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Completely regularly ordered spaces versus T_2-ordered spaces which are completely regular ☆

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Abstract

Schwarz and Weck-Schwarz have shown that a T_2-ordered space $(X, \tau, \leqslant)$ whose underlying topological space (X, τ) is completely regular need not be a completely regularly ordered space (that is, $T_{3.5} + T_2$-ordered $\nRightarrow T_{3.5}$-ordered). We show that a completely regular T_2-ordered space need not be completely regularly ordered even under more stringent assumptions such as convexity of the topology. One example involves the construction of a nontrivial topological ordered space on which every continuous increasing function into the real unit interval is constant.

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1. Introduction

If an ordered topological space $(X, \tau, \leqslant)$ is T_2-ordered and τ is completely regular, then must $(X, \tau, \leqslant)$ be completely regularly ordered? Schwarz and Weck-Schwarz [14] attribute this question to Brümmer and provide a negative answer. Their example, however,

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fails to have a base of order-convex sets. Such a convexity condition is a modest compatibility requirement between the topology and order of an ordered topological space and is widely assumed. This prompted the revised question of whether a T_2-ordered space $(X, \tau, \leqslant)$ with a so-called convex, completely regular topology τ is necessarily completely regularly ordered. In Section 2 we present examples showing that the answer is still negative, but each example suggests a strengthening of the hypotheses which might then yield an affirmative answer. In Section 3 we construct a "handle space" having two points which cannot be separated by any continuous increasing real-valued function, and use this space to construct a nontrivial ordered space with no nonconstant continuous increasing functions (into the real unit interval). Variations of these spaces are used to show there seem to be no obvious conditions weaker than "completely regularly ordered" which, together with "completely regular and T_2-ordered", are sufficient to imply "completely regularly ordered". We show that even regularity conditions are not particularly helpful in this context by imitating some classical constructions (see [7,6]; compare also [10]).

A topological space endowed with a partial order $\leqslant$ is called an *ordered topological space* or simply an *ordered space*. In the following we collect some basic definitions and facts from the theory of ordered topological spaces. For further information we refer the reader to [4,8,11,13].

A mapping $f : X \to Y$ between two ordered spaces is *increasing* (respectively, *decreasing*) if $x \leqslant y$ in X implies $f(x) \leqslant f(y)$ (respectively, $f(y) \leqslant f(x)$) in Y. If $(X, \tau, \leqslant)$ is an ordered space and $A \subseteq X$, the *increasing hull* of A in X is $i_X(A) = \{y \in X\colon \exists a \in A \text{ with } a \leqslant y\}$. If the context is clear, we may write $i(A)$ for $i_X(A)$, and we will write $i(x)$ for $i(\{x\})$. We say A is an *upper set* if $A = i(A)$. The *decreasing hull* $d_X(A)$ of a set $A \subseteq X$ and *lower sets* are defined dually. A *monotone set* is a set that is either an upper set or a lower set. The open upper (lower) sets of $(X, \tau, \leqslant)$ form a topology on X denoted $\tau^\sharp$ ($\tau^\flat$). Observe that $\mathrm{cl}_{\tau^\sharp}(A)$ is the smallest closed lower set containing A. A set $A \subseteq X$ is *convex* if $A = i(A) \cap d(A)$, or equivalently, if $x, z \in A$ and $x \leqslant y \leqslant z$ imply $y \in A$.

We frequently wish to assume some compatibility conditions between the topology and the order of an ordered topological space. An ordered topological space $(X, \tau, \leqslant)$ has a *convex topology* if $\tau^\sharp \cup \tau^\flat$ is a subbase for τ. $(X, \tau, \leqslant)$ is *T_2-ordered* if $a \nleqslant b$ in X implies there exist disjoint neighborhoods U of a and V of b, with U being an upper set and V being a lower set (equivalently, the order relation $\leqslant$ is closed in the topological product $X \times X$). The neighborhoods U and V need not be open. If $a \nleqslant b$ in X implies there exist disjoint *open* neighborhoods U of a and V of b, with U being an upper set and V being a lower set, then we say $(X, \tau, \leqslant)$ is *monotonically separated* (see [9]; in [12] these spaces were called strongly T_2-ordered). An even stronger form of separation of points is the first condition of the definition of completely regularly ordered spaces. $(X, \tau, \leqslant)$ is *completely regularly ordered* if (a) for any $a \nleqslant b$ in X, there exists a continuous increasing real-valued function h on X with $h(a) > h(b)$, and (b) for any $a \in X$ and any closed set $F \subseteq X \setminus \{a\}$, there exist an increasing function $f : X \to [0, 1]$ and a decreasing function $g : X \to [0, 1]$ with $f(a) = 1 = g(a)$ and $\min\{f(x), g(x)\} = 0$ for $x \in F$. In particular, note that a completely regularly ordered space is monotonically separated and has a convex topology.

A T_2-ordered space $(X, \tau, \leqslant)$ is *strongly upper regularly ordered* (see [12]) if for any closed upper set F and any $x \in X \setminus F$, there exists an open upper set U and an open lower set V with $U \cap V = \emptyset$, $F \subseteq U$, and $x \in V$. The dual condition defines *strongly lower regularly ordered*, and $(X, \tau, \leqslant)$ is *strongly regularly ordered* if it is both strongly upper and strongly lower regularly ordered. Note that the latter condition exactly means that the associated bitopological space $(X, \tau^\flat, \tau^\sharp)$ is pairwise regular.

