https://doi.org/10.1080/07468342.2018.1512326

Calculus with Curtains

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Certain optimization problems seen in calculus have well-known solutions. If a wire of length l can be cut and bent into a square and a circle, the maximum enclosed area results from using the entire length to form a circle. The rectangle of largest area inscribed in a circle is a square. A rectangular region to be fenced and partitioned with m horizontal strands and n vertical strands of wire has maximum area when the total length of the horizontal strands equals the total length of the vertical strands. This short note presents another optimization problem with a striking answer, which seems to be less well-known. We start with a specific example.

At a local theater, the stage opening is a rectangle, 20 m wide and 10 m high. A certain event requires the curtain to be draped into triangles on the sides, extending from the top to the floor, with a semicircle in the center, as shown. What radius will minimize the area covered by the curtain, and thus maximize the visible area?

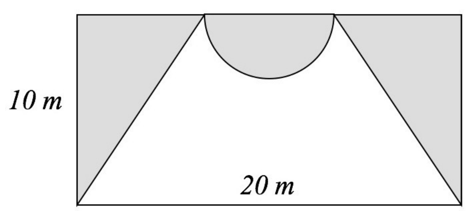

Figure 1. What radius will minimize the draped area?

If the radius of the semicircle is r, then the curtain area is $A(r) = \frac{1}{2}\pi r^2 + 10(10 - r)$ for $r \in [0, 10]$, and $A'(r) \geq 0$ for $r \geq \frac{10}{\pi}$, so the minimum area is achieved when $r = \frac{10}{\pi}$. Thus, when the curtain area is minimized, the arc length of the boundary of the semicircular curtain is $\frac{1}{2}(2\pi r) = \pi(\frac{10}{\pi}) = 10$, which is the height of the stage opening.

This problem generalizes.

Theorem 1. *If the curtain on a rectangular stage opening is to cover n adjacent semicircles across the top and (possibly degenerate) triangles extending to the floor on the left and right sides, then to minimize the covered area, the semicircles should be of equal radius, with the arc length of each equal to the height of the stage opening, unless the width of the stage is too small to accommodate this, in which case the area is minimized by n adjacent semicircles which extend the entire width of the stage.*

Proof. Suppose the rectangular stage opening has height h meters and width w meters.

First, we will show that to minimize the area, the adjacent semicircles must be of equal radius. Suppose adjacent semicircles have radii r_1 and r_2 with $r_1 + r_2 = K$, a constant. Now the combined area of the semicircles is $A(r_1, r_2) = \frac{1}{2}\pi(r_1^2 + r_2^2) = \frac{\pi}{2}(r_1^2 + (K - r_1)^2) = A(r_1)$, a function of r_1 alone. $A(r_1)$ achieves its minimum value at the critical point $r_1 = K/2$, which corresponds to $r_2 = K/2 = r_1$. Thus, in a minimal-area configuration, every pair of adjacent semicircles along the top will have equal radii.

Since the combined area of any collection of triangles with heights h and combined bases $w - 2nr$ is constant, to simplify our configuration we may place the n semicircles across the top starting at the right edge, with only one triangle on the left. The combined area covered by the n semicircles of radius r and the triangle is $A(r) = \frac{n}{2}\pi r^2 + \frac{1}{2}(w - 2nr)h$. The radii may increase until the horizontal span $2nr$ of the n semicircles equals the width w, or until the semicircles touch the floor, so the domain is $r \in (0, min\{h, \frac{w}{2n}\}]$. It is easy to check that $A(r)$ decreases until r reaches the critical point $r = \frac{h}{\pi}$, then increases, so $A(r)$ is minimum at $r = \frac{h}{\pi}$ if this critical number lies in the domain, and is minimum at the right endpoint of the domain otherwise. The arc length πr of each semicircle at the critical point $r = \frac{h}{\pi}$ is h.

