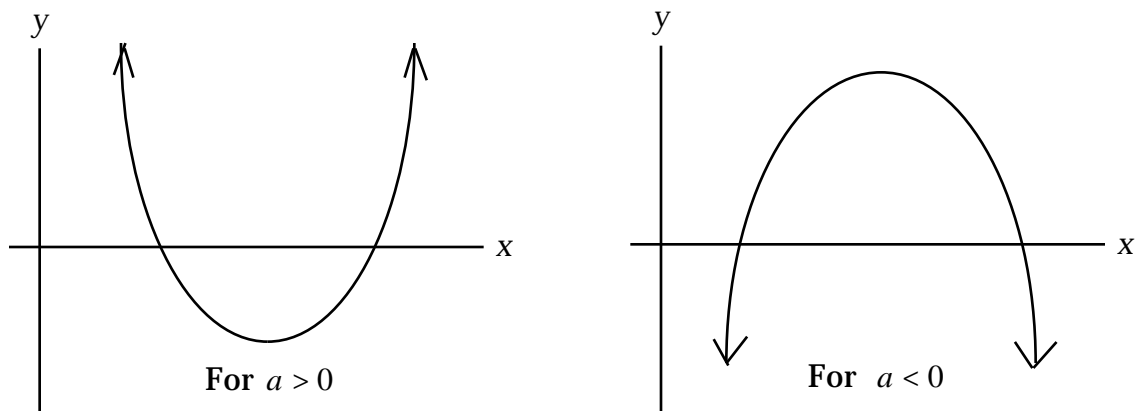


A quadratic function has the form

$$f(x) = ax^2 + bx + c$$

where  $a \neq 0$ . The graph of the function is a parabola that opens upward if  $a > 0$ , and opens downward for  $a < 0$ .



Given such a parabolic function, we will

- (a) Find the vertex.    (b) State the  $y$ -intercept.    (c) Find the  $x$ -intercepts.
- (d) Graph the function.    (e) Solve the equation  $f(x) = d$ .
- (f) Use the solution from (e) to solve the inequalities  $y = d$ ,  $y < d$ ,  $y \geq d$ , or  $y > d$ .

As an example throughout, we will use the function

$$y = 3x^2 - 12x + 6.$$

### The Vertex

The vertex  $(x, y)$  is the point at which the graph changes from decreasing to increasing. The  $x$ -coordinate of the vertex is given by

$$x = \frac{-b}{2a}.$$

We then substitute this  $x$ -value into the function to find the  $y$ -coordinate of the vertex.

For  $y = 3x^2 - 12x + 6$ , the  $x$ -coordinate of the vertex is

$$x = \frac{-b}{2a} = \frac{-(-12)}{2(3)} = \frac{12}{6} = 2.$$

The  $y$ -coordinate is then  $3(2)^2 - 12(2) + 6 = -6$ . So the vertex is the point  $(2, -6)$ .

### The y-intercept

The  $y$ -intercept of the function  $y = ax^2 + bx + c$  is always given by the constant term  $c$ . That is, when  $x = 0$ , then  $y = c$ . For  $y = 3x^2 - 12x + 6$ , the  $y$ -intercept is 6.

### The x-intercepts

To find the  $x$ -intercepts of the quadratic  $y = ax^2 + bx + c$ , we set  $y = 0$  and solve for  $x$ . That is, we must solve the quadratic equation

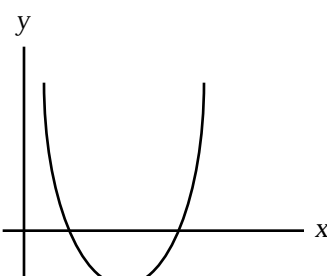
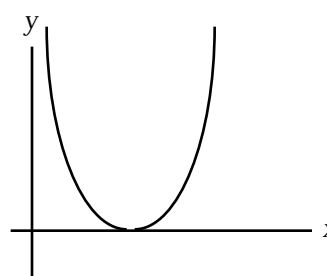
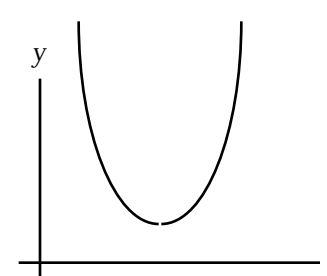
$$ax^2 + bx + c = 0.$$

If the quadratic does not factor, then we can use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

**Note:** The expression under the radical  $b^2 - 4ac$  is called the "discriminant" which determines how many solutions there are to the equation  $ax^2 + bx + c = 0$ .