An *ordered compactification* of an ordered topological space $(X, \tau, \leqslant)$ is a compact T_2-ordered space $(X', \tau', \leqslant')$ which contains (a homeomorphic, order isomorphic copy of) $(X, \tau, \leqslant)$ as a dense subspace. It is well known (see, e.g., [4, Corollary 4.10]) that a T_2-ordered space admits an ordered compactification if and only if it is completely regularly ordered. A T_2-ordered space $(X, \tau, \leqslant)$ with convex topology such that the bitopological space $(X, \tau^\flat, \tau^\sharp)$ is pairwise completely regular is called (compare [11,9]) *strictly completely regularly ordered*. Various conditions have been found that imply this strictly stronger version of complete regular orderedness (see [1,9]).

2. Examples

The following two examples show that in general, a completely regular T_2-ordered space with convex topology need not be completely regularly ordered. Thus, these examples illustrate that a T_2-ordered space with convex topology which admits Hausdorff compactifications need not admit ordered compactifications.

Example 1. Let $X = (\omega_1 + 1) \times \{0\} \oplus [(\omega_1 + 1) \times (\omega_0 + 1) \setminus \{(\omega_1, \omega_0)\}] \times \{1\} \oplus (\omega_0 + 1) \times \{2\}$; that is, X is the topological sum of copies of $\omega_1 + 1$, $\omega_0 + 1$ and the (deleted) Tychonoff plank, all equipped with their usual topology. As a partial order on X choose $\leqslant = \Delta_X \cup \{((\alpha, 0), (\alpha, \omega_0, 1))\colon \alpha \in \omega_1\} \cup \{((\omega_1, n, 1), (n, 2))\colon n \in \omega_0\}$. Obviously $\leqslant$ is closed so that $(X, \leqslant)$ is T_2-ordered. Note that each point has a neighborhood base consisting of open upper sets only or of open lower sets only. Hence the topology of X is convex. Since the unique (Hausdorff) compactification of the Tychonoff plank is the one-point-compactification, we obtain the unique compactification βX of X as the one-point-compactification by adding the point $(\omega_1, \omega_0, 1)$; use, e.g., the reflection property to see this.

Suppose that $\preceq$ is a closed order on βX extending the order $\leqslant$ on X. Then clearly $((\omega_1, 0), (\omega_1, \omega_0, 1))$ and $((\omega_1, \omega_0, 1), (\omega_0, 2))$ belong to $\preceq$, and by transitivity, we have the contradiction that $(\omega_1, 0) \leqslant (w_0, 2)$. We conclude that $(X, \leqslant)$ does not have a T_2-ordered compactification. Hence X is not completely regularly ordered, although the topology of X is convex and locally compact.

In fact, the proof just given can be simplified: In the space X constructed above, $(\omega_1, 0)$ and $(\omega_0, 2)$ are incomparable, but the usual proof that the Tychonoff plank is not normal shows that each open upper set containing $(\omega_1, 0)$ intersects each open lower set containing $(\omega_0, 2)$. Thus, X is not monotonically separated and hence not completely regularly ordered.

This example suggests that we should revise our question by strengthening the T_2-ordered hypothesis to monotonically separated. The revised question is thus: if X is a completely regular, monotonically separated ordered space with a convex topology, is

X completely regularly ordered? In the next section (see Example 4), we will see that the answer is negative, but first let us consider another counterexample to the original question, which suggests another revision of the hypotheses.

Example 2. Let $X = [(\omega_1 + 1) \times (\omega_0 + 1) \times \{0\}] \oplus [(\omega_1 + 1) \times (\omega_0 + 1) \times \{2\}]$ where the factor spaces of the two products are equipped with the usual order topologies and $\oplus$, as above, denotes the topological sum. Hence X is the sum of two Tychonoff planks. We obtain the compact Hausdorff quotient space Y from X by identifying $(\alpha, \omega_0, 0)$ with $(\alpha, \omega_0, 2)$ whenever $\alpha \in \omega_1 + 1$. The corresponding nontrivial equivalence classes will be denoted by $(\alpha, \omega_0, 0/2)$ in the following. Delete the point $(w_1, \omega_0, 0/2)$ from Y to obtain the subspace Z of Y. Construct the space F from Z by adding a copy of a convergent sequence, that is, suppose that $F = Z \oplus [\{\omega_1\} \times (\omega_0 + 1) \times \{1\}]$. Define a partial order $\leqslant$ on F as follows: Set $\leqslant = \Delta_F \cup \{((\omega_1, n, i), (\omega_1, n, j))\colon\, n \in \omega_0,\ i \leqslant j;\ i, j \in \{0, 1, 2\}\}$. Obviously F is a completely regular space that is T_2-ordered by $\leqslant$. It is readily seen that F has a convex topology.

