Papers on General Topology and Applications: Ninth Summer Conference at Slippery Rock University, Susan Andima, et. al, editors, Annals of the New York Academy of Sciences, Vol. 767, New York, 1995, pp. 188-193.

Cardinality and Structure of Semilattices of Ordered Compactifications

Douglas D. Mooney and Thomas A. Richmond
Department of Mathematics
Western Kentucky University
Bowling Green, KY 42101
Mooney@wkuvx1.wku.edu
Tom.Richmond@wku.edu

Abstract

Cardinalities and lattice structures which are attainable by semilattices of ordered compactifications of completely regular ordered spaces are examined. Visliseni and Flachsmeyer have shown that every infinite cardinal is attainable as the cardinality of a semilattice of compactifications of a Tychonoff space. Among the finite cardinals, however, only the Bell numbers are attainable as cardinalities of semilattices of compactifications. It is shown here that all cardinals, both finite and infinite, are attainable as the cardinalities of semilattices of ordered compactifications of completely regular ordered spaces. The last section examines lattice structures which are realizable as semilattices of ordered compactifications, such as chains and power sets.

1991 AMS SUBJECT CLASSIFICATIONS: 54D35, 54F05, 54D40, 06A12

KEY WORDS AND PHRASES: Ordered topological space, completely regular ordered, ordered compactification, lattice, semilattice, cardinality

A *partially ordered topological space*, or simply an *ordered space*, is a triple (X, τ, θ) where X is a set, τ a topology on X, and θ is a partial order on X. If θ is closed in the product $X \times X$, then (X, τ, θ) is T_2*-ordered*. An *ordered compactification* of (X, τ, θ) is a compact T_2-ordered space (X^*, τ^*, θ^*) such that (X^*, τ^*) is a compactification of (X, τ) and $\theta^* \cap (X \times X) = \theta$. An ordered space has an ordered compactification if and only if it is *completely regular ordered* as defined in [8]. Throughout we will assume that all topological spaces are Tychonoff, all compactifications are T_2, all ordered topological spaces are completely regular ordered, and all ordered compactifications are T_2-ordered. $K(X)$ will denote the $\vee$-complete semilattice of compactifications of X, and $K_o(X)$ will denote the set of ordered compactifications of X. Given the natural order, $K_o(X)$ is a $\vee$-complete semilattice with largest element $\beta_o X$, the *Nachbin-* (or *Stone-Čech ordered*) *compactification* (see [1]). The discrete order on X is $\Delta_X = \{(x,x) : x \in X\}$. Every topological space (X,τ) can be considered to be a discretely ordered space (X,τ,Δ_X). If θ is a partial order, we will write $x \leq y$ for $(x,y) \in \theta$.

Ordered compactifications of totally ordered spaces have been studied in [1] and [6], and will be used frequently here. In the latter paper, a filter approach is used and it is shown that all compactification points in an ordered compactification of a totally ordered space (X,τ,θ) correspond to certain nonconvergent filters on X called *singularities*. Singularities are of two types: *simple singularities* introduce exactly one compactification point each; *essential singularities* occur in pairs, and each essential pair introduces either one or two compactification points. Thus, if E is the set of essential pairs of singularities of a totally ordered space X, $K_o(X)$ is isomorphic to the power set lattice 2^E.

1 Cardinalities

In [3] it is shown that there is a one-to-one correspondence between the ordered compactifications of an ordered space (X,τ,θ) and the compatible totally bounded quasi-uniformities on X. Since quasi-uniformities on X are collections of subsets of $X \times X$, we see that $|K_o(X)| \leq 2^{2^{|X|}}$ for any ordered space X. If (X,τ,Δ) is a discretely ordered space, then $|K(X)| \leq |K_o(X)|$. The fact that there are far more ordered compactifications of a discretely ordered space than topological compactifications is made clear by the following proposition.

