving the claim.

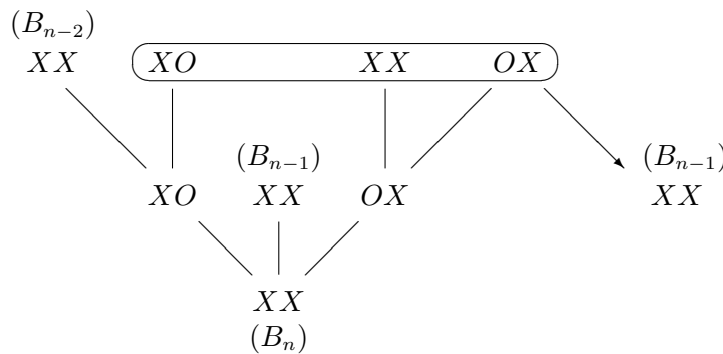


Figure 3. Paths reaching XX on the n^{th} row.

Thus, the sequence B_n satisfies the recurrence relation $x_n = 2x_{n-1} + x_{n-2}$. (Throughout, we will use notation like B_n to denote either the n^{th} term of a sequence or the sequence whose n^{th} term is B_n ; the context will make the usage clear. Also, we understand that satisfying the recurrence relation $x_n = 2x_{n-1} + x_{n-2}$ means satisfying it for every valid choice of $n \in \mathbb{N}$.) It is easy to see that if two sequences satisfy this recurrence relation, then so does any linear combination of these sequences. In particular, if x_n satisfies the recurrence relation, then so does x_{n-1} , and thus so does $x_n - x_{n-1}$. The converse is almost true, as we see in the next result.

Lemma 3. If $s_n = x_n - x_{n-1}$ satisfies the recurrence relation $s_n = 2s_{n-1} + s_{n-2}$, then x_n satisfies $x_n = 2x_{n-1} + x_{n-2} + k$ for some fixed constant k .

Proof. Suppose $s_n = x_n - x_{n-1}$ satisfies the recurrence relation $s_n = 2s_{n-1} + s_{n-2}$. Then

$$x_n - x_{n-1} = 2(x_{n-1} - x_{n-2}) + (x_{n-2} + x_{n-3}).$$

For any n , define k_n to be the number which makes $x_n = 2x_{n-1} + x_{n-2} + k_n$. Substituting this expression and the similar one for x_{n-1} into the left of the equation above gives

$$2x_{n-1} + x_{n-2} + k_n - (2x_{n-2} + x_{n-3} + k_{n-1}) = 2(x_{n-1} - x_{n-2}) + (x_{n-2} + x_{n-3}),$$

which simplifies to $k_n = k_{n-1}$. Thus, the sequence (k_n) is a constant sequence (k) . Now it follows that $x_n = 2x_{n-1} + x_{n-2} + k$. ■

Corollary 4. *Each of the sequences $B_n, D_n, E_n, F_n, G_n,$ and H_n satisfy the recurrence relation $x_n = 2x_{n-1} + x_{n-2}$. Furthermore, $A_n = 2A_{n-1} + A_{n-2} - 1$ and $C_n = 2C_{n-1} + C_{n-2} + 1$.*

Proof. We have shown that B_n satisfies the recurrence relation $x_n = 2x_{n-1} + x_{n-2}$. Since $B_n = E_n = G_n = D_{n-1}$, the sequences $D_n, E_n,$ and G_n satisfy the same recurrence relation. Since E_n and G_n satisfy the recurrence relation, so does $E_n + G_n = H_n - F_n = F_{n+1} - F_n$. Now by Lemma 3, $F_n = 2F_{n-1} + F_{n-2} + k$ for some fixed k , and from the initial terms we see $F_3 = 2F_2 + F_1$, so $k = 0$. (Or, note that the proof of Proposition 2 would apply to F_n as well.) Since $H_n = F_{n-1}$, H_n also satisfies the given recurrence relation. Similar applications of Lemma 3 show that $A_n = 2A_{n-1} + A_{n-2} - 1$ and $C_n = 2C_{n-1} + C_{n-2} + 1$. ■

