LOWER SEPARATION AXIOMS IN BITOPOGENOUS SPACES

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Abstract. Several naturally defined lower separation axioms for bitopological spaces obtained by modifying the axioms T_0, T_1, and R_0 appear in the literature. We introduce and study analogous separation axioms for bitopogenous spaces. In particular, we investigate relationships between the axioms and discuss the conditions under which the relationships are similar to those between the corresponding separation axioms for bitopological spaces.

1. Introduction

Topogenous orders were introduced by Császár [2] to give a unified study of topologies, uniformities, and proximities. A topogenous order on a set X is nothing but a binary relation on the power set of X subject to certain axioms. With every topology τ on a set X, we may associate a topogenous order $\sqsubset$ on X by putting $A \sqsubset B$ if and only if $\mu A \subseteq B$, where μ is the Kuratowski closure operator given by τ. In the same way, we may associate binary relations (more general than topogenous orders) with closure operators more general than the Kuratowski ones. This was done, for categorical closure operators, in [6] to study categorical quasi-uniformities. In [13], binary relations (preorders) ρ are studied that are associated with closure operators μ by putting $A\rho B$ if and only if $A \subseteq \mu B$, hence "dually" to associating topogenous orders. An analogous approach is used in [14] to associate preorders with closure operators on posets.

Lower separation axioms play a significant role in applications of topology to computer science. Machine calculations and graphic displays are based on finite sets and thus cannot be adequately modeled by a Hausdorff topology. A bitopogenous space consists of a set and a pair of topogenous orders on this set. The study of bitopogenous spaces includes the study of bitopological spaces. Ordered topological spaces have form $(X, \tau, \lesssim)$ where τ is a topology on X and $\lesssim$ is a quasiorder on X. There is a one-to-one correspondence between quasisorders on X and Alexandroff topologies (that is, topologies closed under formation of arbitrary intersections) on X given by $x \lesssim y$ if and only if $x \in cl\{y\}$ (see [16]). Thus, an ordered topological space $(X, \tau, \lesssim)$ may be viewed as a bitopological space $(X, \tau, \tau_{\lesssim})$ and thus as a bitopogenous space. A partially ordered topological space $(X, \tau, \leq)$ has an associated bitopological space $(X, \tau^\sharp, \tau^\flat)$ where $\tau^\sharp = \{V \in \tau : \uparrow V = V\}$ and $\tau^\flat = \{V \in \tau : \downarrow V = V\}$, showing another connection between ordered topological spaces and bitopogenous spaces. (As usual, if P is a poset and $A \subseteq P$, $\uparrow A = \{y : y \geq a \text{ for some } a \in A\}$, with $\downarrow A$ defined dually.)

2. Preliminaries

Consider the following conditions which a binary relation $\sqsubset$ on the power set $\mathcal{P}(X)$ of a given set X might satisfy:

(S1) $\emptyset \sqsubset \emptyset$, $X \sqsubset X$.

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(S2) $A \sqsubset B$ implies $A \subseteq B$.
(S3) $A \subseteq A' \sqsubset B' \subseteq B$ implies $A \sqsubset B$.
(S4) $A \sqsubset B$ and $A' \sqsubset B'$ imply $A \cap A' \sqsubset B \cap B'$ and $A \cup A' \sqsubset B \cup B'$.
(S5) $A \sqsubset B$ implies there exists $C \subseteq X$ with $A \sqsubset C \sqsubset B$.

Following Császár's terminology [2], a relation $\sqsubset$ on $\mathcal{P}(X)$ satisfying (S1), (S2), and (S3) is called a *semi-topogenous order* on X and, if it also satisfies (S4), then it is called a *topogenous order* on X. The pair $(X, \sqsubset)$ where X is a set and $\sqsubset$ is a (semi-)topogenous order is called a *(semi-)topogenous space*. A bitopogenous space is a triple $(X, \sqsubset_1, \sqsubset_2)$ where X is a set and $\sqsubset_1$ and $\sqsubset_2$ are topogenous orders on X. If a topogenous order $\sqsubset$ on X satisfies the "interpolating property" (S5), then we will call it a *syntopogenous order* and the pair $(X, \sqsubset)$ will be called a *syntopogenous space*. (In [2], syntopogenous orders are called *simple syntopogenous structures* and, instead of syntopogenous spaces, the concept of topogenous spaces is used.) We will often write $x \sqsubset B$ for $\{x\} \sqsubset B$.

Note that a semi-topogenous order is transitive but need not be reflexive or antisymmetric, even if it is a syntopogenous order. Given an element $x \in X$, we put $\uparrow_{\sqsubset} x = \{U \subseteq X : x \sqsubset U\}$. Note that we always have $x \sqsubset X$.

If $\sqsubset$ is a topogenous order on X, its *complementary* topogenous order $\sqsubset^c$ (on X) is defined by

$$A \sqsubset^c B \iff X - B \sqsubset X - A.$$

A motivating example of a syntopogenous order $\sqsubset_\tau$ on a set X arises from a topology τ on X in the following way: $A \sqsubset_\tau B$ if and only if $A \subseteq int_\tau B$ (where int_τ denotes the interior operator given by τ). The complementary topogenous order to $\sqsubset_\tau$ is given by $A \sqsubset_\tau^c B$ if and only if $cl_\tau A \subseteq B$ (where cl_τ denotes the closure operator given by τ).

A (semi-)topogenous order $\sqsubset$ is *perfect* if $A_i \sqsubset B_i$ for all $i \in I$ implies $\bigcup_{i\in I} A_i \sqsubset \bigcup_{i \in I} B_i$. If τ is a topology, it is easy to see that $\sqsubset_\tau$ is a perfect topogenous order. Indeed, the perfect condition is designed to model the idea that arbitrary unions of open sets are open. Note that $\sqsubset_\tau^c$ is perfect if and only if τ is an Alexandroff topology.

Given a set X, we denote by $TOPOGEN(X)$ the set of all topogenous orders on X partially ordered by set inclusion. For topogenous orders $\sqsubset$ and $\sqsubset'$, Császár [2] shows that $\sqsubset \vee \sqsubset' = \sqsubset^q$ where $A \sqsubset^q B$ if and only if $A = \bigcup_{i=1}^m (A_i \cap A_i')$, $B = \bigcup_{i=1}^m (B_i \cap B_i')$ where $A_i \sqsubset B_i$ and $A_i' \sqsubset' B_i'$ for $i = 1, \ldots, m$. The meet of any nonempty collection of topogenous orders is their intersection, and inclusion is the largest topogenous order on X. Thus, $TOPOGEN(X)$ forms a complete lattice.