Note next that for each completely regularly ordered space X and $x \in X$, the collection of all sets $\operatorname{cl} I \cap \operatorname{cl} D$ where $x \in I \cap D$, I is an open upper set, and D is an open lower set, forms a neighborhood base at x. We want to show that F does not satisfy this condition: Let $x = (\omega_1, \omega_0, 1)$ and consider the open neighborhood $G = F \setminus (\omega_1 \times \{\omega_0\} \times \{0/2\})$ of x in F. If $x \in I \cap D$, where I is an open upper set and D is an open lower set, then there exist $\alpha \in \omega_1$ and $n_0 \in \omega_0$ such that $]\alpha, \omega_1[\ \times\ ]n_0, \omega_0[\ \times \{0\} \subseteq D$ and $]\alpha, \omega_1[\ \times\ ]n_0, \omega_0[\ \times \{2\} \subseteq I$. It clearly follows that $(\alpha + 1, \omega_0, 0/2) \in \operatorname{cl} I \cap \operatorname{cl} D$. We conclude that F is not completely regularly ordered. (It can also be readily seen that the given order cannot be extended to the unique T_2-compactification of the space Z.)

This example suggests that, perhaps, the hypothesis of convexity in the original question should be replaced by the following stronger condition of regularity, which is clearly satisfied in every completely regularly ordered space: For each $x \in X$, the sets of the form $\operatorname{cl}_{\tau^\flat} I \cap \operatorname{cl}_{\tau^\sharp} D$ where $x \in I \cap D$, I is an open upper set and D is an open lower set form a neighborhood base at x. Note that, for instance, each strongly regularly ordered space with a convex topology satisfies the latter condition. Again the construction of the following section will show (see Examples 3 and 5) that the answer remains negative even with these strengthened hypotheses.

3. The handle space

For the following construction we need a completely regular Hausdorff space T that contains two disjoint closed copies of ω_1 that cannot be separated by disjoint open sets and in which the complement K of the union of the two copies of ω_1 is dense in T. In fact in order to simplify the argument we can (and do) also assume that all points in the complement K are isolated.

As a concrete example the reader might wish to consider the space $\Omega = (\omega_1 + 1) \times \omega_1$ with its usual topology. Note first that all the points (α, α) where $\alpha \in \omega_1$ and α is equal to 0 or a successor ordinal are isolated points of Ω. Observe also that the map h from ω_1 to the set of its nonzero limit ordinals defined by $h(\alpha) = \omega + \omega \cdot \alpha$ (where we use addition and multiplication of ordinals) is a continuous bijection that is also closed; hence h is a homeomorphism. Therefore the subspace $\{(\alpha, \alpha)\colon\ \alpha \in \omega_1 \setminus \{0\}$ is a limit ordinal$\}$ of the diagonal and the right edge $\{(\omega_1, \alpha)\colon\ \alpha \in \omega_1\}$ provide us with two appropriate disjoint closed topological copies of ω_1 that cannot be separated by disjoint open sets. Our starting space T is obtained by declaring points not belonging to these two subsets of Ω isolated and not changing the neighborhoods of the remaining points. In particular the complement of the union of the two chosen copies of ω_1 is dense in T.

As another such space T we could choose van Douwen's gap space (see [15]). This space is even separable, and the reader can readily check that working with this space would yield separable examples in Examples 3, 4, and 5 below.

For $n \in \mathbb{Z}$, let E_n be the topological sum of two copies $(A_n \cup B_n \cup X_n)$ and $(C_n \cup D_n \cup Y_n)$ of T, where A_n, B_n, C_n and D_n are copies of ω_1; X_n and Y_n are distinct copies of K, each as a dense set of isolated points in its respective copy of T; and the unions of spaces T involved are disjoint unions.

Put $E = \bigoplus_{n\in\mathbb{Z}} E_n$ and $H = E \cup \{\pm\infty\}$, where the basic neighborhoods of $-\infty$ are of form $\{-\infty\} \cup \bigcup_{n\leqslant k} E_n$ and the basic neighborhoods of ∞ are of form $\{\infty\} \cup \bigcup_{n\geqslant k} E_n$.

The order on H is defined as follows. Recall that A_n, B_{n+1}, C_{n+1} and D_n are copies of ω_1. For every $n \in \mathbb{Z}$, we put the points of A_n pointwise smaller than the points of B_{n+1}; the points of B_{n+1} pointwise smaller than the points of C_{n+1}; and the points of C_{n+1} pointwise smaller than the points of D_n. Thus, any point from a copy of ω_1 in H is contained in a four-element chain. Each point of $\{\pm\infty\} \cup \bigcup_{n\in\mathbb{Z}}(X_n \cup Y_n)$ is incomparable to every other point of H. See Fig. 1.

We will call H the *handle space*, and will now verify some basic properties of H.

H is completely regular: If $x \in E_j$ and F is a closed set not containing x, let $g : E_j \to [0, 1]$ be a continuous function with $g(x) = 0$ and $g(F \cap E_j) = 1$. Extending g to H by taking $g(x) = 1$ for any $x \in H \setminus E_j$ gives a continuous function on H separating x and F. If $x = \infty$ and F is a closed set not containing x, then $H \setminus F$ contains a neighborhood $N = \{\infty\} \cup \bigcup_{n\geqslant k} E_n$ of ∞. The characteristic function on N is a continuous function separating x and F. The case for $x = -\infty$ is analogous.