Proposition 1.1 *If (X,τ,Δ) is a discretely ordered, locally compact, noncompact topological space, then the number of one-point ordered compactifications of X is $|\tau|$.*

Proof. The Alexandroff one-point compactification of X, $X^* = X \cup \{\infty\}$, may be endowed with any order that is closed in $X^* \times X^*$ and is equality on X. All such orders are of the form $\Delta \cup \{\infty\} \times A$ or $\Delta \cup A \times \{\infty\}$ where A is a closed subset of X. □

For an ordered space (X,τ,θ) whose order is not discrete, there is no general relation between $|K(X)|$ and $|K_o(X)|$. For example, if X is the ordinal space ω_1 with the usual topology, but with the discrete order, then by Proposition 1.1 $|K_o(X)| = |\tau|$, so $|K_o(\omega_1)| = 2^{\aleph_1}$. It follows then that $|K(X)| = 1 < 2^{\aleph_1} = |K_o(X)|$. To see that the opposite inequality may also hold, consider $X = \mathbf{R}$, the real line with its usual topology and order. By a footnote in [10], $|\beta X \setminus X| \leq |K(X)|$ whenever $|\beta X \setminus X|$ is infinite, and thus $2^{\mathfrak{c}} \leq |K(\mathbf{R})|$. From [6], $|K_o(\mathbf{R})| = 1$. Thus $|K_o(\mathbf{R})| = 1 < 2^{\mathfrak{c}} \leq |K(\mathbf{R})|$.

Visliseni and Flachsmeyer [10] have shown that every infinite cardinal is attainable as the cardinality of $K(X)$ for some X. The only finite cardinals attainable are the Bell numbers; the n^{th} Bell number is the number of partitions of an n-element set (see [2]). From the results on totally ordered spaces, a cardinal α is attainable as the cardinality of $K_o(X)$ for some totally ordered space X if and only if $\alpha = 2^\gamma$ for some γ. Below we show that every cardinal is attainable as the cardinality of $K_o(X)$ for some ordered space X. Interestingly, these cardinals are all attained with one-point compactifications.

Theorem 1.2 *Suppose that α is an infinite cardinal.*

(1) For any $\rho \leq \alpha$, there exists an ordered space X with $|X| = \alpha$ and $|K_o(X)| = \rho$.

(2) There exists an ordered space X with $|X| = \alpha$ and $|K_o(X)| = 2^\alpha$.

(3) There exists an ordered space X with $|X| = \alpha$ and $|K_o(X)| = 2^{2^\alpha}$.

Proof. (1) Give $A = [0, \alpha)$ and $B = (1, \rho]$ the usual topology and order as an ordinal space and a subspace of an ordinal space respectively. Let X be the direct sum $A \oplus B$ with the direct sum order: $x \leq y$ in X if and only if $x \leq y$ in A or $x \leq y$ in B. (Thus, $x \| y$ if $x \in A, y \in B$.) Note that $|X| = \alpha$. The only topological compactification of X is the one-point compactification $X \cup \{\alpha\}$. Each ordered compactification of X is obtained by putting $\alpha > a$ for all $a \in A$ and $\alpha > b$ for all $b \in (1, \gamma]$ for some $\gamma \in [1, \rho]$ (here it is understood that $(1, 1] = \emptyset$). Since there are ρ choices for γ, $|K_o(X)| = \rho$.

Another such example is $A \times B$ with the product topology and order.

(2) Let X be the following modification of a long line. Starting with the ordinal space α, for each $\gamma \in \alpha$ fill the gap from γ to $\gamma + 1$ with a copy of the set $C = \{\frac{1}{2} \pm \frac{1}{n} : n \in \mathbf{N}, n \geq 2\}$. Give this set X the natural total order and interval topology. Equivalently, let $X = (\alpha + 1) \times C$ with the lexicographic order and interval topology. Now $|X| = \alpha$ and X has α essential pairs of singularities, one for each gap from γ to $\gamma + 1$, so $|K_o(X)| = 2^\alpha$.

(3) Let X be a set of α points given the discrete topology and discrete order. Since X has the discrete topology, $|\beta X \setminus X| = 2^{2^\alpha}$ (see [9, 4U]). Since X has the discrete order, $|K(X)| \leq |K_o(X)|$. Now from the inequalities $|\beta X \setminus X| \leq |K(X)|$ and $|K_o(X)| \leq 2^{2^{|X|}}$, it follows that $|K_o(X)| = 2^{2^\alpha}$. □

The double inequality $|\beta X \setminus X| \leq |K(X)| \leq 2^{|\beta X \setminus X|}$ whenever $|\beta X \setminus X|$ is infinite is given in a footnote of [10]. The analogous inequalities for ordered compactifications need not hold. An easy modification of the construction in the proof of (2) above can be used to produce a totally ordered space X with α essential pairs of singularities and 2^{2^α} simple singularities. For such a space X, $|\beta_o X \setminus X| = 2^{2^\alpha} \not\leq 2^\alpha = |K_o(X)|$. We will delay the proof that $|K_o(X)| \leq 2^{|\beta_o X \setminus X|}$

need not hold until after the proof of Proposition 2.1, since a modification of the construction there gives an easy counterexample.