3. Semi-topogenous orders and closure operators

Recall that a *Kuratowski closure operator* on X is a function $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ which is
grounded: $cl\emptyset = \emptyset$,
extensive: $A \subseteq clA$ for all $A \subseteq X$,
monotonic: $A \subseteq B \subseteq X \Rightarrow clA \subseteq clB$,
additive: $cl(A \cup B) = clA \cup clB$ for all $A, B \subseteq X$, and
idempotent: $cl(cl(A)) = clA$ for all $A \subseteq X$.

It is common to consider closure operators that are more general than the Kuratovski ones, hence not satisfying all of these conditions (which are not independent, since additivity implies monotonicity). In this note, in accordance with [14], *closure operators* are only required to be extensive, monotonic and idempotent. Given a semi-topogenous order $\sqsubset$ on a set X, we put

$$cl_{\sqsubset}(A) \quad = \quad \{x \in X : \uparrow_{\sqsubset} x \subseteq \bigcup_{a \in A} \uparrow_{\sqsubset} a\}$$

for every $A \subseteq X$.

Theorem 3.1. *For every semi-topogenous order $\sqsubset$ on a set X, the function $cl_{\sqsubset} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a grounded closure operator on X.*

Proof. We only show that $cl_{\sqsubset}$ is idempotent. The other properties follow easily from the definitions. Suppose $x \in cl_{\sqsubset}(cl_{\sqsubset}(A))$. Then $\uparrow_{\sqsubset} x \subseteq \bigcup_{c \in cl_{\sqsubset}(A)} \uparrow_{\sqsubset} c$, so $x \sqsubset U \Rightarrow c \sqsubset U$ for some $c \in cl_{\sqsubset}(A)$. But $c \in cl_{\sqsubset}(A) \Rightarrow \uparrow_{\sqsubset} c \subseteq \bigcup_{a \in A} \uparrow_{\sqsubset} a$, so $c \sqsubset U \Rightarrow a \sqsubset U$ for some $a \in A$. Thus, $x \sqsubset U \Rightarrow a \sqsubset U$ for some $a \in A$, so $\uparrow_{\sqsubset} x \subseteq \bigcup_{a \in A} \uparrow_{\sqsubset} a$ and hence $x \in cl_{\sqsubset}(A)$. The converse, $cl_{\sqsubset}(A) \subseteq cl_{\sqsubset}(cl_{\sqsubset}(A))$, follows since $cl_{\sqsubset}$ is extensive. □

Császár ([2], pp. 212–219) defined Kuratowski closure operators associated with syntopogenous orders. Extending his definition to semi-topogenous orders, we get a grounded closure operator cl_{Cs} on X associated with a semi-topogenous order $\sqsubset$ on a set X in the following way: for $A \subseteq X$,

$$cl_{Cs}(A) = \{x : x \sqsubset U \Rightarrow U \cap A \neq \emptyset\}.$$

It is easy to see that $cl_{\sqsubset}(A) \subseteq cl_{Cs}(A)$ for any $A \subseteq X$. The following example shows that the two closure operators are not equal in general.

***Example* 3.2.** Given a fixed $\varepsilon > 0$, on $\mathbb{R}$, define $A \sqsubset_{\varepsilon} B$ if and only if the ε-fattening of A is contained in B, that is, if and only if $a \in A \Rightarrow (a - \varepsilon, a + \varepsilon) \subseteq B$. It is easy to check that $\sqsubset_{\varepsilon}$ is a topogenous order satisfying (S1)–(S4). Taking $A = \{0\} \sqsubset_{\varepsilon} (-\varepsilon, \varepsilon) = B$, we see that (S5) fails, so $\sqsubset_{\varepsilon}$ is not a syntopogenous order.

Let $\varepsilon = 1/2$. Then $cl_{\sqsubset}\{1\} = \{x : x \sqsubset_{1/2} U \Rightarrow 1 \sqsubset_{1/2} U\} = \{1\}$. Using Császár's closure, $cl_{Cs}\{1\} = \{x : x \sqsubset_{1/2} U \Rightarrow 1 \in U\} = (1/2, 3/2)$.

Remark 3.3. It is easy to see that, for every topology τ, $cl_{\sqsubset_{\tau}}$ agrees with cl_{τ}.

The lemma below gives a characterization of point closures which will be useful in working with lower separation axioms.

Theorem 3.4. *If $\sqsubset$ is a semi-topogenous order on X and $x, y \in X$, then*

$$y \in cl_{\sqsubset}\{x\} \iff y \not\sqsubset X - \{x\}.$$

Proof. $(\Rightarrow)$: Suppose $y \in cl_{\sqsubset}\{x\}$, so $\uparrow_{\sqsubset} y \subseteq \uparrow_{\sqsubset} x$. If $y \sqsubset X - \{x\}$, then $X - \{x\} \in \uparrow_{\sqsubset} y \subseteq \uparrow_{\sqsubset} x$, which gives $x \sqsubset X - \{x\}$, contradicting (S2).

$$\begin{aligned} (\Leftarrow): \quad y \not\sqsubset X - \{x\} &\Rightarrow y \not\sqsubset V \text{ for each } V \subseteq X - \{x\} \text{ by (S3)} \\ &\Rightarrow (y \sqsubset V \iff V = X) \\ &\Rightarrow \uparrow_{\sqsubset} y = \{X\} \subseteq \uparrow_{\sqsubset} x \\ &\Rightarrow y \in cl_{\sqsubset}\{x\}. \end{aligned}$$

□

Any topogenous order $\sqsubset$ on X gives a topology on X (see [3]) defined by

$$\mathcal{T}_{\sqsubset} = \{U \subseteq X : x \in U \Rightarrow x \sqsubset U\}.$$

Thinking of $\uparrow_{\sqsubset} x$ as the neighborhood filter of x, this corresponds to the statement that U is open if and only if it is a neighborhood of each of its points. It is easy to see that if $\sqsubset$ is perfect, then $\mathcal{T}_{\sqsubset} = \{U \subseteq X : U \sqsubset U\}$. The example below shows that when $\sqsubset$ is not perfect, U may be open in $\mathcal{T}_{\sqsubset}$ even if $U \not\sqsubset U$.