H is T_2-ordered: Suppose $(x_\lambda, y_\lambda)_{\lambda\in\Lambda}$ is a net in H^2 converging to (x, y) and $x_\lambda \leqslant y_\lambda$ for all $\lambda \in \Lambda$. If $x = y$, then $x \leqslant y$, so assume $x \neq y$. Now $x_\lambda \leqslant y_\lambda$ implies $x_\lambda = y_\lambda$

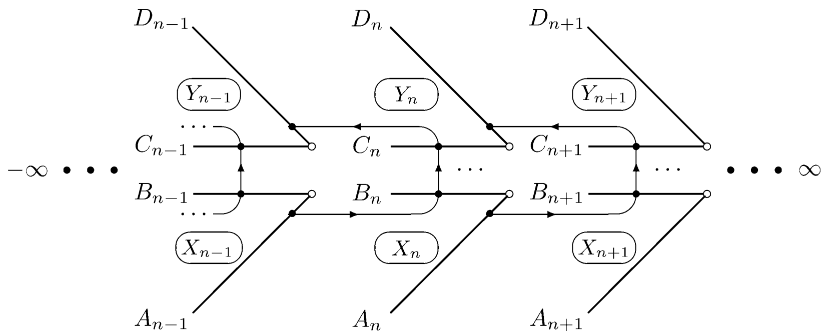

Fig. 1. The handle space H.

or x_λ and y_λ are part of a four-element chain contained in $E_n \cup E_{n+1}$ for some $n \in \mathbb{Z}$. Thus, if $x_\lambda \to x \in \{\pm\infty\}$, then $y_\lambda \to y = x$ as well, contradicting $x \neq y$. Now suppose $(x_\lambda, y_\lambda) \to (x, y) \in E^2$. If $x \in E_j$, then x_λ is eventually in E_j and without loss of generality, we will assume $x_\lambda \in E_j$ for all $\lambda \in \Lambda$. Because the order was defined by putting four copies of ω_1 pointwise smaller than each other, if $x_\lambda \to x$ in E_j and $x_\lambda \leqslant y_\lambda$ for all λ, then y_λ must be the corresponding net x_λ lifted to a higher copy of ω_1, and thus the limit y of (y_λ) must be the lifted copy of x, which is above x, as needed.

The topology of H is convex: Standard neighborhood subbases of any $x \in E$ can be chosen to consist of open monotone sets; $-\infty$ has a neighborhood base of open lower (and also upper) sets of form $\{-\infty\} \cup \bigcup_{n<k} E_n \cup B_k \cup C_k \cup X_k \cup Y_k$ and ∞ has a neighborhood base of open upper (and also lower) sets of form $\{\infty\} \cup \bigcup_{n>k} E_n \cup A_k \cup D_k \cup X_k \cup Y_k$.

Observe that if a closed lower set M in an ordered topological space $(X, \tau, \leqslant)$ can be separated from a closed upper set N in X by a continuous increasing function $f : X \to [0, 1]$ with $f(M) = 0$ and $f(N) = 1$, then there exists a countable dense set $Q \subseteq]0, 1[$ and a collection of open lower sets $\{V_p\colon\ p \in Q\}$ with the properties that $p < q \Longrightarrow \mathrm{cl}_{\tau^\sharp}(V_p) \subseteq V_q$ and $M \subseteq V_p \subseteq X \setminus N$ for all $p, q \in Q$. Indeed, taking $V_p = f^{-1}[0, p[$ for $p \in \mathbb{Q} \cap]0, 1[$ provides the desired collection. Furthermore, the converse holds as well by the Urysohn construction: If Q is a countable dense subset of $]0, 1[$ and $\{V_p\colon\ p \in Q\}$ is a collection of open lower sets with the properties that $p < q \Longrightarrow \mathrm{cl}_{\tau^\sharp}(V_p) \subseteq V_q$ and $M \subseteq V_p \subseteq X \setminus N$ for all $p, q \in Q$, and then $f(x) = \inf\{p \in Q\colon\ x \in V_p\}$ (where as usual $\inf \emptyset = 1$) is a continuous increasing function separating M and N as desired.

Proposition 1. *Let H be the handle space defined above. For any continuous increasing function $f : H \to [0, 1]$, we have $f(\infty) = f(-\infty)$. In particular, H is not completely regularly ordered.*

Proof. Suppose to the contrary. Then either there exists a continuous increasing function $f : H \to [0, 1]$ with $f(-\infty) = 0$ and $f(\infty) = 1$ or there exists such a function with $f(-\infty) = 1$ and $f(\infty) = 0$. We will only consider the former case; the latter case is analogous. Now $M = f^{-1}[0, 0.5[$ and $N = f^{-1}[0.5, 1]$ are lower and upper sets, respectively. Let k be the largest integer such that $\{-\infty\} \cup \bigcup_{n\leqslant k} E_n \subseteq M$. Similarly, let t be the smallest positive integer such that $E_{k+t} \subseteq N$. Since M is a lower set and $D_k \subseteq E_k \subseteq M$, we have that C_{k+1} is a subset of M.

