Ramanujan J.
DOI 10.1007/s11139-012-9457-8
The final publication is available at www.springerlink.com

Cantor Sets Arising from Continued Radicals

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Received: 14 August 2012 / Accepted: 29 November 2012

Abstract If $a_1, a_2, a_3, \ldots$ are nonnegative real numbers and $f_j(x) = \sqrt{a_j + x}$, then $\lim_{n\to\infty} f_1 \circ f_2 \circ \cdots \circ f_n(0)$ is a continued radical with terms $a_1, a_2, a_3, \ldots$. The set of real numbers representable as a continued radical whose terms a_i are all from a set $S = \{a, b\}$ of two natural numbers is a Cantor set. We investigate the thickness, measure, and sums of such Cantor sets.

Keywords Cantor set · Continued radical · Thickness

Mathematics Subject Classification (2010) 40A99 · 11P99 · 26A18 · 37B20 · 11K55 · 28A80

1 Introduction

We consider continued radicals

$$\sqrt{a_1, a_2, \ldots} \equiv \lim_{n\to\infty} \sqrt{a_1 + \sqrt{a_2 + \cdots + \sqrt{a_n}}}$$

whose terms $a_1, a_2, \ldots$ all belong to a finite set S of nonnegative real numbers. If the members of S are too widely spaced, the set $R(S)$ of real numbers representable as a continued radical using terms from S will have recurring gaps. In [14], conditions are given on S for $R(S)$ to form an interval or to form a Cantor set. We will focus on Cantor sets $C(\{a, b\})$ of form $R(S)$ where $S = \{a, b\}$ is a set of two natural numbers.

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The study of continued radicals may be modeled on the study of continued fractions, another similar iterated function system. In [3,2,10], Cantor sets of numbers representable as continued fractions with terms from a restricted set are studied in general, with [5,4,7–9,12] giving particular attention to the question of when the sum $C_1 + \cdots + C_n = \{x_1 + \cdots + x_n : x_i \in C_i \text{ for } i = 1, \ldots, n\}$ of such Cantor sets forms an interval. Here we address similar questions for Cantor sets arising from continued radicals with restricted terms. In the case of continued fractions, the well-known recurrence relations for the numerators and denominators of the partial quotients are fundamental. No such recurrence relations are known for partial expressions of continued radicals. (Indeed, the partial expressions are seldom rational.) Thus, other approaches are required for continued radicals.

Sizer [19] and Laugwitz [16] have shown that a continued radical $\sqrt{a_1, a_2, \ldots}$ converges if and only if the set $\{\sqrt[2^n]{a_n} : n \in \mathbb{N}\}$ is bounded. Herschfeld [11] gave earlier results on convergence. Sizer [19] also noted that any real number can be represented as $a_0 + \sqrt{a_1, a_2, a_3, \ldots}$ where $a_0 \in \mathbb{Z}$ and $a_i \in \{0, 1, 2\}$ for $i \in \mathbb{N}$. The continued radical $\sqrt{n, n, n, \ldots}$ is easily seen to converge to $(1 + \sqrt{4n + 1})/2 \equiv \varphi_n$ (see [11,14]). Notice that φ_1 is the golden ratio.

Ramanujan's well-known example from 1911 that

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}} = 3$$

can be rewritten in the form $\sqrt{a_1, a_2, \ldots}$ by bringing the coefficient of each radical inside the radical. This yields $a_n = n!^2(n-1)!^2(n-2)!^2 \ldots 3!^2 2!^2 1!^2$. Further discussion of this example appears in [6,11,20].

By a *Cantor set*, we mean a compact, perfect, totally disconnected subset of the real line. Astels [3,5] says a Cantor set C is *derived from the interval* I if $C = I \setminus \bigcup \mathcal{G}$ where I is a compact interval and $\mathcal{G}$ is a countable collection of disjoint open intervals contained in I. In practice, Cantor sets are typically constructed iteratively by successively removing a finite number of open intervals. A different iterative sequence of removals may produce the same Cantor set. A sequence of iterative removals producing a Cantor set C is a *derivation* of C. In all Cantor sets C considered here, C will be derived from an interval by iteratively removing 2^{n-1} new *gaps* at each iteration. The connected intervals remaining at each iteration are called *bridges.* Specifically, starting with $I_0 = I$, at the nth iteration, we have $I_n = I_{n-1} \setminus \{G_{n,1}, G_{n,2}, G_{n,3}, \ldots, G_{n,2^{n-1}}\}$ where the gaps $G_{n,i}$ are mutually disjoint open intervals contained in I_{n-1}. Now I_n consists of 2^n bridges $B_{n,1}, B_{n,2}, \ldots, B_{n,2^n}$, and $C = \bigcap_{n \in \mathbb{N}} I_n$. As a convention, we label the new gaps $G_{n,1}, G_{n,2}, \ldots, G_{n,2^{n-1}}$ on level n from left to right. Similarly, the bridges $B_{n,1}, \ldots, B_{n,2^n}$ on level n are labeled from left to right. If $B_{n,i} \cup G_{n,k} \cup B_{n,i+1}$ is an interval, we say the bridges $B_{n,i}$ and $B_{n,i+1}$ are adjacent to the gap $G_{n,k}$.