***Example* 3.5.** On $\mathbb{R}$, define $A \sqsubset B$ if and only if $A \subseteq B$ and A is finite or $B = \mathbb{R}$. It is easy to see that $\sqsubset$ is a syntopogenous order on $\mathbb{R}$. However, $\sqsubset$ is not perfect: $\{x\} \sqsubset [0,2]$ for all $x \in [0,1]$, but $[0,1] = \bigcup_{x\in[0,1]}\{x\} \not\sqsubset [0,2]$. It is easy to see that $\mathcal{T}_\sqsubset = \mathcal{P}(\mathbb{R})$. The set $U = (0,1)$ is $\mathcal{T}_\sqsubset$-open but $U \not\sqsubset U$.

We further note that $A = cl_\sqsubset(A)$ for all $A \subseteq \mathbb{R}$. Indeed, if $x \in cl_\sqsubset(A)$, then $U = \{x\} \in \uparrow_\sqsubset x$ so there exists $a \in A$ with $U = \{x\} \in \uparrow_\sqsubset a$. Thus, $a \sqsubset \{x\}$, so $\{a\} \subseteq \{x\}$ and $x = a \in A$.

It follows that, in this example, $U \in \mathcal{T}_\sqsubset \iff cl_\sqsubset(\mathbb{R} - U) = \mathbb{R} - U$.

Burgess and Fitzpatrick [1] define, in a more general setting, a preorder $\lesssim$ on X by $x \lesssim y \iff x \not\sqsubset X - \{y\}$, or by Theorem 3.4, $x \lesssim y \iff x \in cl_\sqsubset\{y\}$. They require (an analog) of the interpolative property (S5) and do not seem to address any of the questions we consider.

4. The T_0 separation axiom

The following definition may be found in [2].

Definition 4.1. A semi-topogenous space $(X, \sqsubset)$ is T_0 if for distinct points x, y, we have $x \sqsubset X - \{y\}$ or $y \sqsubset X - \{x\}$.

The bitopogenous definitions below generalize the corresponding bitopological definitions, which are discussed in [8].

Definition 4.2. A bitopogenous space $(X, \sqsubset_1, \sqsubset_2)$ is called

(a) *weak pairwise* T_0 if for distinct points x and y,

$$[x \sqsubset_1 X - \{y\} \ \vee \ y \sqsubset_2 X - \{x\}] \vee [y \sqsubset_1 X - \{x\} \ \vee \ x \sqsubset_2 X - \{y\}],$$

(b) *pairwise* T_0 if for distinct points x and y,

$$[x \sqsubset_1 X - \{y\} \ \vee \ y \sqsubset_2 X - \{x\}] \wedge [y \sqsubset_1 X - \{x\} \ \vee \ x \sqsubset_2 X - \{y\}].$$

The condition on distinct points $x \neq y$ used in the definition of pairwise T_0 could be simplified to $[x \sqsubset_1 X - \{y\} \ \vee \ y \sqsubset_2 X - \{x\}]$ since $x \neq y$ implies $y \neq x$. We used the long form to contrast it with weak pairwise T_0 space.

If $(X, \sqsubset_1, \sqsubset_2)$ is a bitopogenous space and P is some separation axiom, the property of $(X, \sqsubset_1, \sqsubset_2)$ being *pairwise-P* is defined case-by-case, typically involving some interaction between $\sqsubset_1$ and $\sqsubset_2$. Following the terminology for bitopological spaces in Lal [7], we say a bitopogenous space $(X, \sqsubset_1, \sqsubset_2)$ is

bi-P if each of the spaces $(X, \sqsubset_1)$ and $(X, \sqsubset_2)$ is P, and
sup-P if the space $(X, \sqsubset_1 \vee \sqsubset_2)$ is P.

Below we will consider possible implications between sup-P, pairwise-P, and bi-P for various separation axioms P applied to a bitopogenous space.

Theorem 4.3. *If $(X, \sqsubset_1, \sqsubset_2)$ is a bitopogenous space, the following implications hold.*

$$\begin{array}{ccc} \textit{pairwise } T_0 & \Rightarrow & \textit{weakly pairwise } T_0 \iff \textit{sup } T_0. \\ \Downarrow & \nearrow\!\!\!\nearrow & \\ \textit{bi } T_0 & & \end{array}$$

Proof. The only implication which does not follow immediately from the definitions is sup $T_0 \Rightarrow$ weakly pairwise T_0. Suppose $(X, \sqsubset_1, \sqsubset_2)$ is sup T_0 and $\sqsubset_1 \vee \sqsubset_2 = \sqsubset^q$. Then for $x \neq y$, $x \sqsubset^q X - \{y\}$ or $y \sqsubset^q X - \{x\}$. Suppose $x \sqsubset^q X - \{y\}$. Then from the characterization of $\sqsubset^q$, we have $\{x\} = \bigcup_{i=1}^m (A_i \cap A_i')$ and $X - \{y\} = \bigcup_{i=1}^m (B_i \cap B_i')$ where $A_i \sqsubset_1 B_i$ and $A_i' \sqsubset_2 B_i'$ for all $i \in \{1, \ldots, m\}$. It follows that $A_i \cap A_i' \subseteq \{x\}$ for all i and $A_j \cap A_j' = \{x\}$ for some $j \in \{1, \ldots, m\}$.

Furthermore, $y \notin B_j \cap B'_j$. Suppose $y \notin B'_j$. Then $\{x\} \subseteq A'_j \sqsubset_2 B'_j \subseteq X - \{y\}$, so $x \sqsubset_2 X - \{y\}$, which shows $(X, \sqsubset_1, \sqsubset_2)$ is weakly pairwise T_0. The other three cases are similar. □

5. The T_1 separation axiom

Definition 5.1. A semi-topogenous space $(X, \sqsubset)$ is T_1 if for distinct points x, y, we have $x \sqsubset X - \{y\}$ (and $y \sqsubset X - \{x\}$).

