MATH 116 Inverse Functions

Given a function $y = f(x)$ that is either strictly increasing or strictly decreasing for x in an interval, we can find the inverse of the function, denoted f^{-1} (called "f inverse"). The inverse "undoes" the original function f by sending a y-value back to the x-value from which it came. So if $f(3) = 9$, then $f^{-1}(9) = 3$.

But in order for f^{-1} to be a function, we can only send y back to *one* x-value. If there are two or more x-values that f sends to one y-value, then we would not know which x-value to assign back to this y. So in this case f^{-1} will not exist. For example, consider $f(x) = x^2$ (which is neither strictly increasing nor strictly decreasing). Because $f(3) = 9 = f(-3)$, there are two x-values that f sends to 9. So how will we define $f^{-1}(9)$? We cannot in this case.

If different x-values are always assigned different y-values (which happens with strictly increasing or strictly decreasing functions), then we can always send a y back to *one* x-value. In this case, the original function is called *one-to-one* and its inverse will exist.

So $f(x) = x^2$ is not one-to-one; in fact it is two-to-one: two different x-values go to the same y-value. Hence, $f(x) = x^2$ has no inverse function. But $f(x) = x^3$ is strictly increasing, so it is one-to-one. It has an inverse given by $f(x) = x^{1/3}$ (the cube root).

When an inverse function exists, the following properties must hold:

(i) Domain f = Range f^{-1} and Range f = Domain f^{-1}.

(ii) $f^{-1}(f(x)) = x$ for all x in Domain f, and $f\left(f^{-1}(x)\right) = x$ for all x in Domain f^{-1}.

(iii) The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric about the line $y = x$.

(iv) If $f(a) = b$, then $f^{-1}(b) = a$. If $f^{-1}(c) = d$, then $f(d) = c$.

To find the inverse of a one-to-one function $y = f(x)$, you can use the following steps:

(i) Switch x and y and write $x = f(y)$. (ii) Solve for y as a function of x.
(iii) Call the resulting function $f^{-1}(x)$. (iv) Specify Domain f^{-1} to be Range f.

Examples. Find the inverses of the following functions. Give the domain and range of f and f^{-1} and verify that $f^{-1}(f(x))$ and $f\left(f^{-1}(x)\right)$ both equal x.

(a) $f(x) = \dfrac{20}{x-4} + 5$, for $x > 4$ (b) $f(x) = 16 - x^2$, for $x \le 0$ (c) $f(x) = 9 - 2x$, for $x \le 6$

Solutions

(a) For $f(x)=\dfrac{20}{x-4}+5$ with Domain f = (4, ∞), then Range f = (5, ∞).

To find f^{-1}, let $x=\dfrac{20}{y-4}+5$ and solve for y → $x-5=\dfrac{20}{y-4}$ → $y-4=\dfrac{20}{x-5}$ → $y=\dfrac{20}{x-5}+4$ → $f^{-1}(x)=\dfrac{20}{x-5}+4$. Then Domain f^{-1} = (5, ∞) and Range f^{-1} = (4, ∞).

Now to verify the inverse property:

$$f^{-1}(f(x)) = \frac{20}{f(x)-5}+4 = \frac{20}{\left(\frac{20}{x-4}+5\right)-5}+4 = \frac{20}{\frac{20}{x-4}}+4 = \frac{x-4}{20}\times 20+4 = x-4+4 = x$$

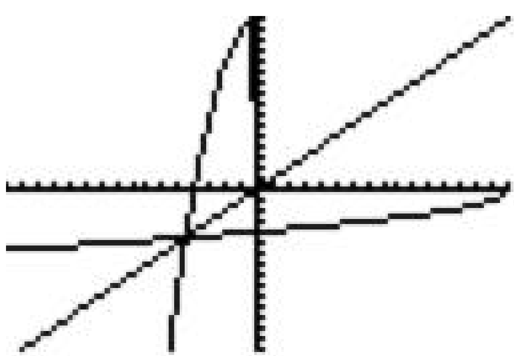
and

$$f\left(f^{-1}(x)\right) = \frac{20}{f^{-1}(x)-4}+5 = \frac{20}{\left(\frac{20}{x-5}+4\right)-4}+5 = \frac{20}{\frac{20}{x-5}}+5 = \frac{x-5}{20}\times 20+5 = x-5+5 = x.$$

Note further: $f(6)=\dfrac{20}{6-4}+5=15$ and $f^{-1}(15)=\dfrac{20}{15-5}+4=6$.