The *thickness* of a Cantor set, introduced by Newhouse [17] and used in [13, 15], is a measure of the relative size of the bridges in relation to the gaps. For a given derivation $\mathcal{D}$ of a Cantor set C, we define the thickness at level n to be

$$\tau(n) = \inf \left(\frac{\min(|B_{n,i}|, |B_{n,i+1}|)}{|G_{n,k}|} \right)$$

where $B_{n,i}, G_{n,k}$, and $B_{n,i+1}$ are adjacent, $|A|$ represents the length of interval A, and the infimum is taken over all gaps on level n. The thickness of the derivation $\mathcal{D}$ of C is then taken to be $\tau(\mathcal{D}) = \inf\{\tau(n) : n \in \mathbb{N}\}$. The thickness of the Cantor set C is defined to be $\tau(C) = \inf\{\tau(\mathcal{D}) : \mathcal{D}$ is a derivation of $C\}$. Astels [3] shows that the thickness τ of a Cantor set C is $\tau(\mathcal{D}_o)$ for any derivation $\mathcal{D}_o$ with the following property: For any bridges A and E of $\mathcal{D}_o$ with $A = B_{n,i} \cup G_{n,k} \cup B_{n,i+1}$ and $E = B_{m,j} \cup G_{m,r} \cup B_{m,j+1}$, if $m \geq n$ and $E \subseteq A$, then $|G_{n,k}| \geq |G_{m,r}|$. Such a derivation is called an *ordered derivation.* If $\tau(C)$ is the thickness of a Cantor set C, then the normalized thickness of C is $\gamma(C) = \frac{\tau(C)}{\tau(C)+1}$.

In the next section, we will present results which will allow us to apply the following theorem of Astels [3] about when the sum of Cantor sets is an interval. Of course, if one Cantor set C_1 is derived from an interval whose length is shorter than a gap of another Cantor set C_2, then their sum cannot span the gap of C_2 to produce an interval. The following condition prevents this type of situation. If, for each $j \in \{1, \ldots, k\}$, C_j is a Cantor set derived from I_j and G^j is the longest gap on any level of C_j, the sequence of intervals $(I_1, \ldots, I_k)$ is *compatible* if $|I_{r+1}| \geq |G^j|$ and $|I_1| + \cdots |I_r| \geq |G^{r+1}|$ for $r = 1, \ldots, k-1$ and $j = 1, \ldots, r$. The Hausdorff dimension of X is denoted $\dim_H(X)$.

Theorem 1 (Astels [3]) *Let k be a positive integer and for $j = 1, \ldots, k$ let $\mathcal{C}_j$ be a Cantor set derived from I_j. Put $S_\gamma = \gamma(\mathcal{C}_1) + \cdots + \gamma(\mathcal{C}_k)$ and assume that $(I_1, \cdots, I_k)$ is compatible. If $S_\gamma \geq 1$, then $\mathcal{C}_1 + \cdots + \mathcal{C}_k = I_1 + \cdots + I_k$. Otherwise, $\gamma(\mathcal{C}_1 + \cdots + \mathcal{C}_k) \geq S_\gamma$ and $dim_H(\mathcal{C}_1 + \cdots + \mathcal{C}_k) \geq \frac{log(2)}{log\left(1+\frac{1}{S_\gamma}\right)}$.*

Since the sum of a set of intervals is an interval, this theorem gives sufficient conditions for the sum of Cantor sets to be an interval.

2 Sums of Cantor Sets from Continued Radicals with Two Terms

From [14], if $S = \{a, b\} \subseteq \mathbb{N}$ with $a < b$, then the set $R(S)$ of real numbers representable as a continued radical with all terms from S is a Cantor set $C(\{a, b\})$ derived from the interval $I = [\varphi_a, \varphi_b]$: Since $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$ implies $\sqrt{x_1, x_2, \ldots} \leq \sqrt{y_1, y_2, \ldots}$, the smallest and largest numbers in $C(\{a, b\})$ are $\sqrt{a, a, a, \ldots} = \varphi_a$ and $\sqrt{b, b, b, \ldots} = \varphi_b$, respectively. The largest number in $C(\{a, b\})$ with first term a is $\sqrt{a, b, b, b, \ldots} = \sqrt{a + \varphi_b} = \sqrt{a, \varphi_b^2}$. The smallest number in $C(\{a, b\})$ with first term b is $\sqrt{b, a, a, a, \ldots} = \sqrt{b + \varphi_a} = \sqrt{b, \varphi_a^2}$. Since $\sqrt{a + \varphi_b} < \sqrt{b + \varphi_a}$, there is a gap $G_{1,1}$ in I caused by the jump from a to b in the first terms. Similarly, gaps

$$(\sqrt{\mathbf{w}, a, b, b, b, \ldots}, \sqrt{\mathbf{w}, b, a, a, a, \ldots}) = (\sqrt{\mathbf{w}, a + \varphi_b}, \sqrt{\mathbf{w}, b + \varphi_a})$$