A semi-topogenous space $(X, \sqsubset)$ is T_1 if $y \sqsubset^c y$ for all y: For $x \neq y$, $\{x\} \subseteq X - \{y\} \sqsubset X - \{y\}$, so $x \sqsubset X - \{y\}$. The converse holds if $\sqsubset$ is perfect: If $\sqsubset$ is T_1 and perfect, $y \sqsubset^c X - \{x\}$ for all $x \neq y$ gives $y \sqsubset^c \bigcap_{x \neq y} X - \{x\} = \{y\}$, so $y \sqsubset^c y$.

The various bitopogenous definitions below generalize the corresponding bitopological definitions from [12, 15, 11], as discussed in [8].

Definition 5.2. A bitopogenous space $(X, \sqsubset_1, \sqsubset_2)$ is called

(a) *Reilly pairwise* T_1 if for distinct points x and y,

$$x \sqsubset_1 X - \{y\} \ \wedge \ y \sqsubset_2 X - \{x\}.$$

(b) *weak pairwise* T_1 if for distinct points x and y,

$$(x \sqsubset_1 X - \{y\} \wedge y \sqsubset_2 X - \{x\}) \bigvee (x \sqsubset_2 X - \{y\} \wedge y \sqsubset_1 X - \{x\}).$$

(c) *MN pairwise* T_1 if for distinct points x and y,

$$x \sqsubset_1 X - \{y\} \ \vee \ x \sqsubset_2 X - \{y\}.$$

Lemma 5.3. *A semi-topogenous space $(X, \sqsubset)$ is T_1 if and only if $cl_{\sqsubset}\{x\} = \{x\}$ for each $x \in X$.*

Proof. If $\sqsubset$ is T_1 and $x \neq y$, then $y \sqsubset X - \{x\}$ and Theorem 3.4 implies $y \notin cl_{\sqsubset}\{x\}$, so $cl_{\sqsubset}\{x\} = \{x\}$. If $cl_{\sqsubset}\{x\} = \{x\}$ for every x, then for $x \neq y$, we have $y \notin cl_{\sqsubset}\{x\}$ and $x \notin cl_{\sqsubset}\{y\}$, and applying Theorem 3.4 shows $\sqsubset$ is T_1. □

It is easy to see that the following implications are the only implications which hold between the various forms of pairwise T_0 and T_1.

$$\begin{array}{ccc}
\text{Reilly pairwise } T_1 & \Longrightarrow & \text{pairwise } T_0 \\
\Downarrow & & \\
\text{weak pairwise } T_1 & & \Big\Downarrow \\
\Downarrow & & \\
\text{MN pairwise } T_1 & \Longrightarrow & \text{weak pairwise } T_0
\end{array}$$

We will employ the following basic result:

Theorem 5.4. *If τ_1 and τ_2 are topologies on X, then the topogenous orders $\sqsubset_{\tau_1}, \sqsubset_{\tau_2} \in TOPOGEN(X)$ satisfy $\sqsubset_{\tau_1} \vee \sqsubset_{\tau_2} = \sqsubset_{\tau_1 \vee \tau_2}$.*

Proof. Let $\sqsubset^q = \sqsubset_{\tau_1} \vee \sqsubset_{\tau_2}$ and $\tau_3 = \tau_1 \vee \tau_2$.

If $A \sqsubset^q B$, then $A = \bigcup_{i=1}^m (A_i \cap A'_i)$, $B = \bigcup_{i=1}^m (B_i \cap B'_i)$ where $A_i \subseteq int_1 B_i$ and $A'_i \subseteq int_2 B'_i$ for $i = 1, \ldots, m$. For $j = 1, 2$, $int_j B \subseteq int_3 B$, so we have $A_i \cap A'_i \subseteq int_3 B_i \cap int_3 B'_i \subseteq int_3(B_i \cap B'_i)$. Thus, $A = \bigcup_{i=1}^m (A_i \cap A'_i) \subseteq \bigcup_{i=1}^m int_3(B_i \cap B'_i) \subseteq int_3 \bigcup_{i=1}^m (B_i \cap B'_i) = int_3 B$, so $A \sqsubset_3 B$.

Conversely, if $A \sqsubset_3 B$, then there exists an τ_3-open set C with $A \subseteq C \subseteq B$. Since $\tau_3 = \tau_1 \vee \tau_2$, $C = \bigcup_{i=1}^m (U_i \cap V_i)$ where $U_i \in \tau_1, V_i \in \tau_2$ for all i. Let $A_i = A \cap U_i, A'_i = A \cap V_i$, $B_i = B \cup U_i$, $B'_i = B \cup V_i$. Now $A_i \subseteq U_i \subseteq B_i$ shows $A_i \sqsubset_1 B_i$ and similarly $A'_i \sqsubset_2 B'_i$. Since $A = \bigcup_{i=1}^m (A_i \cap A'_i)$ and $B = \bigcup_{i=1}^m (B_i \cap B'_i)$, we have $A \sqsubset^q B$. □

***Example* 5.5.** Let X be a set and $TOP(X)$ be the lattice of topologies on X. The subposet $\mathcal{S} = \{\sqsubset_\tau : \tau \in TOP(X)\}$ of $TOPOGEN(X)$ is not lattice isomorphic to $TOP(X)$, even though $\wedge = \cap$ in both $TOP(X)$ and $\mathcal{S} \subseteq TOPOGEN(X)$. Specifically,

$$\sqsubset_{\tau_1 \wedge \tau_2} \neq \sqsubset_{\tau_1} \wedge \sqsubset_{\tau_2} .$$

On $X = \{1,2,3,4\}$, let τ_1 and τ_2 be the topologies generated by the bases $\mathcal{B}_1 = \{\{1,2\},\{3,4\}\}$ and $\mathcal{B}_2 = \{\{1\}\{2,3\},\{4\}\}$, respectively. Now $\{3\} \subseteq int_i\{2,3,4\}$ for $i = 1,2$, so $3 \sqsubset_{\tau_i} \{2,3,4\}$ for $i = 1,2$. Thus, with $\sqsubset = \sqsubset_{\tau_1} \cap \sqsubset_{\tau_2}$, we have $3 \sqsubset \{2,3,4\}$. However, $\tau_1 \wedge \tau_2 = \{\emptyset, X\}$, so $3 \not\sqsubset_{\tau_1 \wedge \tau_2} \{2,3,4\}$ since $\{3\} \not\subseteq \emptyset = int_{\tau_1 \wedge \tau_2}\{2,3,4\}$.