It was seen above that $|\beta_o(X) \setminus X|$ is not related to $|K_o(X)|$ in the way that one might have expected. This section concludes with some additional remarks concerning $|\beta_o(X) \setminus X|$. Kost [7] has shown that for any cardinal α there exists a Tychonoff space X with $|\beta X \setminus X| = \alpha$. More generally, it is known (see [2, 4.17]) that for any Tychonoff space Y, there exists a Tychonoff space X with $\beta X \setminus X$ homeomorphic to Y. This result holds in the ordered setting.

Theorem 1.3 *For any completely regular ordered space Y, there exists a completely regular ordered space X and a function f which is both a homeomorphism and an order isomorphism from $\beta_o X \setminus X$ onto Y.*

Proof. Consider the ordinal space ω_1 with the usual topology and order. Since $\beta_o(\omega_1) = \omega_1 + 1$, it follows from Theorems 1 and 4 of [4] that $B = \beta[\beta_o Y \times \omega_1] = \beta_o Y \times (\omega_1 + 1)$. With the product order, this is an ordered compactification of $\beta_o Y \times \omega_1$. Indeed, the product order on B is the smallest order on B that extends the order on $\beta_o Y \times \omega_1$, so with the product order, $B = \beta_o[\beta_o Y \times \omega_1]$. Now let $X = (\beta_o Y \times (\omega_1 + 1)) \setminus (Y \times \{\omega_1\})$ and observe that $\beta_o Y \times \omega_1 \subseteq X \subseteq \beta_o Y \times (\omega_1 + 1)$. Since $S \subseteq T \subseteq \beta_o S$ implies $\beta_o T = \beta_o S$, it follows that $\beta_o X = \beta_o Y \times (0, \omega_1]$, and thus $\beta_o X \setminus X = Y \times \{\omega_1\}$, which is simultaneously homeomorphic and order isomorphic to Y. $\square$

If D is a countable discrete topological space, then $|\beta D \setminus D| = 2^{2^{\aleph_0}}$. Countable discrete spaces D can be ordered so that $|\beta_o D \setminus D|$ assumes different values. With the discrete order, $\beta_o D$ is simply βD with the discrete order, so $|\beta_o D \setminus D| = 2^{2^{\aleph_0}}$. If D is the set of rational numbers with the usual order and discrete topology, $\beta_o D$ introduces two copies of each real number together with $\pm\infty$, so $|\beta_o D \setminus D| = 2^{\aleph_0}$. Finally if $1 \leq \alpha \leq \omega$, give $D = \omega \times [1, \alpha)$ the product order. By the Glicksberg-type theorem for totally ordered spaces in [5], it follows that $\beta_o(D) = \beta_o \mathbf{N} \times \beta_o[1, \alpha)$, and thus $|\beta_o D \setminus D| = \alpha$.

2 Structures

We now turn our attention to the lattice structures attainable as a semilattice of ordered compactifications of an ordered space.

For a topological space X, it is easy to see that the following are equivalent: a) $K(X)$ is a chain; b) $|\beta X \setminus X| \leq 2$; c) $|K(X)| \leq 2$. If X is totally ordered, it is easy to see that the following are equivalent: a) $|K_o(X)|$ is a chain; b) X has no more than one essential pair of singularities; c) $|K_o(X)| \leq 2$. For partially ordered spaces X, it is also true that $K_o(X)$ is a chain if $|K_o(X)| \leq 2$, but longer chains may be realized as $K_o(X)$ for appropriate X.

Proposition 2.1 *For every compact totally ordered space A, there exists a partially ordered space X with $K_o(X)$ dually isomorphic to A.*

Proof. The proof is constructive. Suppose A is a compact totally ordered space. Let $X = \omega_1 \oplus A$ with the usual ordinal space topology and order on ω_1 and the direct sum topology τ and the direct sum order θ (as described in the proof of Theorem 1.2(1)) on X. Let a be the least element of A, and let $\theta^* = \theta \cup (\omega_1 \times \{a\})$. Now (X, τ, θ^*) has only one topological compactification, and it may be ordered to get an ordered compactification by placing the compactification point $\{\omega_1\}$ above any closed (decreasing) segment $[a, x] \subseteq A$. Now $(K_o(X), \leq) \cong (A, \geq)$. $\square$