arise on level $n+1$ from the jump from a to b in the $n+1$st term, for all 2^n choices of the *direction vector* $\mathbf{w} = (w_1, w_2, \ldots, w_n)$, where each $w_i \in \{a, b\}$. A direction vector $\mathbf{w}$ of length n directs us through the successive iterations I_j, moving to the left if $w_j = a$ and moving to the right if $w_j = b$, to the left endpoint of a bridge on level n, which is split on level $n+1$ by the gap $(\sqrt{\mathbf{w}, a+\varphi_b}, \sqrt{\mathbf{w}, b+\varphi_a})$. The adjacent bridges are $[\sqrt{\mathbf{w}, a+\varphi_a}, \sqrt{\mathbf{w}, a+\varphi_b}\,]$ and $[\sqrt{\mathbf{w}, b+\varphi_a}, \sqrt{\mathbf{w}, b+\varphi_b}\,]$. Throughout this section we will only consider Cantor sets of form $C(\{a,b\}) = R(\{a,b\})$, which is the set of real numbers representable as continued radicals, all of whose terms are a or b ($a, b \in \mathbb{N}$, $a < b$).

If $w \geq 1$, and $t(x) = \sqrt{w+x}$, then rationalizing the numerator of $t(x) - t(y)$ shows that t is a contraction on $[0, \infty)$ with contraction factor $1/(2\sqrt{w})$. Iterating, $g(x) = \sqrt{w_1, \ldots, w_n + x} = \sqrt{\mathbf{w}, x^2}$ is a contraction with contraction factor $1/(2^n\sqrt{w_1 w_2 \cdots w_n})$. Consequently, if two points $\sqrt{x_1, x_2, \ldots}$ and $\sqrt{y_1, y_2, \ldots}$ in $C(\{a,b\})$ agree in the first n terms, then their values differ by less than $2^{-n}(b-a)$. This property can be used to show that each point of $C(\{a,b\})$ is the limit of endpoints of bridges and if $x, y, z \in C(\{a,b\})$ with $x < y < z$, then the interval (x, z) contains an endpoint of a gap. Thus, $C(\{a,b\})$ is perfect and totally disconnected. Since it is clearly compact, $C(\{a,b\})$ is indeed a Cantor set.

Theorem 2 *If $B_{n,j}$ and $B_{n,j+1}$ are the two bridges adjacent to some gap $G_{n,k}$ on level n, then $|B_{n,j}| > |B_{n,j+1}|$. That is, the longer of two bridges adjacent to a gap is the left bridge.*

Proof If $B_{n,j}$ and $B_{n,j+1}$ are adjacent to a gap on level n, then $|B_{n,j}| = \sqrt{\mathbf{w}, a+\varphi_b} - \sqrt{\mathbf{w}, a+\varphi_a}$ and $|B_{n,j+1}| = \sqrt{\mathbf{w}, b+\varphi_b} - \sqrt{\mathbf{w}, b+\varphi_a}$, where $\mathbf{w} = (w_1, \ldots, w_{n-1})$ is a direction vector of length $n-1$. Letting $h(x) = \sqrt{\mathbf{w}, x+\varphi_b} - \sqrt{\mathbf{w}, x+\varphi_a}$, we have $|B_{n,j}| = h(a)$ and $|B_{n,j+1}| = h(b)$, so the desired result will follow if we show $h(x)$ is a strictly decreasing function. Letting $g_{i,x}(y) = \sqrt{w_i, w_{i+1}, \ldots, w_{n-1}, x+y}$, we find that

$$h'(x) = \frac{1}{2^n}\left[\prod_{i=1}^{n-1}\frac{1}{g_{i,x}(\varphi_b)} - \prod_{i=1}^{n-1}\frac{1}{g_{i,x}(\varphi_a)}\right].$$

Now $h'(x) < 0$ follows since each $g_{i,x}(y)$ is a strictly increasing function of y, $0 < \varphi_a < \varphi_b$, and the product of decreasing functions is decreasing. □

Theorem 3
(a) The longest gap on a given level n is the left-most gap $G_{n,1}$.
(b) The shortest gap on a given level n is the right-most gap $G_{n,2^{n-1}}$.
(c) The longest bridge on a given level n is the left-most bridge $B_{n,1}$.
(d) The shortest bridge on a given level n is the right-most bridge $B_{n,2^n}$.

Proof (a) If $G_{n,i}$ is any gap on level n other than the left gap $G_{n,1}$, then we have $|G_{n,i}| = \sqrt{\mathbf{w}, b+\varphi_a} - \sqrt{\mathbf{w}, a+\varphi_b}$ and $|G_{n,1}| = \sqrt{\mathbf{a}, b+\varphi_a} - \sqrt{\mathbf{a}, a+\varphi_b}$, where $\mathbf{w} = (w_1, w_2, \cdots, w_{n-1})$ and $\mathbf{a} = (a_1, a_2, \cdots, a_{n-1})$ are direction vectors of length $n-1$, with each $a_i = a$, and $a_i \leq w_i \in \{a, b\}$ for each i and $a < w_i$ for at least one i. Defining $f(x) = \sqrt{\mathbf{a}, x}$, $g(x) = \sqrt{\mathbf{w}, x}$, $\alpha = a + \varphi_b$, and $\beta = b + \varphi_a$, we have $|G_{n,1}| = f(\beta) - f(\alpha)$ and $|G_{n,i}| = g(\beta) - g(\alpha)$. Thus, to show $|G_{n,1}| > |G_{n,i}|$, it suffices to show

$$\frac{|G_{n,1}|}{|G_{n,i}|} = \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} > 1.$$

