Proceedings of the
24^{th} NATIONAL CONFERENCE of
GEOMETRY and TOPOLOGY
Timişoara, Romania, July 5–9, 1994
pp. 137–143

Ball transitive ordered metric spaces

by
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0. Introduction.

If $(X, \leq)$ is a partially ordered set and $A \subseteq X$, then the *decreasing hull* $d(A)$ of A in X is defined to be $d(A) = \{x \in X : x \leq a \text{ for some } a \in A\}$. If the poset X is not understood from the context, we may write $d_X(A)$. A subset $A \subseteq X$ is a *decreasing set* if $A = d(A)$. The intersection or union of any collection of decreasing sets in X is again a decreasing set in X. The *increasing hull* $i(A)$ of a set A, and *increasing sets* are defined dually. The compliment of a decreasing set is an increasing set, and dually. A subset $A \subseteq X$ is *(order) convex* if $A = i(A) \cap d(A)$, or equivalently, if $a, b \in A$ and $a \leq c \leq b$, then $c \in A$. If $(X, \leq_X)$ and $(Y, \leq_Y)$ are partially ordered sets, then the *product order* on $X \times Y$ is defined by $(a, b) \leq (c, d)$ if and only if $a \leq_X c$ and $b \leq_Y d$. Unless otherwise noted, we will assume that the real line $\mathbf{R}$ carries its usual order and $\mathbf{R}^n$ carries the product order.

If τ is a topology on X and $\leq$ is a partial order on X, then $(X, \tau, \leq)$ is an *ordered topological space*. For $A \subseteq X$, the *closed decreasing hull* $D(A)$ of A is the smallest closed decreasing set that contains A. The decreasing hull operator $d : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ and the closed decreasing hull operator $D : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ are both Kuratowski closure operators. Since $cl(A) \subseteq D(A)$ and $d(A) \subseteq D(A)$, we have $d(cl(A)) \subseteq D(A)$ and $cl(d(A)) \subseteq D(A)$, but in general, equality does not hold. It is known [6], however, that if A is compact, then $d(A) = D(A)$, so that

$cl(d(A)) = D(A)$ and, if X is Hausdorff, $d(cl(A)) = D(A)$. Also note that since $cl(A) \subseteq D(A)$, we have $D(cl(A)) = D(A)$ for any subset $A \subseteq X$.

An ordered topological space $(X, \tau, \leq)$ has a *convex topology* if τ has a subbase of open monotone sets of X. $(X, \tau, \leq)$ is T_2-*ordered* if the graph of the partial order $\leq$ is a closed subset of $X \times X$ with the product topology. If $(X, \tau, \leq)$ is T_2-ordered, then the underlying topological space (X, τ) is T_2, but not conversely. $(X, \tau, \leq)$ is T_4-*ordered* if for any two disjoint closed subsets R and S, the former increasing and the latter decreasing, there exist disjoint open sets U and V with U increasing, V decreasing, $R \subseteq U$, and $S \subseteq V$.

1. Ball Transitivity.

Definition 1. If n is a natural number, we say a metric space (X, δ) with a partial order $\leq$ is $\frac{1}{n}$-*ball transitive* if $x \leq y$ implies $B(x, \frac{\epsilon}{n}) \subseteq d(\, B(y, \epsilon)\,)$ and $B(y, \frac{\epsilon}{n}) \subseteq i(\, B(x, \epsilon)\,)$ for any $\epsilon > 0$. We will say a metric space X with a partial order is *ball transitive* if it is $\frac{1}{n}$-ball transitive for some natural number n.

We will give two examples for illustration. First, the space $C([0,1])$ of continuous real-valued functions on the interval $[0,1]$ given the pointwise order and the sup-norm metric $\delta(f,g) = \sup\{|f(x) - g(x)| : x \in X\}$ has a convex topology and is 1-ball transitive. Next, if Q is the open first quadrant, then the subspace S of $\mathbf{R}^2$ given by $S = ([-1,1] \times [-1,1]) \backslash Q$ is not ball transitive: if $\epsilon < 1$, $B(\,(0,0), \frac{\epsilon}{n}) \not\subseteq d(\, B(\,(0,1), \epsilon)\,)$ for any n, even though $(0,0) \leq (0,1)$. Note that S is compact, T_2-ordered (and hence T_4-ordered), and convex, so none of these properties implies ball transitivity. With ϵ as above, observe that while $B(\,(0,1), \epsilon)$ is open, $d(\, B(\,(0,1), \epsilon)\,)$ is not open. This can not happen in a ball transitive space.