We will now illustrate, as mentioned in the first section, that the inequality $|K_o(X)| \leq 2^{|\beta_o X \setminus X|}$ need not hold. Let $Y = [0, \infty) \setminus \cup\{[n, n+\frac{1}{2}) : n \in \mathbf{N}\}$, with the usual topology and order inherited from the real line. Y is totally ordered with $\aleph_0$ many simple singularities. Suppose A is as in the proof above, and $|A| > 2^{\aleph_0}$. Let $A' = A \oplus Y$, with the total order $\theta_Y \cup \theta_A \cup (Y \times A)$. Letting $X = \omega_1 \oplus A'$ with the order $\theta^* = \theta_{A'} \cup (\omega_1 \times (Y \cup \{a\}))$, it follows as above that $(K_o(X), \leq) \cong (A, \geq)$. In particular, $|K_o(X)| = |A| > 2^{\aleph_0}$. But since $|\beta_o X \setminus X| = \aleph_0$, we have $2^{\aleph_0} < |K_o(X)| \not\leq 2^{|\beta_o X \setminus X|} = 2^{\aleph_0}$.

Characterizing the power set lattices that can be realized as the lattice of ordered compactifications of some ordered space X is easy. Any power set $P(E)$ is lattice isomorphic to $K_o(X)$ where X is a totally ordered space with $|E|$ essential pairs of singularities. Clearly such spaces exist; simply take a long line of length $|E|$ and remove the point $\frac{1}{2}$ from each copy of the unit interval. The characterization of partially ordered spaces X for which $K_o(X)$ is a power set lattice, however, remains unsolved. The following are examples of two such spaces.

Example 2.2 *Suppose A is a finite set with the discrete topology and discrete order, and ω_1 has its usual topology and order as an ordinal space. Let $X = \omega_1 \oplus A$, with the direct sum topology and order. X has only one topological compactification, $X \cup \{\omega_1\}$. The compactification point can be ordered above any subset of A, and thus $(K_o(X), \leq) \cong (P(A), \supseteq)$.* $\square$

Observe that if, in the above example, A were taken to be compact, infinite, and discretely ordered, the proof of Proposition 1.1 would imply that $K_o(X) \cong \tau_X$. As another application of Proposition 1.1 to the structure of $K_o(X)$, we note that if X is almost compact (i.e., $|\beta X \setminus X| \leq 1$), noncompact, and discretely ordered, then $K_o(X) \cong \tau \mathbin{\dot\cup} \tau$, the disjoint union of two copies of τ.

Example 2.3 *Let X be the subset $[0, 1) \times \{1, \ldots, n\}$ of $\mathbf{R}^2$, given the usual topology and order (i.e., the product order). Now $\beta_o X = [0, 1] \times \{1, \ldots, n\}$, so $\beta_o X \setminus X = \{1, \ldots, n\}$ is totally ordered. The ordered compactifications of X correspond to the partitions of $\{1, 2, \ldots, n\}$ into convex sets. There are 2^n such* *partitions. (There are twice as many convex partitions of $\{1,\ldots,n\}$ as there are of $\{1,\ldots,n-1\}$. To each convex partition of $\{1,\ldots,n-1\}$ one may add $\{n\}$ as a new partition element, or include it in the partition element containing $n-1$.) Thus, $|K_o(X)| = 2^n = |P(\{1,\ldots,n\})|$. For each subset A of $\{1,\ldots,n\}$, associate the convex partition of $\{1,\ldots,n\}$ for which j and $j-1$ are in distinct partition elements if and only if $j \in A$. With this correspondence, $K_o(X) \cong P(\{1,\ldots,n\})$.* □

We close this section with a result on the problem of when $K_o(X)$ can be represented as the product $K_o(A) \times K_o(B)$ for subspaces $A, B \subseteq X$.

The following notation will be used in the proof of the next theorem. If p is a point of (X,τ,θ) then $i_X(p)$ will be used to denote the increasing hull of p in X, that is $i_X(p) = \{x \in X : p \leq x\}$. The closed increasing hull of p, denoted by $I_X(p)$, is the intersection of all closed increasing sets containing p. The decreasing hull of p and the closed decreasing hull of p, denoted by $d_X(p)$ and $D_X(p)$ respectively, are defined analogously.