By the well-known result that each bounded real-valued function defined on the topological space ω_1 is constant on a final segment of ω_1 [3, Example 3.1.27], f must be constant on a final segment of D_{k+1} and on a final segment of C_{k+1}. If these constant values were not equal, for any m strictly between the values, $f^{-1}[0, m[$ and $f^{-1}]m, 1]$ would give open sets separating final segments of D_{k+1} and C_{k+1}, contrary to the non-normality of the space T. Thus, f has the same value on final segments of D_{k+1} and $C_{k+1} \subseteq M = f^{-1}[0, 0.5[$, so a final segment of D_{k+1} is contained in M.

Iterating this argument, we see that M contains a final segment of D_{k+t}. We have reached another contradiction, since M is disjoint from N and N was supposed to contain all of E_{k+t}.

We also note that if the left half of the handle space is deleted, leaving $H^+ = \bigoplus_{n\in\mathbb{N}} E_n \cup \{\infty\}$, we still cannot separate ∞ from the closed set C_1 by a continuous increasing real-valued function.

Example 3. The handle space H, by Proposition 1, is not completely regularly ordered, but H is a completely regular, T_2-ordered space in which each point x has a neighborhood base of sets of the form $\mathrm{cl}_{\tau^\flat} I \cap \mathrm{cl}_{\tau^\sharp} D$ where $x \in I \cap D$, I is an open upper set and D is an open lower set. Since we already know that the topology of H is convex, the last statement is a consequence of the following proof which shows that H is strongly regularly ordered:

Indeed, the $\tau^\flat$-neighborhoods at any point $x \in H$ have a neighborhood base consisting of closed lower sets, and dually: If $x \in X_n \cup Y_n$, then $\{x\}$ is such a closed lower neighborhood of x. If x is an element of a copy A_n, B_n, C_n, or D_n of ω_1 and $d(x)$ has $n \in \{1, 2, 3, 4\}$ elements, there is a clopen lower neighborhood of x consisting of copies of a closed interval $[\alpha + 1, x]$ in the n copies of ω_1 below x, together with points from some X_j's and some Y_j's. If $x = \infty$, each open lower neighborhood of ∞ of form $\{\infty\} \cup A_n \cup X_n \cup \bigcup_{k>n} E_k$ contains a closed lower neighborhood of ∞ of form $\{\infty\} \cup A_n \cup \bigcup_{k>n} E_k$. If $x = -\infty$, each open lower neighborhood of $-\infty$ of form $\{-\infty\} \cup \bigcup_{k<n} E_k \cup A_n \cup X_n \cup B_n \cup C_n \cup Y_n$ contains a closed lower neighborhood of $-\infty$ of form $\{-\infty\} \cup \bigcup_{k<n} E_k \cup B_n \cup C_n$. These neighborhoods show that the $\tau^\flat$-neighborhoods of $x \in H$ have a base of closed lower sets. Similarly one shows that each point in H has a $\tau^\sharp$-neighborhood base consisting of closed upper sets.

Example 4. It follows from above that the subspace $X = H \setminus \{-\infty\}$ of the handle space H is completely regular, T_2-ordered, and has a convex topology. We now show that for any two points x, y in X such that $x \nleq y$ there exists a continuous increasing function $f : X \to [0, 1]$ such that $f(x) > f(y)$. In particular, X is monotonically separated. Indeed if $x \neq \infty$ we can clearly find a clopen upper set S containing x, but not y. Similarly, if $y \neq \infty$ we can choose a clopen lower set $X \setminus S$ containing y, but not x. Hence in either case it is possible to define the required function as the characteristic function of S.

It may be worthwhile to mention here that H is also monotonically separated, because any strongly regularly T_2-ordered space X is monotonically separated: In fact, if $x, y \in X$ and $x \nleq y$, then $y \notin i(x)$. Since $i(x)$ is closed and X is strongly upper regularly ordered, there are open sets witnessing monotonic separation of X.

A completely regularly ordered space has "enough" continuous monotone (that is, increasing or decreasing) real-valued functions to separate points and to separate points from closed sets, as described in the definition of "completely regularly ordered". The motivating question for this paper asks whether a T_2-ordered space with a convex topology and enough *continuous* functions to separate points from closed sets already has "enough" continuous *monotone* real-valued functions. Our final example shows emphatically that these conditions are inadequate; such an ordered space may have no continuous increasing functions into the unit interval $[0, 1]$ other than the constant functions, even if the associated bitopological space is pairwise regular.

Example 5. We shall use the notation of the discussion of the handle space H.

Let $S_0 = H$ and $D = \bigcup_{n\in\mathbb{Z}}(X_n \cup Y_n)$. Observe that D is a dense subset of isolated points in H, $\pm\infty \notin D$, and each pair of elements of D is incomparable. Let

$$S_1 = S_0 \oplus \bigoplus_{(a,b)\in D^2,\, a\neq b} H(a,b)$$

where for any $(a,b) \in D^2$ with $a \neq b$, $H(a,b) = H(a,b,1) \oplus H(a,b,2)$ where $H(a,b,i) = H$ for $i \in \{1,2\}$. We will call each $H(a,b)$ a double handle. As a labeling convention, $x \in H(a,b,i)$ will be denoted $x_{(a,b,i)}$. Note that S_1 is a completely regular Hausdorff space as the topological sum of completely regular Hausdorff spaces.