By the Cauchy mean value theorem, there exists $c \in (\alpha, \beta)$ with

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(c)}{g'(c)}.$$

Now

$$\frac{f'(c)}{g'(c)} = \frac{\sqrt{w_1, \ldots, w_{n-1}, c}\,\sqrt{w_2, \ldots, w_{n-1}, c} \cdots \sqrt{w_{n-1}, c}}{\sqrt{a_1, \ldots, a_{n-1}, c}\,\sqrt{a_2, \ldots, a_{n-1}, c}\ \cdots\ \sqrt{a, c}},$$

which is greater than 1 since $a_i \leq w_i$ for each i and $a_i < w_i$ for at least one i. This completes the proof of (a). Notice that the Cauchy mean value theorem allows us to consider (quotients of) products of nested radicals rather than (quotients of) differences of nested radicals. The proofs of the remaining parts are similar. □

While the longest bridge and gap on a level appear on the left and the shortest bridge and gap appear on the right, the lengths of the bridges and gaps on a level do not decrease as one moves from left to right, as seen in the example below.

Example 1 Figure 1 shows the lengths of bridges and gaps, rounded to six decimal places, for the first three iterations I_1, I_2, and I_3 of the Cantor set $C(\{1, 23\})$. (The segments are not drawn to scale.)

Note that the lengths of the bridges and gaps on a level do not decrease, since $|B_{3,7}| > |B_{3,6}|$ and $|G_{3,3}| > |G_{3,2}|$. Further computation would show that $|G_{4,1}| \approx 0.047933 > |G_{3,4}| \approx 0.022299$, so in general, it is not true that all gaps on level $n + 1$ are shorter than all gaps on level n.

Recall that the thickness at level n of the Cantor set C is

$$\tau(n) = \inf \left(\frac{\min(|B_{n,i}|, |B_{n,i+1}|)}{|G_{n,k}|} \right)$$

where $B_{n,i}, G_{n,k}$, and $B_{n,i+1}$ are adjacent and the infimum is taken over all gaps on level n. By Theorem 2, the minimum in the numerator is always obtained by the right bridge $B_{n,i+1}$.

Theorem 4 *The thickness $\tau(n)$ is realized as the quotient $|B_{n,2}|/|G_{n,1}|$. That is, $\tau(n)$ is attained using the left-most gap $G_{n,1}$ on level n and the right bridge $B_{n,2}$ adjacent to that gap.*

Proof As in the proof of Theorem 3, let $f(x) = \sqrt{\mathbf{a}, x}$ and $g(x) = \sqrt{\mathbf{w}, x}$ where $\mathbf{a} = (a_1, a_2, \cdots, a_{n-1})$ and $\mathbf{w} = (w_1, w_2, \cdots, w_{n-1})$ are direction vectors of length $n - 1$, with each $a_i = a$, and $a_i \leq w_i \in \{a, b\}$ for each i and $a < w_i$ for at least one i. Now

$$\frac{|B_{n,2}|}{|G_{n,1}|} = \frac{f(b + \varphi_b) - f(b + \varphi_a)}{f(b + \varphi_a) - f(a + \varphi_b)} \tag{1}$$

and the ratio of the lengths of any other right-adjacent bridge to its adjacent gap has form

$$\frac{g(b + \varphi_b) - g(b + \varphi_a)}{g(b + \varphi_a) - g(a + \varphi_b)}. \tag{2}$$

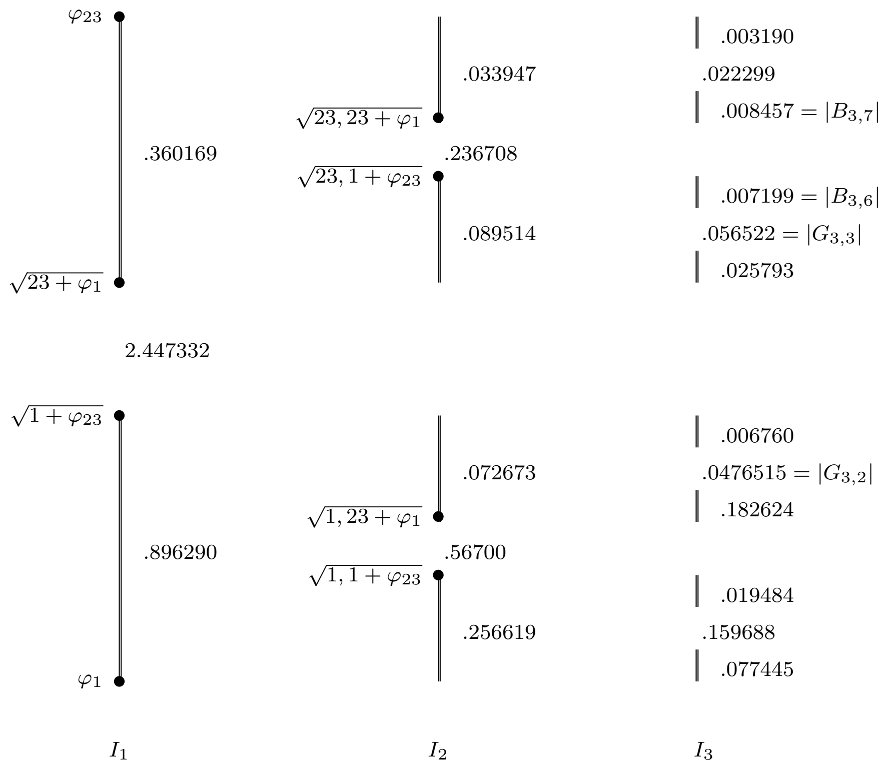

Fig. 1 Bridge and gap length for the Cantor set $C(\{1, 23\})$.