- (i) If  $b^2 - 4ac > 0$ , then there are two  $x$ -intercepts.
- (ii) If  $b^2 - 4ac = 0$ , then there is one  $x$ -intercept.
- (iii) If  $b^2 - 4ac < 0$ , then there are no solutions to  $ax^2 + bx + c = 0$  and no  $x$ -intercepts.
- (iv) If  $b^2 - 4ac$  is a perfect square, then the quadratic  $ax^2 + bx + c$  will factor.

 <p><math>b^2 - 4ac &gt; 0</math></p>	 <p><math>b^2 - 4ac = 0</math></p>	 <p><math>b^2 - 4ac &lt; 0</math></p>
<b>Two <math>x</math>-intercepts</b>	<b>One <math>x</math>-intercept</b>	<b>No <math>x</math>-intercepts</b>

Consider  $y = 2x^2 - x - 10$ . The discriminant is  $b^2 - 4ac = (-1)^2 - 4(2)(-10) = 1 + 80 = 81$ , which is positive and a perfect square. So there are two  $x$ -intercepts that can be found by factoring:  $2x^2 - x - 10 = (2x - 5)(x + 2) = 0$   $2x - 5 = 0$  or  $x + 2 = 0$ . Thus,

$$x = \frac{5}{2} \text{ and } x = -2 \text{ are the } x\text{-intercepts.}$$

We now will find the  $x$ -intercepts of  $y = 3x^2 - 12x + 6$  by using the quadratic formula:

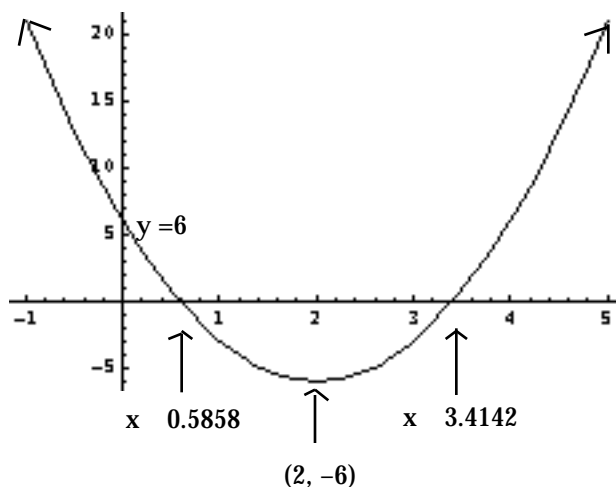
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(6)}}{2(3)} \\ &= \frac{12 \pm \sqrt{72}}{6} = \frac{12}{6} \pm \frac{\sqrt{36 \times 2}}{6} = 2 \pm \frac{6\sqrt{2}}{6} = 2 \pm \sqrt{2} \end{aligned}$$

The approximate decimal values are  $2 - \sqrt{2} \approx 0.585786$  and  $2 + \sqrt{2} \approx 3.4142$ .

### Graphing

Once we have found the vertex, the  $y$ -intercept, and the  $x$ -intercepts, we can graph the parabola by plotting these points.

For  $y = 3x^2 - 12x + 6$ , the vertex is  $(2, -6)$ , the  $y$ -intercept is  $6$  and the  $x$ -intercepts are  $2 \pm \sqrt{2}$ , which are about  $0.585786$  and  $3.4142$ .



The graph of  $y = 3x^2 - 12x + 6$

### Solving Other Equations

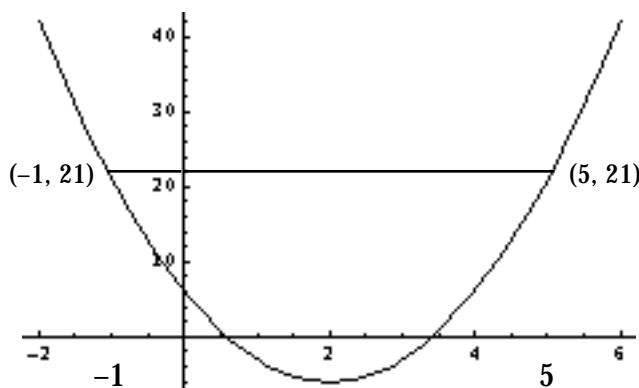
Given the quadratic function  $f(x) = ax^2 + bx + c$ , we also can solve for the  $x$  that makes  $f(x) = d$ . That is, we can solve the equation  $ax^2 + bx + c = d$ . To do so, always set the equation equal to 0 by subtracting the  $d$  term, and then either factor or use the quadratic formula to solve the resulting equation.