Proposition 2. If X is a ball transitive space and U is open in X, then $i(U)$ and $d(U)$ are open in X.

Proof. Suppose $x \in i(U)$. Then there exists $y \in U$ such that $x \geq y$. Choose ϵ so that $B(y, \epsilon) \subseteq U$. Since X is ball transitive, there exists a natural number n such that $B(x, \frac{\epsilon}{n}) \subseteq i(\, B(y, \epsilon)\,) \subseteq i(U)$. Thus, $i(U)$ is open. The dual argument shows that $d(U)$ is open. ∎

In [5], McCartan calls a preordered topological space *continuous* if it satisfies the conclusion of Proposition 2. (More frequently, "continuous" is used to mean

T_2-ordered.) If F is closed in a ball transitive space, $i(F)$ and $d(F)$ need not be closed. (Consider $F = \{(x, \frac{1}{x}) \in \mathbf{R}^2 : x > 0\}$ in $\mathbf{R}^2$.) The condition that $i(F)$ and $d(F)$ be closed for (certain) closed sets $F \subseteq X$ is a frequently used hypothesis. (For example, see *N-spaces* in [7], *anti-continuous spaces* in [5], *C-spaces* in [8], and *c-spaces* in [3].)

Proposition 3. In a ball transitive space, the closure of a decreasing set is decreasing, and the closure of an increasing set is increasing.

Proof. Suppose A is a decreasing set in ball transitive space X. Suppose $y \in cl(A)$ and $z \leq y$. We want to show $z \in cl(A)$. Suppose there exists $\epsilon > 0$ such that $B(z, \epsilon) \cap A = \emptyset$. Noting that $A = d(A)$, it follows that $i(\,B(z, \epsilon)\,) \cap A = \emptyset$. But $y \in i(\,B(z, \epsilon)\,)$, and by Proposition 2, $i(\,B(z, \epsilon)\,)$ is open. Thus, $i(\,B(z, \epsilon)\,)$ is an open neighborhood of y that does not intersect A, contrary to $y \in cl(A)$. The proof for increasing sets is dual. ∎

Proposition 4. In a ball transitive space, the interior of a decreasing set is decreasing, and dually.

Proof. If A is decreasing in ball transitive space X, $X \backslash A$ is increasing, and by Proposition 3, $cl(X \backslash A)$ is also increasing. Thus, $Int(A) = X \backslash cl(X \backslash A)$ is decreasing. ∎

Theorem 5. A ball transitive space is T_4-ordered.

Proof. Suppose R and S are disjoint closed subsets of a $\frac{1}{n}$-ball transitive space X, with R increasing, and S decreasing. For any $r \in R$, r is an element of the open increasing set $X \backslash S$, so there exists ϵ_r such that $i(\,B(r, \epsilon_r)\,) \cap S = \emptyset$. Similarly, for any $s \in S$, there exists ϵ_s such that $d(\,B(s, \epsilon_s)\,) \cap R = \emptyset$. Put

$$\begin{aligned} G_R &= \bigcup_{r \in R} i(\,B(r, \frac{\epsilon_r}{2n})\,), \quad \text{and} \\ G_S &= \bigcup_{s \in S} d(\,B(s, \frac{\epsilon_s}{2n})\,). \end{aligned}$$