Recurrence relations have been extensively studied ([5] or [6]). The recurrence relation $x_n = 2x_{n-1} + x_{n-2}$ is a second-order linear homogeneous relation, and can be solved in a manner analogous to solving a second-order linear homogeneous differential equation. Specifically, we seek solutions of the form $x_n = r^n$. Such a solution must satisfy the characteristic equation $r^2 - 2r - 1 = 0$, so $r = 1 \pm \sqrt{2}$. Thus, the general solution of the recurrence relation is $x_n = s(1 + \sqrt{2})^n + t(1 - \sqrt{2})^n$, where s and t are real numbers. The initial conditions (that is, the values of x_1 and x_2) are used to find the values of s and t .

Using this technique, we find, for example, that $D_n = \frac{\sqrt{2}}{4}(1 + \sqrt{2})^n - \frac{\sqrt{2}}{4}(1 - \sqrt{2})^n$.

Proposition 5. *The number of nonempty connected sets of height n in an $n \times 2$ rectangle is $2D_n + H_n = H_{n+1} = \frac{1}{2}(1 + \sqrt{2})^{n+1} + \frac{1}{2}(1 - \sqrt{2})^{n+1}$.*

Proof. The number D_n tells how many connected sets of height n in an $n \times 2$ rectangle have first row XO , and by symmetry, also how many have first row OX . The number H_n tells how many have first row XX . Adding, we see that the total number of connected sets of height n in an $n \times 2$ rectangle is $2D_n + H_n$. Since $D_n = E_{n+1} = G_{n+1}$ and $H_n = F_{n+1}$, we have $2D_n + H_n = E_{n+1} + F_{n+1} + G_{n+1} = H_{n+1}$. As noted above, the general solution to the recurrence relation satisfied by H_{n+1} has form $s(1 + \sqrt{2})^n + t(1 - \sqrt{2})^n$, and the initial conditions $H_2 = 3$ and $H_3 = 7$ lead to the formula given. ■

Before leaving this section, we mention a connection with continued fractions. A simple finite continued fraction $[a_0; a_1, a_2, \dots, a_n]$ is an expression of form

$$[a_0; a_1, a_2, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n}}},$$

where $a_0 \in \mathbb{Z}, a_1, \dots, a_n \in \mathbb{N}$. Irrational numbers have infinite continued fractions $[a_0; a_1, a_2, \dots] = \lim_{n \rightarrow \infty} [a_0; a_1, \dots, a_n]$. The partial expression $[a_0; a_1, \dots, a_n]$ is called the n^{th} convergent of $[a_0; a_1, a_2, \dots]$, and is a rational number $\frac{p_n}{q_n}$. It is well known ([4], [6]) that the numerators p_n and denominators q_n of the convergents of an infinite continued fraction both satisfy the recurrence relation $x_n = a_n x_{n-1} + x_{n-2}$, with suitable initial conditions ($p_0 = a_0, p_1 = a_0 a_1 + 1, q_0 = 1, q_1 = a_1$). Since the

continued fraction for $\sqrt{2}$ is $[1; 2, 2, 2, 2, \dots]$, it follows that the sequences H_n and D_n are, respectively, the numerators and denominators of convergents of the continued fraction representation of $\sqrt{2}$. (See A001333 and A000129 of [7].) In particular,

$$\lim_{n \rightarrow \infty} \frac{H_n}{D_n} = \sqrt{2}.$$

All connected sets in an $n \times 2$ rectangle

The nonempty connected subsets in an $n \times 2$ rectangle may have height k where $1 \leq k \leq n$, and a connected set of height k may be positioned in the $n \times 2$ rectangle in $n + 1 - k$ ways. Since there are H_{k+1} connected rectangles of height k , we see that the number of nonempty connected subsets in an $n \times 2$ rectangle is

$$\text{Con}(n) = \sum_{k=1}^n (n + 1 - k)H_{k+1}.$$

Some of these numbers are given in Table 2. This sequence appears as A059020 in [7], in the same context as it arises here.

n	1	2	3	4	5	6	7	8	9	10	11
$\text{Con}(n)$	3	13	40	108	275	681	1664	4040	9779	23637	57096

Table 2. The number $\text{Con}(n)$ of nonempty connected subsets of an $n \times 2$ rectangle.