Consider again the function  $f(x) = 3x^2 - 12x + 6$ . When does  $y = 21$ ?

Here we must solve the equation  $3x^2 - 12x + 6 = 21$ . By subtracting 21 we obtain

$$3x^2 - 12x - 15 = 0$$

If we divide by 3, we obtain  $x^2 - 4x - 5 = 0$  which factors as  $(x - 5)(x + 1) = 0$ . So the solutions are  $x = -1$  and  $x = 5$ .



### Solving Inequalities

Using the solution from above, solve the inequalities  $y = 21$ ,  $y < 21$ ,  $y > 21$ , and  $y \geq 21$ . From the graph, we see that

$y = 21$ when $-1 \leq x \leq 5$ i.e., for $x$ in $[-1, 5]$	$y < 21$ when $-1 < x < 5$ i.e., for $x$ in $(-1, 5)$
$y > 21$ when $x < -1$ or $x > 5$ i.e., for $x$ in $(-\infty, -1) \cup (5, \infty)$	$y \geq 21$ when $x \leq -1$ or $x \geq 5$ i.e., for $x$ in $(-\infty, -1] \cup [5, \infty)$

Exercises

1. Let  $y = 3x^2 - 4x - 6$ .

- (a) Find the vertex.
- (b) Find the  $y$ -intercept.
- (c) Find the  $x$ -intercepts.
- (d) Graph. Show the intercepts and vertex.
- (e) Solve the equation  $y = -2$  by factoring.
- (f) For which  $x$  is  $y < -2$ ? For which  $x$  is  $y > -2$ ?

2. Let  $y = -2x^2 + 12x - 10$ .

- (a) Find the vertex.
- (b) Find the  $y$ -intercept.
- (c) Find the  $x$ -intercepts.
- (d) Graph. Show the intercepts and vertex.
- (e) Solve the equation  $y = 6$  by factoring.
- (f) For which  $x$  is  $y > 6$ ? For which  $x$  is  $y < 6$ ?

## Solutions

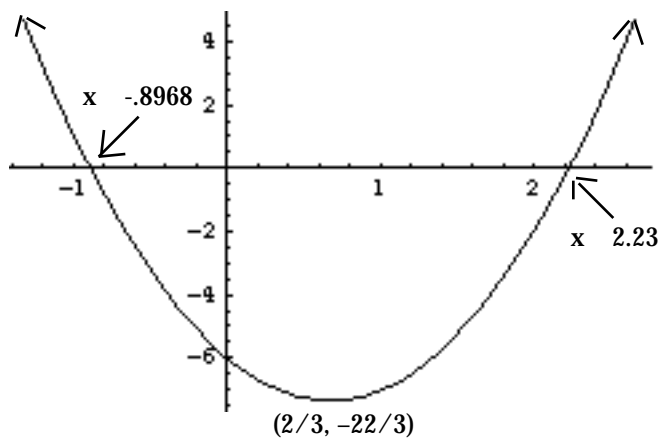
1 (a) The  $x$ -coordinate of the vertex is  $x = \frac{-b}{2a} = \frac{-(-4)}{2 \times 3} = \frac{4}{6} = \frac{2}{3}$ . The  $y$ -coordinate is  $3(2/3)^2 - 4(2/3) - 6 = -\frac{22}{3}$ . So the vertex is  $\frac{2}{3}, -\frac{22}{3}$ .

(b) When  $x = 0$ , the  $y$ -intercept is  $y = -6$ .