By Proposition 2 and properties of open and monotone sets, G_R is an open increasing neighborhood of R and G_S is an open decreasing neighborhood of S. To show that G_R and G_S are disjoint, suppose the contrary. If $x \in G_R \cap G_S$, then there exist points $r \in R$ and $s \in S$ such that $x \in i(\,B(r, \frac{\epsilon_r}{2n})\,) \cap d(\,B(s, \frac{\epsilon_s}{2n})\,)$. Thus, there exist points $r' \in B(r, \frac{\epsilon_r}{2n})$ and $s' \in B(s, \frac{\epsilon_s}{2n})$ with $r' \leq x \leq s'$. In case $\epsilon_s \geq \epsilon_r$, we have

$$r \in B(r', \frac{\epsilon_r}{2n}) \subseteq B(r', \frac{\epsilon_s}{2n}) \subseteq d(\,B(s', \frac{\epsilon_s}{2})\,) \subseteq d(\,B(s, \epsilon_s)\,),$$

contrary to $R \cap d(\,B(s, \epsilon_s)\,) = \emptyset$. In case $\epsilon_r \geq \epsilon_s$, we have

$$s \in B(s', \frac{\epsilon_s}{2n}) \subseteq B(s', \frac{\epsilon_r}{2n}) \subseteq i(\,B(r', \frac{\epsilon_r}{2})\,) \subseteq i(\,B(r, \epsilon_r)\,),$$

contrary to $S \cap i(\,B(r, \epsilon_r)\,) = \emptyset$. Now G_R and G_S are the desired open monotone separating neighborhoods of R and S. ∎

The proposition below shows that, using the natural product construction, the product of ball transitive spaces is ball transitive.

Proposition 6. If $(X, m_X, \leq_X)$ is a $\frac{1}{n}$-ball transitive space and $(Y, m_Y, \leq_Y)$ is a $\frac{1}{k}$-ball transitive space, then $X \times Y$ with the metric $m_X + m_Y$ and the product order is a $\frac{1}{n+k}$-ball transitive space.

Proof. Suppose $(a, b) \leq (s, t)$ in $X \times Y$ and $(x, y) \in B(\,(a, b)\,, \epsilon)$. Then $m_X(x, a) + m_Y(y, b) < \epsilon$, so $x \in B_X(a, \epsilon)$ and $y \in B_Y(b, \epsilon)$. From the hypotheses, it follows that $x \in d(\,B_X(s, n\epsilon)\,)$ and $y \in d(\,B_Y(t, k\epsilon)\,)$. Thus, there exist points $s' \in B_X(s, n\epsilon)$ and $t' \in B_Y(t, k\epsilon)$ with $(x, y) \leq (s', t')$. Since $(s', t') \in B_{X \times Y}(\,(s, t), (n+k)\epsilon\,)$, it follows that

$$B(\,(a, b), \epsilon) \subseteq d(\,B(\,(s, t), (n+k)\epsilon)\,) \text{ in } X \times Y.$$

With the dual argument, this proves $X \times Y$ is $\frac{1}{n+k}$-ball transitive. ∎

Given the usual metric and order, it is easy to see that the real line $\mathbf{R}$ is 1-ball transitive (and almost as easy to see that $\mathbf{R}^n$ is 1-ball transitive). Thus, from the previous propositions, we have the following important result.

Theorem 7. Given the usual metric and order, $\mathbf{R}^n$ is T_4-ordered (for any natural number n).

The following definition of a metric lattice is from Birkhoff [1].

Definition 8. A lattice L is a *metric lattice* if there exists a strictly increasing function $m : L \longrightarrow \mathbf{R}$ satisfying $m(x)+m(y) = m(x\vee y)+m(x\wedge y)$ for all $x, y \in L$. Then $\delta(x, y) = m(x \vee y) - m(x \wedge y)$ is a metric on L.

Birkhoff gives several properties of metric lattices, including the fact that every metric lattice is a modular lattice, and that for any a, x, y in a metric lattice L, $\delta(a \vee x, a \vee y) + \delta(a \wedge x, a \wedge y) \leq \delta(x, y)$.

Proposition 9. Any metric lattice is ball transitive.