Since the empty set is connected, $1 + \text{Con}(n)$ is the total number of connected subsets of an $n \times 2$ rectangle.

A formula for $\text{Con}(n)$ which does not involve a sum is given below.

Proposition 6. *The number of nonempty connected subsets in an $n \times 2$ rectangle is*

$$\text{Con}(n) = \frac{7 + 5\sqrt{2}}{4}(1 + \sqrt{2})^n + \frac{7 - 5\sqrt{2}}{4}(1 - \sqrt{2})^n - 2n - \frac{7}{2}.$$

Proof. Since $\text{Con}(n) = \sum_{k=1}^n (n + 1 - k)H_{k+1}$ and H_j satisfies the recurrence relation $x_n = 2x_{n-1} + x_{n-2}$, we may ask whether $\text{Con}(n)$ also satisfies this recurrence relation. It does not, but the values of $\text{Con}(n) - (2\text{Con}(n - 1) + \text{Con}(n - 2))$ suggest the solution. Using the summation formula

$$\text{Con}(n) = n(H_2) + (n - 1)H_3 + (n - 2)H_4 + \dots + 3H_{n-1} + 2H_n + H_{n+1},$$

we find that

$$\begin{aligned} 2\text{Con}(n - 1) + \text{Con}(n - 2) &= 2(n - 1)H_2 + (n - 2)[2H_3 + H_2] + (n - 3)[2H_4 + H_3] + \\ &\quad \dots + 2[2H_{n-1} + H_{n-2}] + [2H_n + H_{n-1}] \\ &= 2(n - 1)H_2 + (n - 2)H_4 + (n - 3)H_5 + \dots + 2H_n + H_{n+1}. \end{aligned}$$

Subtracting and substituting $H_2 = 3, H_3 = 7$, we get $\text{Con}(n) - (2\text{Con}(n - 1) + \text{Con}(n - 2)) = nH_2 + (n - 1)H_3 - 2(n - 1)H_2 = 4n - 1$. Thus, $\text{Con}(n)$ satisfies the second-order linear non-homogeneous recurrence relation $\text{Con}(n) = 2\text{Con}(n - 1) + \text{Con}(n - 2) + 4n - 1$. As in the theory of linear non-homogeneous differential equations, since the non-homogeneous forcing term $4n - 1$ is a linear function, using the method of undetermined coefficients, we guess a solution of form $\text{Con}(n) = s(1 + \sqrt{2})^n + t(1 - \sqrt{2})^n + kn + l$, where the s and t terms give the general solution to the homogenous equation. Substituting the first four initial values of $\text{Con}(n)$ and solving the resulting system of four linear equations in s, t, k, l gives the result in the statement. ■

Connected subsets of an $n \times 2$ rectangle with connected complements

Now we turn to counting the partitions of an $n \times 2$ rectangle into two connected sets, that is, the number of ways to cut an $n \times 2$ note card into two pieces, cutting only along the $n \times 2$ grid lines. Note that the partitions $\{A, B\}$ and $\{B, A\}$ are equal. If we were counting *labeled* partitions, say with the first set black and the second set red, then each (non-labeled) partition $\{A, B\}$ would have two labelings, and we would double the number below.

Proposition 7. *The number of partitions of an $n \times 2$ rectangle into two connected sets is $p_2(n) = 2n^2 - n$.*

Proof. Since we are not counting labeled partitions, we will assume the top left corner is X , so the top row is XO or XX .