(c) To find the  $x$ -intercepts, we will solve  $3x^2 - 4x - 6 = 0$  with the quadratic formula. For our equation,  $a = 3$ ,  $b = -4$ , and  $c = -6$ ; hence, the solutions are

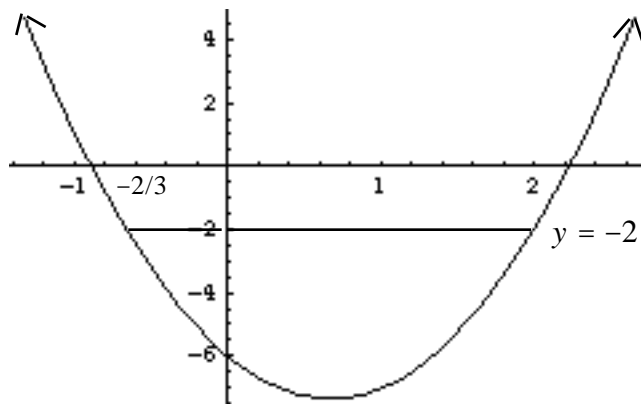
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-6)}}{2(3)} \\ &= \frac{4 \pm \sqrt{88}}{6} = \frac{4 \pm 2\sqrt{22}}{6} = \frac{4}{6} \pm \frac{2\sqrt{22}}{6} = \frac{2}{3} \pm \frac{\sqrt{22}}{3} \end{aligned}$$

The approximate numerical solutions are  $x = 2.23$  and  $x = -0.8968$ .



(e) Now solve  $3x^2 - 4x - 6 = -2$ , which is equivalent to  $3x^2 - 4x - 4 = 0$ . We can factor as  $(3x + 2)(x - 2) = 0$ . So  $(3x + 2) = 0$  or  $(x - 2) = 0$ . The solutions to  $y = 2$  are then  $x = -2/3$  and  $x = 2$ .

(f) When  $-2/3 < x < 2$ , then  $y < -2$ .  
When  $x < -2/3$  or  $x > 2$ , then  $y > -2$ .

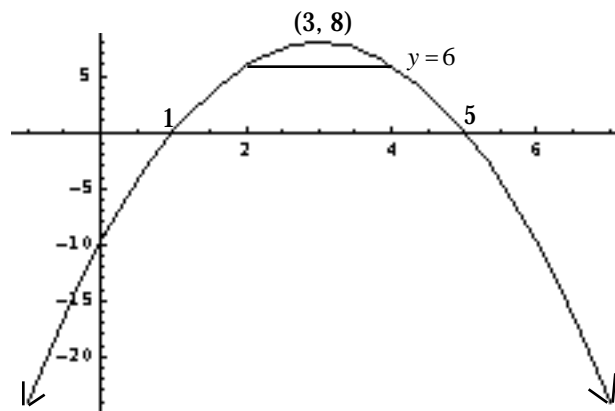


2.  $y = -2x^2 + 12x - 10$  (a) The  $x$ -coordinate of the vertex is  $x = \frac{-b}{2a} = \frac{-12}{2(-2)} = 3$ . The  $y$ -coordinate is  $-2(3)^2 + 12(3) - 10 = 8$ . So the vertex is  $(3, 8)$ . (b) When  $x = 0$ , we obtain  $y = -10$  for the  $y$ -intercept.

(c) To find the  $x$ -intercepts, we must solve  $-2x^2 + 12x - 10 = 0$ . Dividing by  $-2$ , we instead can solve  $x^2 - 6x + 5 = 0$ , which factors as  $(x - 5)(x - 1) = 0$ , so  $x = 1$  and  $x = 5$  are the  $x$ -intercepts.

Or, using the quadratic formula on the original equation  $-2x^2 + 12x - 10 = 0$ , we obtain

$$\begin{aligned} x &= \frac{-12 \pm \sqrt{12^2 - 4(-2)(-10)}}{2(-2)} \\ &= \frac{-12 \pm \sqrt{64}}{-4} = \frac{-12 \pm 8}{-4} = 3 \pm 2 \end{aligned}$$



(d) Now we must solve  $-2x^2 + 12x - 10 = 6$ , which is equivalent to  $-2x^2 + 12x - 16 = 0$ . Dividing by  $-2$ , we have  $x^2 - 6x + 8 = 0$ , which factors as  $(x - 4)(x - 2) = 0$ . Thus,  $x = 4$  and  $x = 2$  are the solutions to  $y = 6$ .

(e) We have  $y > 6$  when  $2 < x < 4$ . And  $y < 6$  when  $x < 2$  or  $x > 4$ .