(b) For $f(x)=16-x^2$ with Domain f = (–∞, 0], then Range f = (–∞, 16]. Now let $x=16-y^2$ and solve for y: $y^2=16-x$ → $y=\pm\sqrt{16-x}$. But the range here must be the domain of the original f which are negative numbers. Hence, $f^{-1}(x)=-\sqrt{16-x}$ with Domain f^{-1} = (–∞, 16] and Range f^{-1} = (–∞, 0].

Now for $x\le 0$, $f^{-1}(f(x)) = -\sqrt{16-f(x)} = -\sqrt{16-(16-x^2)} = -\sqrt{x^2} = -(-x) = x$. Then for $x \le 16$, we have $f\left(f^{-1}(x)\right)=16-\left(f^{-1}(x)\right)^2 = 16-\left(-\sqrt{16-x}\right)^2 = 16-(16-x)=x$.



Notice the symmetry of the graphs of f and f^{-1} about the line $y=x$.

(c) For $f(x) = 9 - 2x$ with Domain (–∞, 6], the range is [–3, ∞). Now let $x = 9 - 2y$ and solve for y → $2y = 9 - x$ → $f^{-1}(x) = \dfrac{9-x}{2} = -\dfrac{1}{2}x + \dfrac{9}{2}$ for $x \geq -3$ with range = (–∞, 6].

$$\text{Then } f^{-1}\big(f(x)\big) = \frac{9 - f(x)}{2} = \frac{9-(9-2x)}{2} = \frac{2x}{2} = x$$

and

$$f\Big(f^{-1}(x)\Big) = 9 - 2f^{-1}(x) = 9 - 2\left(\frac{9-x}{2}\right) = 9-(9-x) = x\,.$$

Exercises

For each function:

(i) Graph f and label two specific points.
(ii) State the domain and range of f.
(iii) Find f^{-1} and state its domain and range.
(iv) Graph f^{-1}. Show the inverse image of the two specific points graphed in (i).
(v) Evaluate and simplify $f^{-1}\big(f(x)\big)$ and $f\Big(f^{-1}(x)\Big)$ *with no scratch outs*.

1. $f(x) = 4 - \dfrac{1}{2}x$, for $x \leq 12$

2. $f(x) = -\sqrt{x-4} - 6$

3. $f(x) = x^2 - 9$, for $x \geq 2$

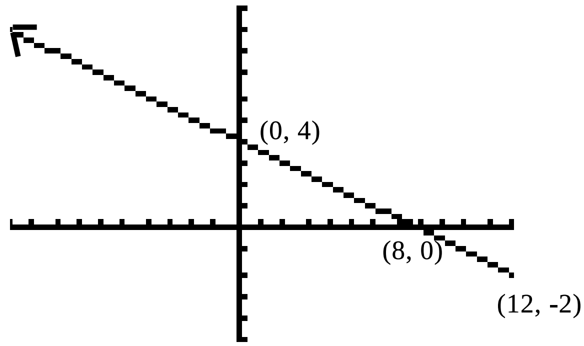
4. $f(x) = \dfrac{10}{x-4} + 2$, for $x < 4$

Solutions

1. $f(x) = 4 - \frac{1}{2}x$, for $x \le 12$

Domain f: $(-\infty, 12]$

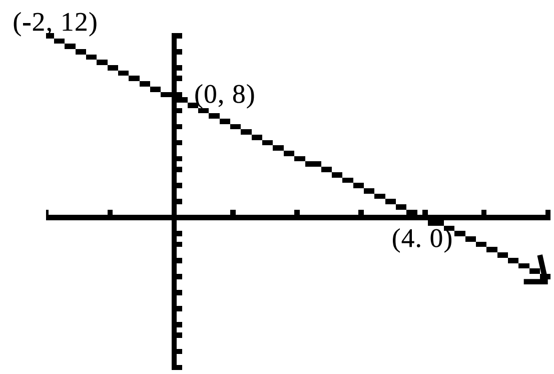
Range f: $[-2, \infty)$



Switch x and y $\rightarrow$ $x = 4 - \frac{1}{2}y$.