If X has at least three points and $p \in X$, the particular point topology $\{U \in X : p \in U\} \cup \{\emptyset\}$ and the excluded point topology $\{U \in X : p \notin U\} \cup \{X\}$ give topogenous orders with $x \sqsubset_p X - \{y\}$ if and only if $y \neq p$ and $x \sqsubset_e X - \{y\}$ if and only if $y = p$ when x and y are distinct. By Theorem 5.4, $\sqsubset_p \vee \sqsubset_e$ corresponds to the discrete topology, which is T_1. Now with $x \neq y$ in $X - \{p\}$, we see $(X, \sqsubset_p, \sqsubset_e)$ is MN T_1 but not weak pairwise T_1. In particular, sup-$T_1 \not\Rightarrow$ weak pairwise T_1.

Theorem 5.6. *For a bitopogenous space* $(X, \sqsubset_1, \sqsubset_2)$,
bi-$T_1 \iff$ *Reilly pairwise* T_1 *and* *MN pairwise* $T_1 \iff$ *sup-*T_1.

Proof. The first equivalence follows from the definitions. Since $\sqsubset_1 \cup \sqsubset_2 \subseteq \sqsubset_1 \vee \sqsubset_2$, it follows that MN pairwise T_1 implies sup-T_1. The converse follows by repeating the proof of Theorem 4.3. □

6. The R_0 separation axiom

The R_0 topological condition, introduced by Davis [4], is useful because a topological space (X, τ) is T_1 if and only if it is T_0 and R_0.

Recall that a topological space (X, τ) is R_0 if it satisfies any one of the following equivalent conditions.

(a) If F is a closed set and $x \notin F$, then there exists an open set U with $F \subseteq U$ and $x \notin U$.
(b) If U is an open set and $x \in U$, then $cl\{x\} \subseteq U$.
(c) $\{cl\{x\} : x \in X\}$ is a partition of X.

The equivalence of these statements (and six others) may be found in [4] and [5] (see [8]).

***Example* 6.1.** If (X, τ) is R_0 and τ' is coarser or finer than τ, then τ' need not be R_0. Let $X = \{1,2\}$ and $\tau = \{\emptyset, \{2\}, \{1,2\}\}$. Now $\{cl\{x\} : x \in X\} = \{\{1\},\{1,2\}\}$ is not a partition of X so by (c), τ is not R_0. Again using (c), it is easy to see that the discrete topology τ_d and the indiscrete topology τ_i on X are R_0. Since $\tau_i \subseteq \tau \subseteq \tau_d$, this shows the result.

The defining conditions (a)–(c) above may be interpreted in the (semi)-topogenous setting as follows:

(a') For every $x \in X$ and every $U \subseteq X$, if $x \sqsubset U$ then there exists $V \subseteq X - \{x\}$ such that $X - U \sqsubset V$.

(b') For every $x \in X$ and every $U \subseteq X$, if $x \sqsubset U$ then $x \sqsubset^c U$, that is, for all $x \in X, \uparrow_{\sqsubset} x \subseteq \uparrow_{\sqsubset^c} x$.

(c') $\{cl_{\sqsubset}\{x\} : x \in X\}$ is a partition of X.

Since we hope T_0 and R_0 together will imply T_1 in the topogenous setting, the example below shows that (a') and (b') are generally not a suitable way to define R_0 in a topogenous space.

***Example* 6.2.** On $\mathbb{R}$, define a syntopogenous order by $A \sqsubset B$ if and only if $A \subseteq B$ and A is finite or $A = \mathbb{R}$, as in Example 3.5. Then the syntopogenous space $(\mathbb{R}, \sqsubset)$ is T_1. Furthermore, $1 \sqsubset U = [0,2]$ but $\mathbb{R} - U \sqsubset V$ if and only if $V = \mathbb{R} \not\subseteq \mathbb{R} - \{x\}$, so $\sqsubset$ does not satisfy condition (a'). Also, $1 \not\sqsubset^c [0,2]$ since $\mathbb{R} - [0,2] \not\sqsubset \mathbb{R} - \{1\}$, so $\sqsubset$ does not satisfy condition (b'). Significantly, note that $\sqsubset$ is not perfect.

Theorem 6.3. *Suppose $(X, \sqsubset)$ is a semi-topogenous space, conditions $(a'), (b')$, (c') are as given above, and (c'') is the condition that, for all $x, y \in X$, we have $x \in cl_\sqsubset\{y\} \Rightarrow y \in cl_\sqsubset\{x\}$. Then*

$$(a') \iff (b') \Longrightarrow (c') \iff (c'').$$

Furthermore, all these conditions are equivalent if $(X, \sqsubset)$ is perfect.

Proof. Suppose $\sqsubset$ satisfies (a'). Now $x \sqsubset U$ implies that there exists $V \subseteq X$ such that $X - U \sqsubset V \subseteq X - \{x\}$, so by (S3), $X - U \sqsubset X - \{x\}$, or $x \sqsubset^c U$. Thus $(a') \Rightarrow (b')$. Conversely, suppose $\sqsubset$ fails condition (b'). Then there exist $x \in X$ and $U \subseteq X$ such that $x \sqsubset U$, yet $X - U \not\sqsubset X - \{x\}$. Then $X - U \not\sqsubset V$ for any $V \subseteq X - \{x\}$, so $\sqsubset$ fails condition (a'). Thus, $(a') \iff (b')$.

Suppose $\sqsubset$ satisfies (b') and $y \notin cl_\sqsubset\{x\}$. By Theorem 3.4, $y \sqsubset X - \{x\}$, and by (b'), $x \sqsubset X - \{y\}$. Now Theorem 3.4 gives $x \notin cl_\sqsubset\{y\}$. Thus, $\sqsubset$ satisfies (c').