To show that the positive expression of Eq. (1) is always smaller than that of Eq. (2), we may consider their quotient and show that

$$1 > \frac{\frac{f(b+\varphi_b)-f(b+\varphi_a)}{g(b+\varphi_b)-g(b+\varphi_a)}}{\frac{f(b+\varphi_a)-f(a+\varphi_b)}{g(b+\varphi_a)-g(a+\varphi_b)}} = \frac{\frac{f'(c_1)}{g'(c_1)}}{\frac{f'(c_2)}{g'(c_2)}}, \tag{3}$$

for some $c_1 \in (b+\varphi_a, b+\varphi_b)$ and some $c_2 \in (a+\varphi_b, b+\varphi_a)$, after two applications of the Cauchy mean value theorem. Equation (3) is equivalent to

$$\frac{f'(c_1)}{g'(c_1)} < \frac{f'(c_2)}{g'(c_2)},$$

and since $c_2 < c_1$, it suffices to show that $f'(x)/g'(x)$ is a strictly decreasing function. Now

$$\begin{aligned}\frac{f'(x)}{g'(x)} &= \left(\frac{\sqrt{w_1,\ldots,w_{n-1},x}}{\sqrt{a_1,\ldots,a_{n-1},x}}\right)\left(\frac{\sqrt{w_2,\ldots,w_{n-1},x}}{\sqrt{a_2,\ldots,a_{n-1},x}}\right)\cdots\left(\frac{\sqrt{w_{n-1},x}}{\sqrt{a_{n-1},x}}\right)\\ &= \prod_{j=1}^{n-1} h_j(x) \quad \text{where } h_j(x) = \frac{\sqrt{w_j,\ldots,w_{n-1},x}}{\sqrt{a_j,\ldots,a_{n-1},x}}.\end{aligned}$$

Now $h_j'(x) = A_j/B_j$ where

$$\begin{aligned}A_j &= (a_j + \sqrt{a_{j+1},\ldots,a_{n-1},x})(\sqrt{a_{j+1},\ldots,a_{n-1},x}\cdots\sqrt{a_{n-1},x})\\ &\quad -(w_j + \sqrt{w_{j+1},\ldots,w_{n-1},x})(\sqrt{w_{j+1},\ldots,w_{n-1},x}\cdots\sqrt{w_{n-1},x})\end{aligned}$$

and

$$\begin{aligned}B_j = 2^{n-j+1}&\left(\sqrt{w_j,\ldots,w_{n-1},x}\cdots\sqrt{w_{n-1},x}\sqrt{x}\right)\cdot\\ &\left(\sqrt{a_j,\ldots,a_{n-1},x}\cdots\sqrt{a_{n-1},x}\right)\left(\sqrt{a_j,\ldots,a_{n-1},x}\right)^2.\end{aligned}$$

Now $B_j > 0$ and, since $a_j \le w_j$ and $a_j < w_j$ for at least one j, $A_j \le 0$ and $A_j < 0$ for at least one j. It follows that each $h_j(x)$ is decreasing and one is strictly decreasing, and thus $f'(x)/g'(x)$ is strictly decreasing, as needed. □

Theorem 5 *For $n \in \mathbb{N}$, we have $\tau(n) > \tau(n+1)$. That is, the thickness on level n decreases as n increases.*

Proof Let $\mathbf{a} = (a_1,\ldots,a_{n-1})$ with $a_i = a$ for all i, and $f(x) = \sqrt{\mathbf{a},x}$ as in the proof of Theorem 4, and let $s(x) = \sqrt{a,\mathbf{a},x}$. Now to show $\tau(n) > \tau(n+1)$, or equivalently, $1 < \tau(n)/\tau(n+1)$, by Theorem 4 and two applications of the Cauchy mean value theorem we must show

$$1 < \frac{\frac{f(b+\varphi_b)-f(b+\varphi_a)}{s(b+\varphi_b)-s(b+\varphi_a)}}{\frac{f(b+\varphi_a)-f(a+\varphi_b)}{s(b+\varphi_a)-s(a+\varphi_b)}} = \frac{\frac{f'(c_1)}{s'(c_1)}}{\frac{f'(c_2)}{s'(c_2)}}$$

for some $c_1 \in (b+\varphi_a, b+\varphi_b)$ and some $c_2 \in (a+\varphi_b, b+\varphi_a)$. Thus, it suffices to show that

$$\frac{f'(c_1)}{s'(c_1)} > \frac{f'(c_2)}{s'(c_2)}.$$

Now since each $a_i = a$, we have

$$\frac{f'(x)}{s'(x)} = 2\sqrt{a_1,\ldots,a_n,x},$$

which is a strictly increasing function, and the desired result follows since $c_1 > c_2$. □