Solve for y $\rightarrow$ $\frac{1}{2}y = 4 - x$ $\rightarrow$ $y = 8 - 2x$.

So $f^{-1}(x) = 8 - 2x$ for x in $[-2, \infty)$. Then Range $f^{-1} = (-\infty, 12]$.



$$f^{-1}(f(x)) = 8 - 2f(x) = 8 - 2\left(4 - \frac{1}{2}x\right) = 8 - 8 + x = x.$$

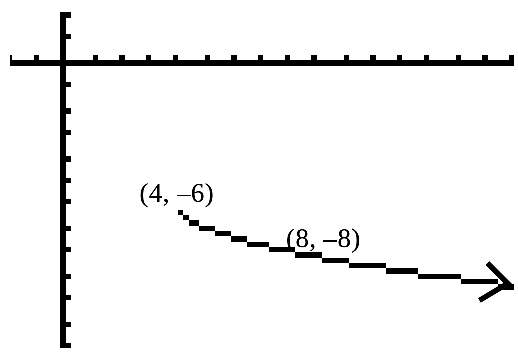
$$f\left(f^{-1}(x)\right) = 4 - \frac{1}{2}f^{-1}(x) = 4 - \frac{1}{2}(8 - 2x) = 4 - 4 + x = x.$$

---

2. $f(x) = -\sqrt{x-4} - 6$

Domain f: $[4, \infty)$

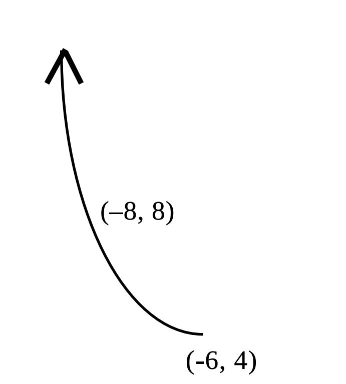
Range f: $(-\infty, -6]$



Switch x and y $\rightarrow$ $x = -\sqrt{y-4} - 6$.

Solve for y $\rightarrow$ $x + 6 = -\sqrt{y-4}$ $\rightarrow$ $(x+6)^2 = y - 4$ $\rightarrow$ $y = (x+6)^2 + 4$.

So $f^{-1}(x) = (x+6)^2 + 4$ for x in $(-\infty, -6]$. Then Range $f^{-1} = [4, \infty)$.



$$f^{-1}\big(f(x)\big) = (f(x)+6)^2+4 = (-\sqrt{x-4}-6+6)^2+4 = (-\sqrt{x-4})^2+4 = x-4+4 = x.$$

For $x \le -6$ note that $x+6 \le 0$, so $\sqrt{(x+6)^2} = -(x+6)$ Thus,

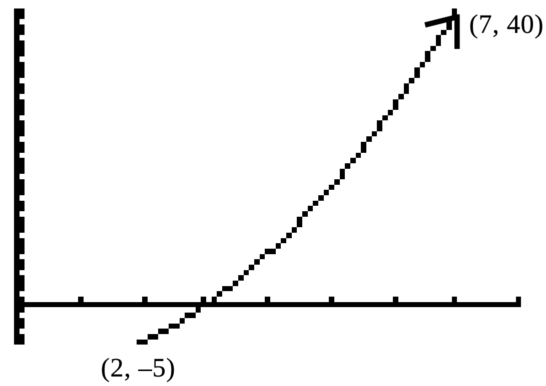
$$f\left(f^{-1}(x)\right) = -\sqrt{f^{-1}(x)-4}-6 = -\sqrt{(x+6)^2+4-4}-6$$
$$= -\sqrt{(x+6)^2}-6 = --(x+6)-6 = x+6-6 = x.$$