Proof. Suppose L is a metric lattice and $a \leq c$ in L. To show $B(a,\epsilon) \subseteq d(\, B(c,\epsilon)\,)$, suppose $x \in B(a,\epsilon)$. Using the inequality given in the paragraph above, $\delta(c\vee x, c\vee a) + \delta(c \wedge x, c \wedge a) \leq \delta(x, a) < \epsilon$. Now $a \leq c$ implies $c \vee a = c$, so $\delta(c \vee x, c) < \epsilon$, or $c \vee x \in B(c,\epsilon)$. Now $x \leq c \vee x$ implies $x \in d(\, B(c,\epsilon)\,)$. A similar argument shows that $B(c,\epsilon) \subseteq i(\, B(a,\epsilon)\,)$, and thus L is 1-ball transitive. ∎

While Proposition 6 showed that ball transitivity was productive (with the natural product construction), ball transitivity is not hereditary. For example, although $\mathbf{R}^2$ with the usual metric and order is ball transitive, the subspace S given before Proposition 2 is not ball transitive. An obstacle that prevents ball transitivity from being hereditary is the fact that, in general, $d_X(A)\cap C \neq d_C(A)$. Also, $D_X(A) \cap C \neq D_C(A)$ in general. If the latter equality holds, McCartan [4] would say A is a τ-*compatibly ordered* subspace of X. We note one special case where ball transitivity is hereditary. The proof of Proposition 9 remains valid if L is any sublattice of a metric lattice, so ball transitivity is hereditary to sublattices of a metric lattice.

The weaker version of ball transitivity given in the definition below is based on ball transitivity not for every ϵ-ball, but for a basis of ϵ-balls at each point. This weaker version has a nice hereditary property, but this gain is compensated by the apparent loss of the T_4-ordered property.

Definition 10. A metric space X with partial order is *basically* $\frac{1}{n}$*-ball transitive* if for every $x \in X$ there exists $\epsilon_x > 0$ such that for $0 < \epsilon \leq \epsilon_x$

$$x \leq y \quad \text{implies} \quad B(y, \frac{\epsilon}{n}) \subseteq i(\, B(x,\epsilon)\,) \quad \text{and}$$
$$y \leq x \quad \text{implies} \quad B(y, \frac{\epsilon}{n}) \subseteq d(\, B(x,\epsilon)\,).$$

If X is basically $\frac{1}{n}$-ball transitive for some n, we say X is *basically ball transitive.*

Note that ball transitivity implies basic ball transitivity, but not conversely (with S as given before Proposition 2, consider $Int(S)$). We summarize some properties of basically ball transitive spaces below.

Proposition 11. If X is basically ball transitive, then

a) $i(U)$ and $d(U)$ are open whenever U is open.

b) The closure of a decreasing set is decreasing, and dually.

c) The interior of an decreasing set is decreasing, and dually.

d) If Y is also basically ball transitive, then $X \times Y$ with the sum metric and product order is basically ball transitive.

e) If U is an open subset of X, the subspace U is basically ball transitive.

Proof. Modifications of the proofs of Propositions 2, 3, 4, and 6 can be used to prove a), b), c), and d), respectively. For e), suppose U is an open subset of basically ball transitive space X. For $x \in U \subseteq X$, there exists an ϵ_x as in Definition 10, and there exists an $\epsilon_x{}^* > 0$ such that $B_X(x, \epsilon_x{}^*) \subseteq U$. Take $\epsilon'_x = \min\{\epsilon_x, \epsilon_x{}^*\}$. Now for $0 < \epsilon \leq \epsilon'_x$ and $y \leq x$, we have $B_X(y, \frac{\epsilon}{n}) \subseteq d_X(\, B_X(x, \epsilon)\,)$, so that

$$B_U(y, \frac{\epsilon}{n}) = U \cap B_X(y, \frac{\epsilon}{n}) \subseteq U \cap d_X(\, B_X(x, \epsilon)\,) = d_U(\, B_U(x, \epsilon)\,),$$

where the last equality follows from the inclusion $B_X(x, \epsilon) \subseteq U$. A similar argument shows $B_U(y, \frac{\epsilon}{n}) \subseteq i_U(\, B_U(x, \epsilon)\,)$ whenever $0 < \epsilon \leq \epsilon'_x$ and $x \leq y$. ∎

We note that the specific proof given for Proposition 4 does not generalize to the setting of a basically ball transitive space, though a counterexample–a basically ball transitive space that is not T_4-ordered–is not known.

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