If the top row is XO , suppose the partition has the top k rows ($k = 1$ to n) being XO . The remaining rows must be all XX or OO . In the case $k = n$, completing the “remaining” rows with XX or with OO gives the same partition, so there are $2n - 1$ partitions with top row XO .

If the top row is XX , suppose the partition has the top k rows ($k = 1$ to $n - 1$) being XX . The remaining $n - k$ rows form a partition of an $(n - k) \times 2$ rectangle with top row XO, OX , or OO . By the previous paragraph, the number of partitions of the bottom $n - k$ rows starting with XO is $2(n - k) - 1$, and summing from $k = 1$ to $n - 1$ gives $n^2 - 2n + 1$ such partitions. By symmetry, there will be the same number of partitions of the bottom $n - k$ rows starting with OX . If the bottom $n - k$ rows start with OO , then all remaining rows must be OO , so as k ranges from 1 to $n - 1$, there are $n - 1$ such partitions. Adding, the total number of partitions of an $n \times 2$ rectangle with top row XX is $2(n^2 - 2n + 1) + (n - 1) = 2n^2 - 3n + 1$.

Adding the number of partitions with top row XO and XX found in the previous two paragraphs, we find that the total number of partitions into two connected sets is $2n^2 - n$. ■

While the number $\text{Con}(n)$ of nonempty connected sets in an $n \times 2$ rectangle grows exponentially with n , the number of partitions into two connected sets grows quadratically. Table 3 gives some of the numbers $p_2(n)$. A direct combinatorial proof showing that these are the hexagonal numbers (A000384 in [7]) would be of interest.

Partitions of an $n \times 2$ rectangle into 3 connected sets

In this section, we prove the following result.

n	1	2	3	4	5	6	7	8	9	10	11	12
$p_2(n)$	1	6	15	28	45	66	91	120	153	190	231	276

Table 3. The number $p_2(n)$ of partitions of an $n \times 2$ rectangle into two connected sets.

Proposition 8. *If $n \geq 2$, an $n \times 2$ rectangle can be partitioned into three connected sets in $p_3(n) = (4n^4 - 8n^3 + 11n^2 - 13n + 6)/6$ ways.*

Table 4 shows the values of $p_3(n)$ for $n = 2$ to 12.

n	2	3	4	5	6	7	8	9	10	11	12
$p_3(n)$	4	29	107	286	630	1219	2149	3532	5496	8185	11759

Table 4. The number $p_3(n)$ of partitions of an $n \times 2$ rectangle into three connected sets.

Proof. Suppose an $n \times 2$ rectangle R is partitioned into three sets A, B, C . Without loss of generality, we will let A be the set containing the upper left corner of R and let B be the second set encountered. Then the first row must be AB or AA .

Case 1: *The first row of R is AB and no subsequent row is AA , so A is a $k \times 1$ set for some k with $1 \leq k \leq n$.*

If the k^{th} row is AC , then the bottom $n - k$ rows must be CC , and B is a $j \times 1$ set to the right of A for some $j = 1$ to $k - 1$, as depicted on the left in Figure 4. Then A, B , and C are determined by the k positions for the bottom of A and $k - 1$ positions for the bottom of B . There are $\sum_{k=1}^n (k - 1) = (n^2 - n)/2$ such partitions.

If the k^{th} row is AB , as seen on the right in Figure 4, then $1 \leq k \leq n - 1$ (for $k = n$ would imply $C = \emptyset$), and the bottom $n - k$ rows contain no A s. Either the bottom $n - k$ rows are all C s, or the bottom $n - k$ rows are partitioned into two connected B and C . By Proposition 7, there are $2(n - k)^2 - (n - k)$ ways to partition the bottom $n - k$ rows into two sets. Adding the one way to partition them into one set C and summing over $k = 1$ to $n - 1$, we see that there are $\sum_{k=1}^{n-1} (2(n - k)^2 - (n - k) + 1) = (4n^3 - 9n^2 + 11n - 6)/6$ such sets.