Order S_1 by keeping the order on S_0 and on each $H(a,b,i)$, and for any $a,b \in D$, $a \neq b$, putting

$$-\infty_{(a,b,1)} \geqslant a \geqslant -\infty_{(a,b,2)}$$

and

$$\infty_{(a,b,1)} \leqslant b \leqslant \infty_{(a,b,2)}$$

and any additional order between points $\pm\infty_{(a,b,i)}$ required by transitivity. Observe that on S_0 and on the added handles the new partial order agrees with the original one. See Fig. 2.

S_1 is T_2-ordered: We show that the graph of the order on S_1 is closed in the product $S_1 \times S_1$. If $x_\lambda \leqslant y_\lambda$ for all $\lambda \in \Lambda$ and $x_\lambda \to x$, $y_\lambda \to y$, then since S_1 consists of a large topological sum, eventually x_λ is in one summand and eventually y_λ is in one summand. If these nets are eventually in the same summand, then $x \leqslant y$ since the order on each summand is closed. If x_λ and y_λ are eventually in distinct summands $H(a,b,i)$ and $H(c,d,j)$ or in S_0, then it follows that the net (x_λ, y_λ) is constantly one of the pairs $(-\infty_{(a,b,2)}, a)$, $(a, -\infty_{(a,b,1)})$, $(\infty_{(a,b,1)}, b)$, or $(b, \infty_{(a,b,2)})$ used to define the order on S_1, or (by transitivity), a pair of form $(-\infty_{(a,b,2)}, \infty_{(c,a,2)})$, $(-\infty_{(a,b,2)}, -\infty_{(a,c,1)})$, $(\infty_{(b,a,1)}, \infty_{(c,a,2)})$, or $(\infty_{(b,a,1)}, -\infty_{(a,c,1)})$ for appropriate points $a,b,c \in D$. As an eventually constant net in the graph of the order, the limit is also in the graph of the order.

Turning to the convexity of the topology on S_1, let N be an open neighborhood of $-\infty_{(a,b,2)}$ contained in $H(a,b,2) \setminus \{\infty_{(a,b,2)}\}$. Without loss of generality, N is an open lower neighborhood of $-\infty_{(a,b,2)}$ in the handle $H(a,b,2)$ (see the discussion above showing that H has a convex topology), and for such a neighborhood N, we have $d(N) = N$ in S_1, and it follows that $\infty_{(a,b,2)}$ has a subbase of open monotone sets in S_1. Similar arguments show that each point of $H(a,b,1) \cup H(a,b,2)$ has a neighborhood subbase of open monotone sets.

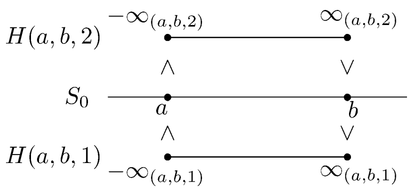

Fig. 2. Adjoining double handles.

For each subset U of S_0 we have

$$i(U) = i_{S_0}(U) \cup \{\infty_{(a,b,2)} \in S_1\colon b \in U\} \cup \{-\infty_{(a,b,1)} \in S_1\colon a \in U\},$$
$$d(U) = d_{S_0}(U) \cup \{-\infty_{(a,b,2)} \in S_1\colon a \in U\} \cup \{\infty_{(a,b,1)} \in S_1\colon b \in U\}.$$

Since H is monotonically separated and $-\infty$ and ∞ are not comparable in H, there are open lower neighborhoods L_∞ and $L_{-\infty}$ of ∞ and $-\infty$, respectively, and open upper neighborhoods I_∞ and $I_{-\infty}$ of ∞ and $-\infty$, respectively, such that $L_\infty \cap I_{-\infty} = \emptyset$ and $L_{-\infty} \cap I_\infty = \emptyset$. Consider now $L \cap I$ where L and I are an open lower set and an open upper set of S_0, respectively. According to the formulas above, $d(L)$ and $i(I)$ will not be open in general, but

$$L' = L \cup \bigcup\{L_{-\infty_{(a,b,2)}}\colon -\infty_{(a,b,2)} \in S_1,\ a \in L\}$$
$$\cup \bigcup\{L_{\infty_{(a,b,1)}}\colon \infty_{(a,b,1)} \in S_1,\ b \in L\}$$

and

$$I' = I \cup \bigcup\{I_{\infty_{(a,b,2)}}\colon \infty_{(a,b,2)} \in S_1,\ b \in I\}$$
$$\cup \bigcup\{I_{-\infty_{(a,b,1)}}\colon -\infty_{(a,b,1)} \in S_1,\ a \in I\}$$

will be an open lower set and an open upper set in S_1, respectively. (Here we are using a self-explanatory notation to denote copies of neighborhoods in the added handles.)

Clearly, $L' \cap I' = L \cap I$. Hence, we finally conclude that each point of S_0 has a subbase of open monotone sets in S_1, because the topology of the subspace S_0 had the corresponding property. We have shown that S_1 has a convex topology.