Theorem 6 *Given a bridge $B_{n-1,j}$ on level $n-1$, the gap $G_{n,k}$ removed from this bridge is longer than both gaps $G_{n+1,m}, G_{n+1,m+1} \subset B_{n-1,j}$ 'below it' on the next level. Thus, the iterative construction of the Cantor set $C(\{a,b\})$ is an ordered derivation of $C(\{a,b\})$.*

Proof First we show that, of the two gaps $G_{n+1,m}$ and $G_{n+1,m+1}$ two levels below the bridge $B_{n-1,j}$, the longer one is on the left, that is, $|G_{n+1,m}| > |G_{n+1,m+1}|$. The lengths of the gaps two levels below the bridge whose left endpoint is determined by $\mathbf{w}$ are $\ell(x) = \sqrt{\mathbf{w}, x, b+\varphi_a} - \sqrt{\mathbf{w}, x, a+\varphi_b}$ where x is either a (for the left gap) or b (for the right gap). Now

$$\ell'(x) =$$

$$\left[2^n \sqrt{w_1, \ldots, w_{n-1}, x, b+\varphi_a}\sqrt{w_2, \ldots, w_{n-1}, x, b+\varphi_a} \cdots \sqrt{x, b+\varphi_a}\right]^{-1}$$

$$- \left[2^n \sqrt{w_1, \ldots, w_{n-1}, x, a+\varphi_b}\sqrt{w_2, \ldots, w_{n-1}, x, a+\varphi_b} \cdots \sqrt{x, a+\varphi_b}\right]^{-1}.$$

Now since $b+\varphi_a > a+\varphi_b$, for each $j \in \{1, \ldots, n-1\}$ and any $x > 0$ we have

$$[\sqrt{w_j, \ldots, w_{n-1}, x, b+\varphi_a}]^{-1} < [\sqrt{w_j, \ldots, w_{n-1}, x, a+\varphi_b}]^{-1}.$$

It follows that $\ell'(x) < 0$ for all $x > 0$, and in particular, $\ell(a) > \ell(b)$. This proves our first claim.

Now to show that a gap $G_{n,k}$ on level n is longer than either of the two gaps below it, we need only show that it is longer than the left gap below it. That is, we must show that for any direction vector $\mathbf{w}$ determining the left endpoint of the bridge $B_{n-1,j}$, we have $|G_{n,k}| > |G_{n+1,m}|$. Considering the endpoints of these gaps and dividing, we wish to show that the ratio

$$r(\mathbf{w}) \equiv \frac{\sqrt{\mathbf{w}, b+\varphi_a} - \sqrt{\mathbf{w}, a+\varphi_b}}{\sqrt{\mathbf{w}, a, b+\varphi_a} - \sqrt{\mathbf{w}, a, a+\varphi_b}} > 1 \tag{4}$$

for any direction vector $\mathbf{w}$ of length $n-1$.

We show that the ratio $r(\mathbf{w})$ is minimized when $\mathbf{w} = \mathbf{a} = (a, a, \ldots, a)$. Letting $f(x) = \sqrt{\mathbf{a}, x}$ and $g(x) = \sqrt{\mathbf{w}, x}$ as in the proof of Theorem 4, we wish to show

$$1 > \frac{r(\mathbf{a})}{r(\mathbf{w})} = \frac{\frac{f(b+\varphi_a)-f(a+\varphi_b)}{g(b+\varphi_a)-g(a+\varphi_b)}}{\frac{f(a+\sqrt{b+\varphi_a})-f(a+\sqrt{a+\varphi_b})}{g(a+\sqrt{b+\varphi_a})-g(a+\sqrt{a+\varphi_b})}} = \frac{\frac{f'(c_1)}{g'(c_1)}}{\frac{f'(c_2)}{g'(c_2)}}$$

where $a+\sqrt{a+\varphi_b} < c_2 < a+\sqrt{b+\varphi_a} < a+\varphi_b < c_1 < b+\varphi_a$. Thus it suffices to show that $f'(x)/g'(x)$ is a strictly decreasing function. This was shown in the proof of Theorem 4.

Now since $r(\mathbf{w})$ is minimized at $\mathbf{w} = \mathbf{a}$, Eq. (4) will follow if we show $r(\mathbf{a}) > 1$. With $f(x) = \sqrt{\mathbf{a}, x}$ and $s(x) = \sqrt{a, \mathbf{a}, x}$ as in the proof of Theorem 5, we have

$$r(\mathbf{a}) = \frac{f(b+\varphi_a) - f(a+\varphi_b)}{s(b+\varphi_a) - s(a+\varphi_b)} = \frac{f'(c)}{s'(c)} = 2\sqrt{a, a, \ldots, a, c} \geq 2\varphi_a > 1,$$

where c is some point in $(a+\varphi_b, b+\varphi_a)$ guaranteed by the Cauchy mean value theorem. □

Note that Example 1 showed that all gaps on level $n+1$ need not be shorter than all gaps on level n, but the previous result shows that if we start with a bridge and consider all gaps below it, then the gap lengths decrease as the level increases.