Adding the options from the two paragraphs above, there are $(2n^3 - 3n^2 + 4n - 3)/3$ partitions in Case 1.

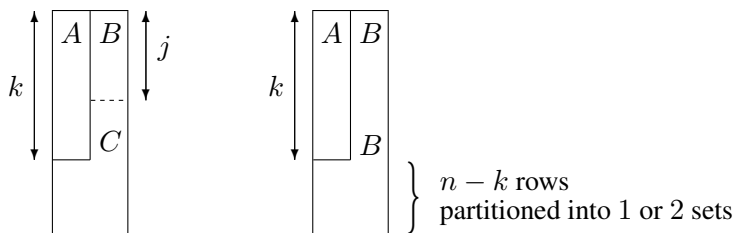


Figure 4. Case 1: partitions of an $n \times 2$ rectangle into three sets if A is a $k \times 1$ set.

Case 2: *The first row of R is AB and there is a subsequent row AA after the initial row AB . Suppose the first row which is AA is the $(k + 1)^{\text{st}}$ row ($1 \leq k \leq n - 1$).*

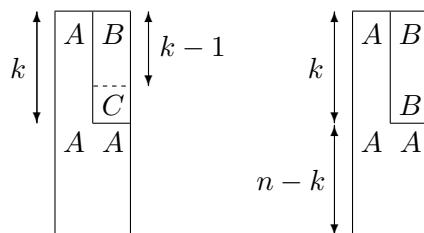


Figure 5. Case 2: partitions of an $n \times 2$ rectangle with initial row AB and first AA at row $k + 1$.

If row k is AC , as shown on the left in Figure 5, then $2 \leq k \leq n - 1$, rows $k + 1$ through n must all be AA , and $B \cup C$ is a $k \times 1$ rectangle. In particular, B is a $j \times 1$ rectangle for some $j = 1$ to $k - 1$. Summing, there are $\sum_{k=2}^{n-1} (k - 1) = (n^2 - 3n + 2)/2$ such partitions.

If row k is AB , as shown on the right in Figure 5, then B is a $k \times 1$ rectangle, and $k \leq n - 2$. Removing the top k rows leaves an $(n - k) \times 2$ rectangle with first row AA which is partitioned into $A \cup C$. From the third paragraph of the proof of Proposition 7, there are $2(n - k)^2 - 3(n - k) + 1$ ways to form such a partition. Summing as k goes from 1 to $n - 2$ gives $(4n^3 - 15n^2 + 17n - 6)/6$ such partitions.

Adding the options from the two paragraphs above, there are $(2n^3 - 6n^2 + 4n)/3$ partitions in Case 2.

Case 3: *The first row of R is AA .*

Let us assume the first k rows ($1 \leq k \leq n - 1$) are AA and the $(k + 1)^{\text{st}}$ row is not AA .

If the $(k + 1)^{\text{st}}$ row does not contain an A , then the bottom $n - k$ rows are partitioned by $B \cup C$. By Proposition 7, there are $2(n - k)^2 - (n - k)$ such partitions, and summing as k goes from 1 to $n - 1$ gives $(4n^3 - 9n^2 + 5n)/6$ such partitions.

If the $(k + 1)^{\text{st}}$ row does contain an A , then $k \leq n - 2$ and the bottom $n - k$ rows give an $(n - k) \times 2$ rectangle with top row AB or BA which must be partitioned into three sets. Such partitions were counted in Cases 1 and 2. Combining the totals in those two cases for a rectangle of height $n - k$, we find that this can be done in $(4(n - k)^3 - 9(n - k)^2 + 8(n - k) - 3)/3$ ways. Summing as k goes from 1 to $n - 2$ and doubling, to account for the equal number starting from BA , we find that there are $(2n^4 - 10n^3 + 19n^2 - 17n + 6)/3$ such partitions.