Suppose $(X, \sqsubset)$ satisfies condition (c') and $x \in cl_\sqsubset\{y\}$. Now $\{x\} \in cl_\sqsubset\{x\} \cap cl_\sqsubset\{y\}$ implies $cl_\sqsubset\{x\} = cl_\sqsubset\{y\}$, so $y \in cl_\sqsubset\{x\}$ and $(X, \sqsubset)$ satisfies (c''). Conversely, suppose $\sqsubset$ satisfies (c'') and $z \in cl_\sqsubset\{x\} \cap cl_\sqsubset\{y\}$. Then $z \in cl_\sqsubset\{x\} \Rightarrow x \in cl_\sqsubset\{z\}$, so we have $\uparrow_\sqsubset z \subseteq \uparrow_\sqsubset x \subseteq \uparrow_\sqsubset z$, and thus $cl_\sqsubset\{x\} = \{w : \uparrow_\sqsubset w \subseteq \uparrow_\sqsubset x\} = \{w : \uparrow_\sqsubset w \subseteq \uparrow_\sqsubset z\} = cl_\sqsubset\{z\}$. Similarly, $cl_\sqsubset\{y\} = cl_\sqsubset\{z\}$, so $cl_\sqsubset\{x\} \cap cl_\sqsubset\{y\} \neq \emptyset \Rightarrow cl_\sqsubset\{x\} = cl_\sqsubset\{y\}$. Clearly each $cl_\sqsubset\{x\}$ is nonempty and the union of such point-closures is X. Thus, $(c') \iff (c'')$.

Finally, suppose $\sqsubset$ is perfect and satisfies (c'') and $x \sqsubset U$. For any $y \notin U$, we have $U \subseteq X - \{y\}$ so $x \sqsubset X - \{y\}$. Applying (c'') and Theorem 3.4, we have $y \sqsubset X - \{x\}$. Since $\sqsubset$ is perfect, it follows that $X - U = \bigcup_{y \in X - U}\{y\} \sqsubset X - \{x\}$, so $\sqsubset$ satisfies (b'). □

We take conditions (c'') as our definition of R_0.

Definition 6.4. A semi-topogenous space $(X, \sqsubset)$ is R_0 if and only if $x \in cl_\sqsubset\{y\} \Rightarrow y \in cl_\sqsubset\{x\}$ whenever $x, y \in X$.

The next theorem shows that our definition does what it should.

Theorem 6.5. *A semi-topogenous space $(X, \sqsubset)$ is T_1 if and only if it is R_0 and T_0.*

Proof. Suppose $(X, \sqsubset)$ is R_0 and T_0 and $x \neq y$. Say $x \sqsubset X - \{y\} = U$. Then $X - \{y\} \in \uparrow_\sqsubset x$ but by (S2), $X - \{y\} \notin \uparrow_\sqsubset y$. Thus $\uparrow_\sqsubset x \not\subseteq \uparrow_\sqsubset y$, so $x \notin cl_\sqsubset\{y\}$. By R_0, $y \notin cl_\sqsubset\{x\}$. By Theorem 3.4, $y \notin cl_\sqsubset\{x\} \Rightarrow y \sqsubset X - \{x\}$ which completes the proof that $\sqsubset$ is T_1.

Now suppose $(X, \sqsubset)$ is T_1. Clearly it is T_0, and Lemma 5.3 shows it is R_0. □

In [8], nine conditions equivalent to the topological R_0 condition were given, and it was noted that none had direct analogs to both the ordered topological setting and the bitopological setting. That fact prompted the reconsideration of the standard definition of T_1-ordered, and subsequently, of R_0-ordered in [8]. Among the seven equivalent conditions Misra and Dube [9] give to define pairwise R_0 in the bitopological setting, we have selected the following convenient condition, which extends properly to the topogenous setting: A bitopological space (X, τ_1, τ_2) is *pairwise* R_0 if and only if $y \in cl_{\tau_1}(x) \iff x \in cl_{\tau_2}(y)$ for all $x, y \in X$.

Definition 6.6. A bitopogenous space $(X, \sqsubset_1, \sqsubset_2)$ is *pairwise* R_0 if, for all $x, y \in X$,

$$x \in cl_{\sqsubset_1}\{y\} \iff y \in cl_{\sqsubset_2}\{x\}.$$

Definition 6.6 and all the remaining results of this section hold for "bi-topogenous" spaces $(X, \sqsubset_1, \sqsubset_2)$ where $\sqsubset_1$ and $\sqsubset_2$ are only semi-topogenous orders, with the exception of the results below concerning $\sqsubset_1 \vee \sqsubset_2 = \sqsubset^q$, where this supremum is taken among topogenous orders instead of semi-topogenous orders.

Suppose $(X, \tau, \leq)$ is a partially ordered topological space and $\tau^\flat, \tau^\sharp$ are as defined in Section 1. Following the notation of [8], $I(A) = cl_{\tau^\flat}(A), D(A) = cl_{\tau^\sharp}(A)$, and $C(A) = I(A) \cap D(A)$. Now

$(X, \tau^\flat, \tau^\sharp)$ is pairwise R_0 if $x \in I(y) = cl_{\tau^\flat}\{y\} \iff y \in D(x) = cl_{\tau^\sharp}\{x\}$, and it is easy to see that $x \in C(y)$ if and only if $y \in C(x)$, so by Definition 12 of [8], this is equivalent to $(X, \tau, \leq)$ being R_0^K-ordered, which is Künzi's version of ordered R_0 and plays a role in characterizing those ordered topological spaces whose Wallman ordered compactifications are not only T_1, but also satisfy an ordered version of the T_1 property (namely, Künzi's T_1^K-ordered property). This discussion suggests the following result, which is easy to prove. (Compare Theorem 13 of [8].)

Theorem 6.7. *Suppose $(X, \sqsubset_1, \sqsubset_2)$ is a bitopogenous space and $C(x) = cl_{\sqsubset_1}(x) \cap cl_{\sqsubset_2}(x)$. Then the following are equivalent.*

(a) $(X, \sqsubset_1, \sqsubset_2)$ is pairwise R_0.
(b) $y \in C(x) \iff x \in C(y)$.
(c) $\{C(x) : x \in X\}$ is a partition of X.

$(X, \sqsubset_1, \sqsubset_2)$ being pairwise R_0 is independent of $(X, \sqsubset_1)$ and $(X, \sqsubset_2)$ being R_0. That is, pairwise R_0 neither implies nor is implied by bi-R_0. For example, let $\sqsubset_i$ and $\sqsubset_d$ be, respectively, the topogenous orders from the indiscrete and discrete topologies on $\mathbb{R}$. Now $(\mathbb{R}, \sqsubset_i)$ and $(\mathbb{R}, \sqsubset_d)$ are both R_0, but $x \in cl_{\sqsubset_2}\{y\} = \mathbb{R} \not\Rightarrow y \in cl_{\sqsubset_1}\{x\} = \{x\}$.