Now observe that any continuous increasing function f on S_1 into the unit interval $[0, 1]$ must be constant on the dense subset D of S_0: If not, then there exist $a, b \in D$, with $f(a) \neq f(b)$. However, we have (referring to Fig. 2)

$$f(a) \geqslant f(-\infty_{(a,b,2)}) = f(\infty_{(a,b,2)}) \geqslant f(b)$$
$$\geqslant f(\infty_{(a,b,1)}) = f(-\infty_{(a,b,1)}) \geqslant f(a)$$

so that $f(a) = f(b)$, a contradiction.

As f is constant on D and D is dense in S_0, we find that any continuous increasing real-valued function on S_1 is constant on S_0.

Without further modifications, such functions may be nonconstant on $H(a, b, i) \setminus \{\pm\infty_{(a,b,i)}\}$. To rectify this, we iterate the process, forming a space S_{n+1} from S_n by adding double handles between each pair of points in the copy of the dense subset D of each handle added in the construction of S_n. Then each continuous real-valued function defined on S_{n+1} will be constant on S_n: The proof is by induction on n. Indeed any continuous increasing real-valued function on S_{n+1} must be constant, say equal to r, on S_{n-1} by our induction hypothesis. Furthermore on each handle H belonging to $S_n \setminus S_{n-1}$ such a function must be equal to some constant r_H because of the new handles added in $S_{n+1} \setminus S_n$ and the argument given above. But the values of such a function on the points $+\infty$ and $-\infty$ of H must be equal to the values on the two points of S_{n-1} from which that handle H originated so that

$r_H = r$. We conclude that any increasing real-valued function on S_{n+1} must be constant on S_n.

After countably many iterations, we get a space S_∞ with the property that every continuous increasing real-valued function into the unit interval $[0,1]$ is constant. Specifically, let $S_\infty = \bigcup_{n=0}^{\infty} S_n$, and take the union of the topologies on each S_n as a base for the topology on S_∞. Similarly we define the order on S_∞ as the union of the orders on the subsets S_n.

The space S_∞ is a topological sum $\bigoplus_{\lambda\in\Lambda} H_\lambda$ of copies H_λ $(\lambda \in \Lambda)$ of the handle space H. Let ∞_λ and $-\infty_\lambda$ denote the endpoints of the handle H_λ, let R_λ denote the copy of $R = \bigcup_{n\in\mathbb{Z}}(A_n \cup B_n \cup C_n \cup D_n)$ in H_λ, and let D_λ denote the copy of $D = \bigcup_{n\in\mathbb{Z}}(X_n \cup Y_n)$ in H_λ. The order on S_∞ consists of 4-element chains involving points of R_λ within any given handle H_λ in S_∞, together with 3-element chains of form

$$-\infty_{(a,b,1)} \geqslant a \geqslant -\infty_{(a,b,2)} \quad \text{or} \quad \infty_{(a,b,1)} \leqslant b \leqslant \infty_{(a,b,2)}$$

linking points $a, b \in D_\lambda \subseteq S_m$ to the endpoints of handles in the next higher iteration. Note that each element of R_λ is contained in a single 4-element chain, and each element $\pm\infty_{(a,b,i)} \in H_\lambda \setminus S_0$ is contained in a single 3-element chain, but elements $a, b \in D_\lambda$ are the middle points of infinitely many 3-element chains.

From the arguments above, each S_n and S_∞ is T_2-ordered.

As a topological sum of completely regular spaces, S_∞ is completely regular. However, S_∞ is clearly not completely regularly ordered, for no pair of distinct points can be separated by a continuous increasing real-valued function.

By induction it follows from the convexity argument given above, that each point of any S_n has a subbase of monotone open sets in S_n. Furthermore any open lower set L_n and any open upper set I_n of S_n can be (canonically, see above) extended to an open lower set L_{n+1} and an open upper set I_{n+1} of S_{n+1}, respectively, such that $L_{n+1} \cap I_{n+1} = L_n \cap I_n$. Then $L_\infty = \bigcup_{k\geqslant n} L_k$ and $I_\infty = \bigcup_{k\geqslant n} I_k$ are an open lower set and an open upper set in S_∞, respectively, such that $L_n \cap I_n = L_\infty \cap I_\infty$. Indeed if $x \in L_s \cap I_t$ then $x \in L_m \cap I_m$ where $m = \max\{s,t\}$. But $L_m \cap I_m = L_n \cap I_n$ by induction. We conclude that each point of S_∞ has a subbase of open monotone sets in S_∞. Therefore we have shown that S_∞ has a convex topology.