Now, in view of the previous two theorems, the thickness $\tau(a,b)$ of the Cantor set $C(\{a,b\})$ of all real numbers representable as a continued radical using terms a and b is

$$\tau(a,b) = \inf_{n\in\mathbb{N}} \tau(n) = \lim_{n\to\infty} \tau(n) = \lim_{n\to\infty} \frac{|B_{n,2}|}{|G_{n,1}|} = \lim_{n\to\infty} \frac{\sqrt{\mathbf{a}_n, b+\phi_b} - \sqrt{\mathbf{a}_n, b+\phi_a}}{\sqrt{\mathbf{a}_n, b+\phi_a} - \sqrt{\mathbf{a}_n, a+\phi_b}},$$

where $\mathbf{a}_n = (a, a, \ldots, a)$ is a vector of length n.

An exact evaluation of this limit would be ideal. The corresponding situation for continued fractions is tractable using the recurrence relations for partial quotients. No corresponding techniques have been found and we are only able to present numerical evidence in particular cases. Before applying Theorem 1, we address the compatibility conditions needed.

Remark 1 (a) If Cantor sets C_j are derived from intervals I_j $(j = 1, \ldots, k)$, then the sequence of intervals is compatible if the shortest interval is larger than the longest gap. Furthermore, by Theorem 6, the longest gap in any Cantor set $C(\{a,b\})$ is $G_{1,1}$, the gap on level 1.
(b) If $C_1 = C_2 = \ldots = C_k$, then $I_1, I_2, \ldots, I_k$ are compatible.

We now present a sampling of numerical data. In Table 1, we give the thickness $\tau(C(\{a,b\}))$ and the normalized thickness $\gamma(C(\{a,b\}))$ to ten decimal places for $a = 1$ and $a = 4$ and $a < b \le 21$.

Numerical evidence such as that from Table 1 prompts our conjecture that for a fixed value of a, the function $\tau(a,x)$ is decreasing in x.

Table 2 gives the normalized thickness, to six decimal places, of $C(\{a,b\})$ for $1 \le a < b \le 7$. For example, we see that $\gamma(C(\{2,5\})) \approx 0.233486$. Now $4 \cdot \gamma(C(\{2,5\})) < 1$ but $5 \cdot \gamma(C(\{2,5\})) > 1$, so Theorem 1 tells us that the sum of five copies of $C(\{2,5\})$ equals the sum of five copies of the interval $[\varphi_2, \varphi_5]$, which is an interval, namely $[5\varphi_2, 5\varphi_5]$.

Table 3 gives the number of copies of $C(\{a,b\})$ required to guarantee, by Theorem 1, that their sum is an interval. Recall from Remark 1 that when using copies of the same Cantor set, compatibility is ensured. In our next example, we consider sums of different Cantor sets.

Example 2 Consider the Cantor sets $C(\{1,4\})$, $C(\{5,8\})$, and $C(\{6,10\})$ derived from the interval $[\varphi_1, \varphi_4]$, $[\varphi_5, \varphi_8]$, and $[\varphi_6, \varphi_{10}]$, respectively. The shortest of these intervals is $[\phi_5, \phi_8]$ with length approximately 0.580993. Furthermore, the longest gap among the three Cantor set is $G_{1,1}^{1,4}$, that is, gap $G_{1,1}$ of $C(\{1,4\})$, with length approximately 0.136643. By Remark 1, any collection of these Cantor sets is compatible. The normalized thicknesses of these sets are approximately $\gamma(C(\{1,4\})) = 0.268473$, $\gamma(C(\{5,8\})) = 0.179621$, and $\gamma(C(\{6,10\})) = 0.161069$. Now using Theorem 1, we can conclude, for example, that the sum of two copies of

Table 1 Thickness of Cantor sets $C(\{1,b\})$ and $C(\{4,b\})$.

b	$\tau(C\{1,b\})$	$\gamma(C\{1,b\})$	$\tau(C\{4,b\})$	$\gamma(C\{4,b\})$
2	0.5563765281	0.3574819576		
3	0.4378241825	0.3045046730		
4	0.3670028380	0.2684726233		
5	0.3191832762	0.2419552173	0.2858162541	0.2222839020
6	0.2843826219	0.2214158126	0.2597708368	0.2062048344
7	0.2577374624	0.2049215119	0.2392003072	0.1930279599
8	0.2365731983	0.1913135418	0.2224487779	0.1819698149
9	0.2192878968	0.1798491540	0.2084826899	0.1725160746
10	0.2048590589	0.1700274048	0.1966197677	0.164312652
11	0.1926010008	0.1614965950	0.1863894396	0.1571064554
12	0.1820353889	0.1540016404	0.1774554799	0.1507109890
13	0.1728175773	0.1473524789	0.1695705608	0.1449853190
14	0.1646924588	0.1414042458	0.1625482542	0.1398206514
15	0.1574668411	0.1360443647	0.1562450868	0.1351314601
16	0.1509915242	0.1311838715	0.1505486485	0.1308494419
17	0.1451493061	0.1267514247	0.1453694833	0.1269192915
18	0.1398467384	0.1226890718	0.1406354194	0.1232956797
19	0.1350083245	0.1189491933	0.1362875184	0.1199410503
20	0.1305723528	0.1154922570	0.1322771242	0.1168239836
21	0.1264878467	0.1122851410	0.1285636769	0.1139179645

$C(\{1,4\})$, one copy of $C(\{5,8\})$, and two copies of $C(\{6,10\})$ is an interval since the corresponding sums of the normalized thicknesses for these five compatible Cantor sets exceeds 1. Similarly the sum of any collection of five of these Cantor sets including exactly two or exactly three copies of $C(\{1,4\})$ is an interval, as is the sum of any collection of six of these Cantor sets including exactly one copy of $C(\{1,4\})$. If no copies of $C(\{1,4\})$ are used, the sum of any collection of six of the other sets including at least two copies of $C(\{5,8\})$ is an interval, or any collection of seven of the others if fewer than two copies of $C(\{5,8\})$ are included.