---

3. $f(x) = x^2-9$, for $x \ge 2$

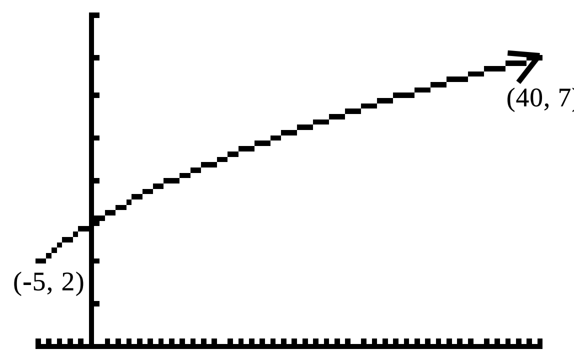
Domain f: $[2, \infty)$

Range f: $[-5, \infty)$



Switch x and y $\rightarrow$ $x = y^2-9$.
Solve for y $\rightarrow$ $y^2 = x+9$ $\rightarrow$ $y = \pm\sqrt{x+9}$
But the range must be positive numbers $[2, \infty)$.

So $f^{-1}(x) = \sqrt{x+9}$ for x in $[-5, \infty)$ and Range $f^{-1} = [2, \infty)$.



$$f^{-1}\big(f(x)\big) = \sqrt{f(x)+9} = \sqrt{x^2-9+9} = \sqrt{x^2} = x, \text{ for } x \ge 2.$$

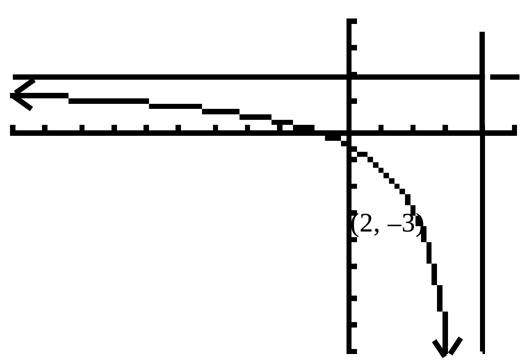
$$f\left(f^{-1}(x)\right) = \left(f^{-1}(x)\right)^2-9 = \left(\sqrt{x+9}\right)^2-9 = x+9-9 = x$$

---

4. $f(x) = \dfrac{10}{x-4}+2$, for $x < 4$

Domain f: $(-\infty, 4)$

Range f: $(-\infty, 2)$

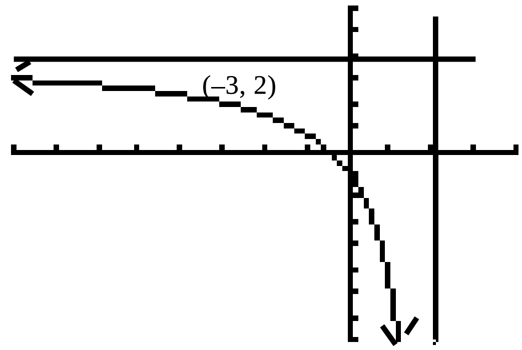


Switch x and y → $x = \dfrac{10}{y-4} + 2$.

Solve for y → $x - 2 = \dfrac{10}{y-4}$ →

$y - 4 = \dfrac{10}{x-2}$ → $y = \dfrac{10}{x-2} + 4$

So $f^{-1}(x) = \dfrac{10}{x-2} + 4$ for x in (–∞, 2) and Range f^{-1} = (–∞, 4).



$$f^{-1}\big(f(x)\big) = \frac{10}{f(x)-2} + 4 = \frac{10}{\dfrac{10}{x-4} + 2 - 2} + 4 = \frac{10}{\dfrac{10}{x-4}} + 4 = \frac{x-4}{10} \times 10 + 4 = x - 4 + 4 = x.$$

$$f\big(f^{-1}(x)\big) = \frac{10}{f^{-1}(x) - 4} + 2 = \frac{10}{\dfrac{10}{x-2} + 4 - 4} + 2 = \frac{10}{\dfrac{10}{x-2}} + 2 = \frac{x-2}{10} \times 10 + 2 = x - 2 + 2 = x.$$