Theorem 5.8 in [9] shows that if a bitopololgical space (X, τ_1, τ_2) is pairwise-R_0 and $\tau_1 \subseteq \tau_2$, then τ_2 is R_0. The proof given there does not carry over directly, but the corresponding result still holds:

Theorem 6.8. *If a bitopogenous space $(X, \sqsubset_1, \sqsubset_2)$ is pairwise-R_0 and $\sqsubset_1 \subseteq \sqsubset_2$, then $\sqsubset_2$ is R_0.*

Proof. A direct application of the definitions shows that if $\sqsubset_1 \subseteq \sqsubset_2$, then $cl_{\sqsubset_2}\{x\} \subseteq cl_{\sqsubset_1}\{x\}$ for all $x \in X$. Thus, under the hypothesis, $C(x) = cl_{\sqsubset_2}(x)$ for all $x \in X$. By Theorem 6.7, $\{cl_{\sqsubset_2}\{x\} : x \in X\}$ is a partition of X, so $\sqsubset_2$ is R_0. ☐

We note that if a bitopolgoical space (X, τ_1, τ_2) is pairwise R_0 then it is sup-R_0, that is, $(X, \tau_1 \vee \tau_2)$ is R_0 (Proposition 9 of [10]).

Theorem 6.9. *For a bitopogenous space $(X, \sqsubset_1, \sqsubset_2)$,*

$$\textit{bi-}R_0 \Rightarrow \textit{pairwise } R_0 \Rightarrow \text{sup-}R_0,$$

and no other implications hold between these three properties.

Proof. Suppose $(X, \sqsubset_1, \sqsubset_2)$ is bi-R_0. Applying Theorem 6.7, we have $y \in C(x) \iff (y \in cl_{\sqsubset_1}(x) \wedge y \in cl_{\sqsubset_2}(x)) \iff (x \in cl_{\sqsubset_1}(y) \wedge x \in cl_{\sqsubset_2}(y)) \iff x \in C(y)$, so $(X, \sqsubset_1, \sqsubset_2)$ is pairwise R_0.

Suppose $(X, \sqsubset_1, \sqsubset_2)$ is pairwise R_0, $\sqsubset^q = \sqsubset_1 \vee \sqsubset_2$, and $x \sqsubset^q X - \{y\}$. We wish to show $y \sqsubset_q X - \{x\}$. From the characterization of $\sqsubset^q$ given in [2], we have that $\{x\} = \bigcup_{i=1}^m (A_i^1 \cap A_i^2)$ and $X - \{y\} = \bigcup_{i=1}^m (B_i^1 \cap B_i^2)$ where $A_i^1 \sqsubset_1 B_i^1$ and $A_i^2 \sqsubset_2 B_i^2$ for all $i = 1, \ldots, m$. Now $\{x\} \in A_i^1 \cap A_i^2$ and $y \notin B_i^1 \cap B_i^2$ for all $i = 1, \ldots, m$. Thus, for any i, we have

$$\begin{array}{ccc} y \notin B_i^1 & \text{or} & y \notin B_i^2 \\ \{x\} \subseteq A_i^1 \sqsubset_1 B_i^1 \subseteq X - \{y\} & \text{or} & \{x\} \subseteq A_i^2 \sqsubset_2 B_i^2 \subseteq X - \{y\} \\ x \sqsubset_1 X - \{y\} & \text{or} & x \sqsubset_2 X - \{y\} \end{array}$$

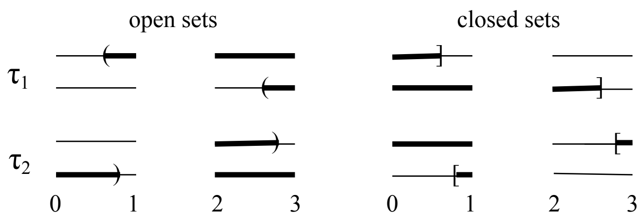
and applying the pairwise R_0 condition,

$$\begin{array}{ccc} C_i^2 \equiv \{y\} \sqsubset_2 X - \{x\} \equiv D_i^2 & \text{or} & C_i^1 \equiv \{y\} \sqsubset_1 X - \{x\} \equiv D_i^1 \\ C_i^1 \equiv \{y\} \sqsubset_1 X \equiv D_i^1 & & C_i^2 \equiv \{y\} \sqsubset_2 X \equiv D_i^2. \end{array}$$

This gives, for any $i = 1, \ldots, m$, sets $C_i^1 \sqsubset_1 D_i^1$ and sets $C_i^2 \sqsubset_2 D_i^2$ with $C_i^1 \cap C_i^2 = \{y\}$ and $D_i^1 \cap D_i^2 = X - \{x\}$. Now $\{y\} = \bigcup_{i=1}^m (C_i^1 \cap C_i^2) \sqsubset^q \bigcup_{i=1}^m (D_i^1 \cap D_i^2) = X - \{x\}$, so $\sqsubset^q$ is R_0.

The counterexamples that follow complete the proof. ☐

***Example* 6.10.** sup-R_0 does not imply pairwise R_0. Let $I_1 = (0,1), I_2 = (2,3), X = I_1 \cup I_2$, and let τ_1 be the right ray topology restricted to $(X, \leq) \subseteq (\mathbb{R}, \leq)$. Let τ_2 be the left ray topology on $(X, \leq')$ where $\leq'$ restricted to I_j is the usual order from the real line, but with $I_2 < I_1$. Open and closed sets in τ_1 and τ_2 are shown in bold in the following figure.



Define $A \sqsubset_i B$ if and only if $A \subseteq int_{\tau_i} B$ for $i = 1, 2$. By Remark 3.3, $cl_{\sqsubset_i} = cl_{\tau_i}$. For $y = .5$ and $x = 1.5$, we have $y \in cl_{\sqsubset_2}(x)$ but $x \notin cl_{\sqsubset_1}(y)$, so (X, τ_1, τ_2) is not pairwise R_0.

Now $\tau_1 \vee \tau_2$ is the Euclidean topology $\mathcal{E}$, so $cl_{\sqsubset_1 \vee \sqsubset_2} = cl_{\mathcal{E}}$ by Theorem 5.4 and Remark 3.3. Thus, $x \in cl_{\mathcal{E}}\{y\} \iff y \in cl_{\mathcal{E}}\{x\}$, so (X, τ_1, τ_2) is sup-R_0.