3 Measure

The measure of the familiar Cantor middle-thirds set is widely known to be zero. By removing gaps which are successively smaller percentages of the bridges from which they are removed, one may construct Cantor sets of positive measure. (See, for example, Aliprantis and Birkinshaw [1, pp. 115–116].)

Theorem 7 *For any natural numbers $a < b$, the measure of $C(\{a,b\})$ is zero.*

Table 2 Normalized thickness $\gamma(C(\{a, b\}))$ for small a, b.

a \ b	2	3	4	5	6	7
1	0.357482	0.304505	0.268473	0.241955	0.221416	0.204922
2		0.288237	0.256910	0.233486	0.215116	0.200212
3			0.248781	0.227225	0.210196	0.196298
4				0.222284	0.206205	0.193028
5					0.202864	0.190242
6						0.187824

Table 3 Minimum Number of Cantor sets $C(\{a, b\})$ required to guarantee their sum is an interval.

a \ b	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	3	4	4	5	5	5	6	6	6	7	7	7	8	8
2		4	4	5	5	5	6	6	6	7	7	7	8	8
3			5	5	5	6	6	6	7	7	7	7	8	8
4				5	5	6	6	6	7	7	7	7	8	8
5					5	6	6	6	7	7	7	7	8	8
6						6	6	6	7	7	7	7	8	8
7							6	6	7	7	7	8	8	8
8								7	7	7	7	8	8	8
9									7	7	7	8	8	8
10										7	7	8	8	8
11											7	8	8	8
12												8	8	8
13													8	8
14														8

Proof The length of a bridge $B_{n,j}$ on level n is $\sqrt{\mathbf{w}, \varphi_b^2} - \sqrt{\mathbf{w}, \varphi_a^2}$ where $\mathbf{w} = (w_1, \ldots, w_n)$ is the direction vector of length n (with each $w_i \in \{a, b\}$) which determines the left endpoint of the bridge. Recalling the contraction factor $2^{-n}(w_1 w_2 \cdots w_n)^{-1/2}$ for $\sqrt{w_1, \ldots, w_n + x} = \sqrt{\mathbf{w}, x^2}$, we have

$$|B_{n,j}| = \sqrt{\mathbf{w}, \varphi_b^2} - \sqrt{\mathbf{w}, \varphi_a^2} < \frac{1}{2^n \sqrt{w_1 w_2 \cdots w_n}} (\varphi_b - \varphi_a).$$

If $a > 1$, then recalling that the longest bridge on level n is the left-most bridge $B_{n,1}$ arising from the direction vector $\mathbf{w} = \mathbf{a} = (a, a, \ldots, a)$, we see that the measure of the nth iteration I_n is

$$|I_n| = \sum_{i=1}^{2^n} |B_{n,i}| < 2^n |B_{n,1}| < 2^n (\frac{1}{2^n \sqrt{a^n}})(\varphi_b - \varphi_a).$$

The right hand side of this expression converges to zero as n approaches infinity if $a > 1$, so it only remains to consider the case $a = 1 < b$. Then

$$|B_{n,j}| < \frac{1}{2^n\sqrt{w_1 w_2 \cdots w_n}}(\varphi_b - \varphi_1) = \frac{(\varphi_b - \varphi_1)}{2^n\sqrt{b^k}},$$

where k is the number of coordinates $w_i \in \{1, b\}$ which are equal to b. Of the 2^n direction vectors $\mathbf{w}$ of length n determining endpoints of bridges on level n, there are $\binom{n}{k}$ of them with exactly k coordinates equal to b. Therefore, the sum of the lengths of the 2^n bridges on level n is less than

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(\varphi_b - \varphi_1)}{2^n\sqrt{b^k}} = (\varphi_b - \varphi_1)\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^n}\frac{1}{\sqrt{b^k}}. \tag{5}$$

The last sum in Eq. (5) is, by the binomial theorem, $((1+c)/2)^n$ for $c = 1/\sqrt{b}$. Since $b \geq 2$, we have $c < 3/4$ and $(1+c)/2 < 7/8$, so the positive expression in Eq. (5) is bounded above by $(\varphi_b - \varphi_1)(7/8)^n$ and thus converges to zero. Thus the Cantor set $C(\{1, b\})$ has measure zero. □

We mention that other proofs that the expression in Eq. (5) bounding the measure of $C(\{1, b\})$ converges to 0 may be based on Sterling's approximation (see [18])

$$\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1,$$

using bridges on levels $2n$ and the fact that $\binom{2n}{k} \leq \binom{2n}{n}$, or by viewing the sum in Eq. (5) as a weighted average of $\{b^{-k/2}\}_{k=0}^{n}$.

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