Adding the options from the two paragraphs above, there are $(4n^4 - 16n^3 + 29n^2 - 29n + 12)/6$ partitions in Case 3.

Combining Cases 1, 2, and 3, we find that there are $(4n^4 - 8n^3 + 11n^2 - 13n + 6)/6$ partitions of an $n \times 2$ rectangle into three connected sets, proving Proposition 8. ■

Areas for further study

Counting connected subsets of an $n \times 2$ pixelated rectangle and partitions of such a rectangle into connected sets leads to some interesting sequential patterns and recursively generated sequences. These initial studies suggest many opportunities for further investigation and discovery.

A general formula for the number of partitions of an $n \times m$ pixelated rectangle into k connected subsets is not known to us, and would be welcomed. Already for $m = 3$, the row-by-row techniques presented here become difficult, since the tables

corresponding to those of Table 1 would have $2^3 - 1 = 7$ columns for possible bottom rows.

In our enumeration above of partitions of $n \times 2$ rectangles into three connected sets, it would be an easy step to count the labeled partitions, say into one black, one red, and one yellow set. Once a partition into 3 sets is found, there are $3! = 6$ ways to label the three sets, so there are 6 times as many labeled partitions.

A more difficult question would be to enumerate the symmetry classes of the connected sets or connected partitions. For example, among the $p_3(3) = 29$ partitions of a 3×2 rectangle into three connected sets shown in Figure 6, observe that the first four are obtained through rigid motions of the first, and thus constitute a single symmetry class.

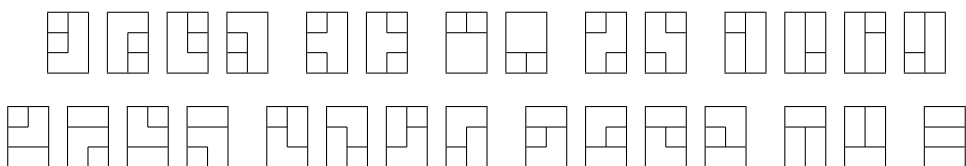


Figure 6. The $p_3(3) = 29$ partitions of a 3×2 rectangle into three connected sets, grouped by symmetry classes.

The group of symmetries of a rectangle has four elements, or actions: rotations of 0° and 180° , and reflections through vertical and horizontal lines. Applying the four actions of this group to a partition into three sets will not result in four distinct partitions if the partition already shows some symmetry, that is, if the partition is invariant under some of the actions. Recall that if G is a group of permutations of a set S , the permutations in G that leave a subset of S invariant is a subgroup of the group of G , and the order of a subgroup divides the order of the group. Thus, a partition of an $n \times 2$ rectangle may remain invariant under 1, 2, or 4 of the symmetries of the rectangle. If only the identity permutation leaves the partition fixed, the symmetry class will have 4 elements. If two actions in the group of symmetries of the rectangle leave the permutation fixed, the symmetry class will have two elements, and if all four actions leave the partition fixed, the symmetry class has only one element. In Figure 6, the 29 permutations are grouped into their 10 symmetry classes.

For a 3×2 rectangle, a visual check shows that among the $\text{Con}(3) = 40$ connected sets, there are 15 symmetry classes; among the $p_2(3) = 15$ partitions into two connected sets, there are 6 symmetry classes; among the $p_3(3) = 29$ partitions into three connected sets, there are 10 symmetry classes. Formulas for the number of symmetry classes are not known to us.

Besides these questions of an algebraic flavor, the topic suggests further investigation into connections with continued fractions and with generating functions [8].

Summary: Suppose an $n \times m$ grid is printed on an $n \times m$ note card. Cutting only along grid lines, in how many ways may a connected piece be cut from the card? In how many ways may the card be cut into two connected pieces? Three connected pieces? We answer these questions for $m = 2$ using recurrence relations.

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