***Example* 6.11.** sup-R_0 does not imply bi-R_0. Let $\sqsubset_i$ be defined by $A \sqsubset_i B$ if and only if $A \subseteq int_{\tau_i} B$ where $\tau_1 = \{\emptyset, \{1\}, \{1,2\}\}$ and $\tau_2 = \{\emptyset, \{2\}, \{1,2\}\}$. Now $A \sqsubset_1 B$ for every $A \subseteq B \subseteq \{1,2\}$ except $A = B = \{2\}$, and since $\{2\} \sqsubset_2 \{2\}$, with $\sqsubset^q = \sqsubset_1 \vee \sqsubset_2$, we have $A \sqsubset^q B$ if and only if $A \subseteq B$. Thus, $x \sqsubset^q X - \{y\} \iff x \neq y \iff y \sqsubset^q X - \{x\}$, so $(X, \sqsubset^q)$ is R_0. However, $\sqsubset_1$ is not R_0 since $1 \sqsubset_1 X - \{2\}$ but $2 \not\sqsubset_1 X - \{1\}$.

***Example* 6.12.** Pairwise R_0 does not imply bi-R_0. Define $\sqsubset_1$ by $A \sqsubset_1 B$ if and only if $A \subseteq int_{\tau_1} B$ where τ_1 is the left-ray topology $\{(-\infty, x) : x \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ on $\mathbb{R}$. It is easy to see that $x \sqsubset_1 \mathbb{R} - \{y\} \iff x < y$. Thus, $x \sqsubset_1 \mathbb{R} - \{y\}$ does not imply $y \sqsubset_1 \mathbb{R} - \{x\}$, so $\sqsubset_1$ is not R_0, and thus $(\mathbb{R}, \sqsubset_1, \sqsubset_2)$ is not bi-R_0. By Theorem 3.4, we see that $cl_{\sqsubset_1}(x) = [x, \infty)$. Let $\sqsubset_2$ be the dual topogenous order arising from the right-ray topology. Now $C(x) = cl_{\sqsubset_1}(x) \cap c_{\sqsubset_2}(x) = \{x\}$, so by Theorem 6.7, $(\mathbb{R}, \sqsubset_1, \sqsubset_2)$ is pairwise R_0.

For another example, let $\sqsubset_1$ be Császár's $\sqsubset_\varepsilon$ with $\varepsilon = 1$. That is, $A \sqsubset_1 B$ if and only if $(-\infty, \sup A + 1) \subseteq B$. Let $\sqsubset_2$ be the dual of $\sqsubset_1$, with $A \sqsubset_2 B$ if and only if $(\inf A - 1, \infty) \subseteq B$. Now $x \sqsubset_1 \mathbb{R} - \{y\} \iff (-\infty, x+1) \subseteq \mathbb{R} - \{y\} \iff x + 1 \leq y$, which does not imply $(y+1 < x \iff y \sqsubset_1 \mathbb{R} - \{x\})$. Thus, $\sqsubset_1$ is not R_0. Similarly, $\sqsubset_2$ is not R_0. Since $y \sqsubset_2 \mathbb{R} - \{x\} \iff y - 1 \geq x \iff x \sqsubset_1 \mathbb{R} - \{y\}$, we see that $(\mathbb{R}, \sqsubset_1, \sqsubset_2)$ is pairwise R_0. Furthermore, $(\mathbb{R}, \sqsubset_1, \sqsubset_2)$ is *sup*-R_0. Indeed, we will show that in this case, $x \sqsubset^q X - \{y\}$ if and only if $x \sqsubset_i X - \{y\}$ for $i = 1$ or $i = 2$. Suppose $x \sqsubset^q X - \{y\}$. Then $x = \bigcup_{i=1}^m (A_i^1 \cap A_i^2)$ and $X - \{y\} = \bigcup_{i=1}^m (B_i^1 \cap B_i^2)$ where $A_i^j \sqsubset_j B_i^j$ $(i = 1, \ldots, m,\ j = 1, 2)$. Now $x \sqsubset_1 A_i^1 \sqsubset_1 B_i^1 \Rightarrow x \sqsubset_1 B_i^1 \Rightarrow (-\infty, x+1) \subseteq B_i^1$. Using $\sqsubset_2$, we get $(x - 1, \infty) \subseteq B_i^2$. Now $y \notin B_i^1 \cap B_i^2$ implies either $y \geq x + 1$ or $y \leq x - 1$, and thus $x \sqsubset_1 X - \{y\}$ or $x \sqsubset_2 X - \{y\}$. Conversely, if $x \sqsubset_1 X - \{y\}$ or $x \sqsubset_2 X - \{y\}$ then $x \sqsubset^q X - \{y\}$ since $\sqsubset_1, \sqsubset_2 \subseteq \sqsubset^q$.

The result below suggests that Reilly pairwise T_1 is the appropriate choice in bitopogenous spaces.

Theorem 6.13. *A bitopogenous space $(X, \sqsubset_1, \sqsubset_2)$ is Reilly pairwise T_1 if and only if it is pairwise R_0 and pairwise T_0.*

Proof. Suppose $(X, \sqsubset_1, \sqsubset_2)$ is pairwise R_0 and pairwise T_0 and $x \neq y$. We wish to show $x \sqsubset_1 X - \{y\}$ and $y \sqsubset_2 X - \{x\}$. Suppose to the contrary $x \not\sqsubset_1 X - \{y\}$ or $y \not\sqsubset_2 X - \{x\}$. The cases are dual, so suppose $x \not\sqsubset_1 X - \{y\}$. By pairwise T_0, $y \sqsubset_2 X - \{x\}$ so Theorem 3.4 gives $y \notin cl_{\sqsubset_2}\{x\}$. Now pairwise R_0 gives $x \notin cl_{\sqsubset_1}\{y\}$, and Theorem 3.4 gives the contradiction $x \sqsubset_1 X - \{y\}$.

Conversely, suppose $(X, \sqsubset_1, \sqsubset_2)$ is Reilly pairwise T_1. Then $(X, \sqsubset_1, \sqsubset_2)$ is pairwise T_0. To see it is pairwise R_0, suppose $x \in cl_{\sqsubset_1}\{y\}$. Then $x \not\sqsubset_1 X - \{y\}$ and Reilly pairwise T_1 implies $x = y$, so $y \in cl_{\sqsubset_2}\{x\}$. □

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