Theorem 2.4 *If X is an ordered space and there exists a point $p \in X$ such that $i_X(p) \cup d_X(p) = X$, then $K_o(X) \cong K_o(i_X(p)) \times K_o(d_X(p))$.*

Proof. Suppose there exists such a point p. For any $\alpha X \in K_o(X)$, $i_{\alpha X}(p) = I_{\alpha X}(p)$ and $d_{\alpha X}(p) = D_{\alpha X}(p)$ are compact sets such that $i_{\alpha X}(p) \cup d_{\alpha X}(p) = \alpha X$ and $i_{\alpha X}(p) \cap d_{\alpha X}(p) = \{p\}$. Since $i_X(p)$ is dense in $i_{\alpha X}(p)$, it follows that $i_{\alpha X}(p) \in K_o(i_X(p))$. Similarly $d_{\alpha X}(p) \in K_o(d_X(p))$. Thus to each $\alpha X \in K_o(X)$, we can associate $i_{\alpha X}(p) \times d_{\alpha X}(p) \in K_o(i_X(p)) \times K_o(d_X(p))$.

On the other hand, suppose $Y \in K_o(i_X(p))$ and $Z \in K_o(d_X(p))$. Every compactification point $y \in Y \setminus i_X(p)$ must be greater than p. (For any net (y_γ) in Y converging to y, (p, y_γ) is a net in the graph of the closed order on Y, so $p \leq y$.) Similarly, every compactification point $z \in Z \setminus d_X(p)$ must be less than p. Identify the two copies of p in the disjoint union of Y and Z to get a compactification αX of X. Given the closed order $\theta_Y \cup \theta_Z \cup [Z \times Y]$, where θ_Y and θ_Z are the orders on Y and Z respectively, αX is an ordered compactification of X with $i_{\alpha X}(p) = Y$ and $d_{\alpha X}(p) = Z$. It follows that $K_o(X) \cong K_o(i_X(p)) \times K_o(d_X(p))$ where the latter poset has the product order.□

The utility of the theorem above lies perhaps not so much in finding spaces with such a segregating point p, but in constructing them. If (Y, τ_Y, θ_Y) and (Z, τ_Z, θ_Z), are ordered spaces, then the direct sum $X = Y \oplus \{p\} \oplus Z$ given the order $\theta_Y \cup \theta_Z \cup [(Y \cup \{p\}) \times (Z \cup \{p\})]$ satisfies the hypotheses of Theorem 2.4. Furthermore, since $\{p\}$ is topologically isolated and maximum in $Y \cup \{p\}$, it follows that $K_o(d_X(p)) = K_o(Y \cup \{p\}) = K_o(Y)$, and similarly, $K_o(i_X(p)) = K_o(Z)$. Thus Theorem 2.4 yields the following corollary.

Corollary 2.5 *For any ordered spaces Y and Z, there exists an ordered space X with $K_o(X) \cong K_o(Y) \times K_o(Z)$.*

Corollary 2.6 *For any integer $n \geq 1$, there exists an ordered space X with $K_o(X) \cong [0,1]^n$.*

Proof. Apply Corollary 2.5 and Proposition 2.1. □

References

[1] J. Blatter. Order compactifications of totally ordered topological spaces. *J. Approximation Theory*, 13:56–65, 1975.

[2] R. Chandler. *Hausdorff Compactifications.* Marcel Dekker, Inc., New York, 1976.

[3] P. Fletcher and W. Lindgren. *Quasi-uniform spaces.* Marcel Dekker, Inc., New York, 1982.

[4] I. Glicksberg. Stone-Čech compactifications of products. *Trans. Amer. Math. Soc.*, 90:369–382, 1959.

[5] D. Kent, D. Liu, and T. Richmond. On the Nachbin compactification of products of totally ordered spaces. *To appear, International J. Math. and Math. Sci.*

[6] D. Kent and T. Richmond. Ordered compactification of totally ordered spaces. *International J. Math. and Math. Sci.*, 11(4):683–694, 1988.

[7] F. Kost. α-point compactifications. *Portugaliea Mathematica*, 32:133–137, 1973.

[8] L. Nachbin. *Topology and Order.* Van Nostrand, New York, 1965.

[9] J. Porter and R.G. Woods. *Extensions and Absolutes of Hausdorff Spaces.* Springer–Verlag, 1988.

[10] J. Visliseni and J. Flachsmeyer. The power and structure of the lattice of all compact extensions of a completely regular space. *Soviet Math. Dokl.*, 6:1423–1425, 1965. *Dokl. Akad. Nauk SSSR*, 165:258–260, 1965.