Now we will show that S_∞ is strongly regularly ordered. Suppose U is an open upper neighborhood of $x \in H_\lambda \subseteq S_n \setminus S_{n-1}$, where we use the convention that $S_{-1} = \emptyset$. Let M_n be an open (in H_λ) upper (in H_λ) neighborhood of x, and let N_m be a closed (in H_λ) upper (in H_λ) neighborhood of x, with $M_n \subseteq N_m \subseteq H_\lambda \cap U$. The existence of such neighborhoods follows from the remarks of Example 3. Now $i(M_n) = M_n \cup PM_n$ where

$$\begin{aligned} PM_n = {} & \{-\infty_{(a,b,1)}\colon a \in M_n \cap D_{\lambda_0},\ b \in D_{\lambda_0} \text{ for some } \lambda_0\} \\ & \cup \{\infty_{(a,b,2)}\colon b \in M_n \cap D_{\lambda_0},\ a \in D_{\lambda_0} \text{ for some } \lambda_0\}, \end{aligned}$$

and $i(N_n) = N_n \cup PN_n$ where

$$\begin{aligned} PN_n = {} & \{-\infty_{(a,b,1)}\colon a \in N_n \cap D_{\lambda_0},\ b \in D_{\lambda_0} \text{ for some } \lambda_0\} \\ & \cup \{\infty_{(a,b,2)}\colon b \in N_n \cap D_{\lambda_0},\ a \in D_{\lambda_0} \text{ for some } \lambda_0\}. \end{aligned}$$

Now $i(M_n)$ may not be open. We will carefully add segments around each point of PM_n to make it open, but the resulting set may not be an upper set, so we iterate this process on M_n and N_n. It is important to note that each $z \in PM_n$ (or PN_n) is on a separate handle in $S_{n+1} \setminus S_n$. For $z \in PM_n$, let H_{λ_z} be the handle containing z and pick an open (in H_{λ_z}) upper (in H_{λ_z}) neighborhood M_z of z, contained in a closed (in H_{λ_z}) upper (in H_{λ_z}) neighborhood N_z of z, with $M_z \subseteq N_z \subseteq H_{\lambda_z} \cap U$. Now put

$$M_{n+1} = \bigcup_{z \in PM_n} M_z \quad \text{and} \quad N_{n+1} = \bigcup_{z \in PN_n} N_z.$$

Note that $M_{n+1} \subseteq N_{n+1} \subseteq S_{n+1} \setminus S_n \subseteq U$. M_{n+1} is a union of open sets ($\emptyset$ or M_z) from each summand of the topological sum S_∞, and thus is open in S_∞. Similarly, N_{n+1} is closed. Furthermore, since the summands used in N_n are distinct from those used in N_{n+1}, $N_n \cup N_{n+1}$ is closed.

We may now iterate the process: Take $i(M_n \cup M_{n+1})$. The "new" points in $i(M_n \cup M_{n+1}) \setminus (M_n \cup M_{n+1})$ are points of form $\pm\infty$ from handles in $S_{n+2} \setminus S_{n+1}$. Choose an open upper (in its handle H_z) neighborhood M_z of each such new point z, with M_z contained in a closed upper (in its handle) neighborhood N_z of z, with $N_z \subseteq H_z \cap U$. Let M_{n+2} be the union of all such M_z's and N_{n+2} be the union of all such N_z's added at this iteration. Again, $M_n \cup M_{n+1} \cup M_{n+2}$ consists of an open segment from each of the summands of S_∞ and is thus open. Similarly, $N_n \cup N_{n+1} \cup N_{n+2}$ is closed. Continuing and noting that if $V \subseteq H_\lambda$ is an upper set in H_λ where H_λ was a handle added in the nth iteration $S_n \setminus S_{n-1}$, then $i(V)$ only contains points of S_{n+1}, and that the added points are from distinct handles, we see that the process avoids "clustering" of the infinite union of closed sets in N_k ($k \geqslant n$) to a set which is not closed. Thus $M = \bigcup_{k \geqslant n} M_k$ and $N = \bigcup_{k \geqslant n} N_k$ provide an open upper neighborhood M of x contained in a closed upper neighborhood N of x contained in U. Now M and $S_\infty \setminus N$ provide monotone open sets separating x and $S_\infty \setminus U$, showing that S_∞ is strongly lower regularly ordered. The dual argument shows that S_∞ is strongly upper regularly ordered.

Remark 1. After this article had been completed, the authors discovered that already several years before Schwarz and Weck-Schwarz, Hommel [5, Remarks 1.3.2(2) and 2.2.3(1)] discussed two examples of T_2-ordered spaces with completely regular topology that are not completely regularly ordered. Another such space due to Saint-Raymond was later studied in an article of Edwards [2, p. 74]. However none of these examples has a convex topology.

Hommel's article also contains a remarkably simple idea to construct a completely regularly ordered space that is not strictly completely regularly ordered (compare [9,10]). Since some inaccuracies in [5] obscure the argument, we close with the description of such an example.

Example 6 (*see Hommel* [5, 2.2.3(3)]). Let $X = [0,1] \times [0,1]$ be equipped with its usual compact topology and ordered by $(x_1, y_1) \leqslant (x_2, y_2)$ provided that $x_1 \leqslant x_2$ and $y_1 = y_2$. Then the partial order is closed, and since compact T_2-ordered spaces are completely regularly ordered [13], we have defined a completely regularly ordered space. Consider the subspace $E = ([0,1[\times (\mathbb{Q} \cap [0,1])) \cup F$ where $F = \{1\} \times ([0,1] \setminus \mathbb{Q})$ of X. Of course, E is a completely regularly ordered space. Furthermore F is a closed lower set in E. Evidently, there does not exist an open upper set containing $(0,0)$ and an open lower set containing

F that are disjoint in E. Hence E is not strongly regularly ordered. It is also interesting to note that E is an I-space (that is, $d(G)$ and $i(G)$ are open whenever G is open in E).

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