Since the critical point $\frac{h}{\pi}$ is always less than h, it falls in the domain unless $\frac{h}{\pi} > \frac{w}{2n}$, that is, unless $w < 2n(\frac{h}{\pi})$, which is the horizontal distance spanned across the top by n semicircles of the optimal radius $r = \frac{h}{\pi}$. If the stage is too narrow to fit n such semicircles, then the minimum occurs when $r = \frac{w}{2n}$, so each semicircle spans $\frac{w}{n}$, and the n semicircles span the entire width w, leaving no room for triangles. ■

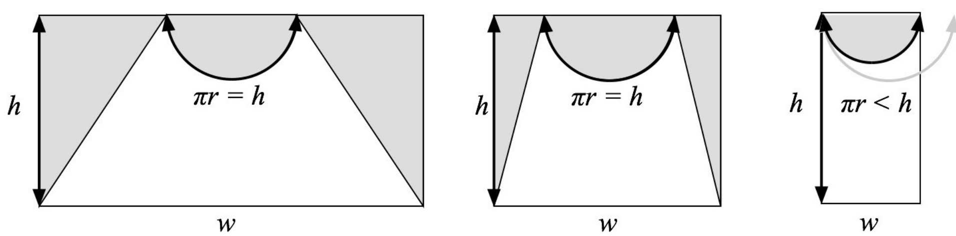

Figure 2. For one semicircular drape, the optimal radius has $\pi r = h$, independent of width w, unless w is too small to accommodate this.

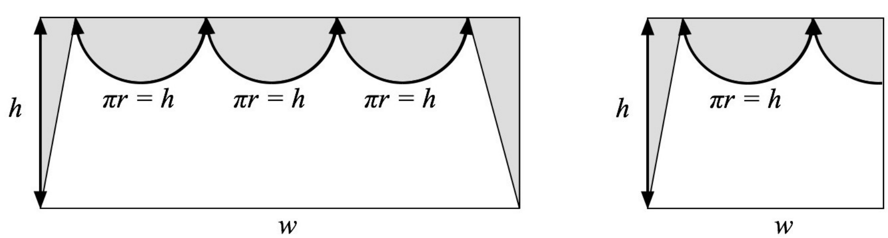

Figure 3. For n or $n + \frac{1}{2}$ semicircular drapes, the optimal radius again has $\pi r = h$, independent of width, unless w is too small to accommodate this.

Generalizations. If we wish to drape $n + \frac{1}{2}$ semicircles and a triangle (or two) in a rectangular opening, then we still have the same critical point, independent of width, assuming the stage is wide enough to accommodate the circles. This is seen by sliding the circles to the right and putting one triangle on the left. Minimizing the covered area corresponds to minimizing the covered area in two copies of the configuration, and two copies of the configuration is again a configuration as in the theorem, involving complete semicircles.

It is also easy to check that the result holds even if the stage opening is a parallelogram with the top edge parallel to the floor, assuming that the parallelogram is shaped reasonably so that it will contain a copy of the optimal semicircle of radius $\frac{h}{\pi}$, where h is the vertical height of the stage opening. For example, in a parallelogram with top corresponding to $[a, b] \subseteq \mathbb{R}$ and bottom corresponding to $[a', b']$, there is no problem if $[a, b] \subseteq [a', b']$. However, if $[a, b] = [0, 1]$ and $[a', b'] = [100, 103]$, then the skew shape of the parallelogram will restrict the values of r for which a semicircle of radius r can be draped over the stage opening.

Instead of a semicircular drape in the center spanning a horizontal distance of $2r$ across the top, we may wish to drape some other specified shape spanning $2r$ across the top, which may be scaled to larger or smaller similar shapes. By the similar scaling, the area of the shape will be kr^2 (with $k = \frac{\pi}{2}$ for the semicircle). It is easy to show that the total area $A(r) = kr^2 + \frac{1}{2}(w - 2r)h$ of the center shape and side triangles decreases to the critical point $\frac{h}{2k}$ then increases, and this critical point is independent of the width, so long as the stage is wide enough to include the critical point $\frac{h}{2k}$ in the domain $(0, \min\{\frac{w}{2}, H\}]$, where H is the value of r which makes the vertical extent of the specified shape equal h (and thus, makes the drape touch the floor). However, the result that the arc-length of the draped regions equals the height is only known to hold for semicircular or quarter-circular regions.

Summary. If a $w \times h$ rectangular stage opening is to be draped in bunting forming triangles on the left and right sides and n semicircles in the remaining gap across the top, it is shown that the viewing area is maximized when the length πr of each semicircle equals the height h of the stage opening, if the width w is large enough to accommodate this.