**MATH 116** **Equation of a Line given Two Points**

Two points determine a line. So if we have points $(x_1, y_1)$ and $(x_2, y_2)$, then we can find the equation of the line $y = mx + b$.

First, we need the slope $m$ which is the "change in $y$ over change in $x$" given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Once we have the slope, then we can use one of the points, say $(x_1, y_1)$, to solve for $b$ in the equation $y_1 = mx_1 + b$. Alternately, we can use the point/slope formula

$$(y - y_1) = m(x - x_1) \text{ which gives } y = m(x - x_1) + y_1.$$

***Example 1***. (a) Find the equation of the line through points (–6, 3) and (2, –7).

(b) Evaluate the line at $x = 10$. (c) For which $x$ is $y$ *at least* –10?

*Solution.* (a) The slope is $m = \frac{\Delta y}{\Delta x} = \frac{-7-3}{2-(-6)} = \frac{-10}{8} = -\frac{5}{4}$.

Now using the first point, $x$ = –6, $y$ = 3, solve for $b$ in $y = -\frac{5}{4}x + b$:

$$3 = -\frac{5}{4}(-6) + b \;\rightarrow\; 3 = \frac{15}{2} + b \text{ which gives } b = 3 - \frac{15}{2} = -\frac{9}{2}.$$

$$\text{So } y = -\frac{5}{4}x - \frac{9}{2}, \text{ for all } x.$$

Alternately,

$$y - 3 = -\frac{5}{4}\big(x - (-6)\big) \;\rightarrow\; y = -\frac{5}{4}(x + 6) + 3 \;\rightarrow\; y = -\frac{5}{4}x - \frac{5}{4}\times 6 + 3 \;\rightarrow\; y = -\frac{5}{4}x - \frac{9}{2}$$

(b) When $x = 10$, then $y = -\frac{5}{4}\times 10 - \frac{9}{2} = -\frac{25}{2} - \frac{9}{2} = -17$.

(c) Solve $-\frac{5}{4}x - \frac{9}{2} \ge -10 \;\rightarrow\; -\frac{5}{4}x \ge -10 + \frac{9}{2} \;\rightarrow\; -\frac{5}{4}x \ge -\frac{11}{2}$

(reverse inequality here when multiplying by –) $\rightarrow\; x \le -\frac{11}{2}\left(-\frac{4}{5}\right) = \frac{44}{10} = 4.4$

$$\text{So } y \ge -10 \text{ for } x \le 4.4.$$

***Example 2***. For full-time undergraduates at CFU (12 to 20 hrs), tuition $T$ is a linear function of credit hours enrolled $x$. For 15 hours, the tuition is \$4937.50. For 18 hours, the tuition is \$5875.

(a) Write the tuition $T$ as a linear function of hours enrolled $x$. Give domain and range.
(b) What is the tuition for 16 hours?
(c) Graph the tuition function and label the axes by name, symbol, and unit.
(d) What hours of full-time undergrads keep $T$ *at most* \$4781.25?
(e) Write the hours enrolled $x$ as a linear function of tuition. Give domain and range.

*Solution.* (a) We have two points (15, 4937.50) and (18, 5875). The slope is given by

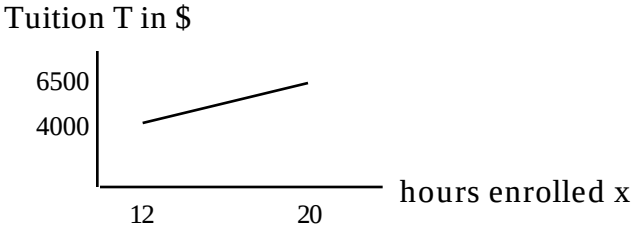
$$m = \frac{(5875 - 4937.50)}{(18 - 15)} = \frac{937.5}{3} = 312.50 .$$

Using $m = 312.50$, we can solve for $b$ in $T = 312.50x + b$ using point (18, 5875): $5875 = 312.50 \times 18 + b \;\rightarrow\; b = 5875 - 312.50 \times 18 = 250$. So $T = 312.50x + 250$ with the domain specified as $12 \le x \le 20$ for full-time undergrads.

When $x = 12$ hrs, the tuition is \$4000. For $x = 20$ hrs, the tuition is \$6500. So the range is $4000 \le T \le 6500$, or [4000, 6500].

(b) For $x$ = 16 hrs, $T(16) = 312.50 \times 16 + 250 = \$5250$.

(c)




(d) Solve $312.50x + 250 \le 4781.25 \;\rightarrow\; 312.50x \le 4531.25 \;\rightarrow\; x \le \frac{4531.25}{312.50} = 14.5$ So we must say $12 \le x \le 14.5$ due to the domain.

(e) We have $T = 312.50x + 250$. Now solve for $x$ as a function of $T$ →

$$T - 250 = 312.50\,x \quad \rightarrow \quad x = \frac{T - 250}{312.50} \quad \rightarrow \quad x = 0.0032\,T - 0.8$$

So the number of hours is given by $x = 0.0032\,T - 0.8$ for $4000 \le T \le 6500$. The range is now [12, 20] hrs.

## Exercises

1. (a) Find the equation of the line through points (15, –4) and (9, 18).
(b) Evaluate the line at $x = 36$.
(c) For which $x$ is $y$ at most –15?

2. For certain models of American made cars weighing from 2000 to 4000 lbs, the mpg $M$ is a function of the weight of the car $w$. Here are two measurements:

If the weight is 3360 lbs, then the mpg is 27.
If the weight is 3960 lbs, then the mpg is 22.

(a) Write the mpg $M$ as a linear function of weight $w$. Give domain and range.
(b) Graph the mpg function and label the axes by name, symbol, and unit.
(c) What is the mpg for a weight of 3600 lbs?
(d) If the mpg is 28, then what is the weight?
(e) Write the weight $w$ as a linear function of the mpg. Give domain and range.

3. Your fine $F$ is a linear function of the mph $M$ your were traveling above the speed limit. If you were 10 mph over the limit, then your fine is $100. If you were 12 mph over the limit, then your fine is $105. You'll only get a ticket if you're *at least* 5 mph over the limit.

(a) Write the fine $F$ as a linear function of mph over the limit. Give domain and range.
(b) Write mph over limit $M$ as a linear function of the fine. Give domain and range.

**Solutions**

1. (a) The slope is $m = \dfrac{\Delta y}{\Delta x} = \dfrac{18-(-4)}{9-15} = \dfrac{22}{-6} = -\dfrac{11}{3}$.

Now using the second point, $x = 9$, $y = 18$, solve for $b$ in $y = -\dfrac{11}{3}x + b$:

$$18 = -\frac{11}{3} \times 9 + b, \text{ which gives } b = 18 + \frac{11}{3} \times 9 = 51.$$

$$\text{So } y = -\frac{11}{3}x + 51, \text{ for all } x.$$

Alternately, $y - 18 = -\dfrac{11}{3}(x-9) \rightarrow y = -\dfrac{11}{3}(x-9) + 18$ or $y = -\dfrac{11}{3}x + 51$, for all $x$.

(b) When $x = 36$, then $y = -81$.

(c) Solve $-\dfrac{11}{3}x + 51 \le -15 \rightarrow -\dfrac{11}{3}x \le -66 \rightarrow x \ge -66 \times \left(-\dfrac{3}{11}\right)$ (reverse inequality here)

$\rightarrow x \ge 18$ or $[18, \infty)$ in interval notation.

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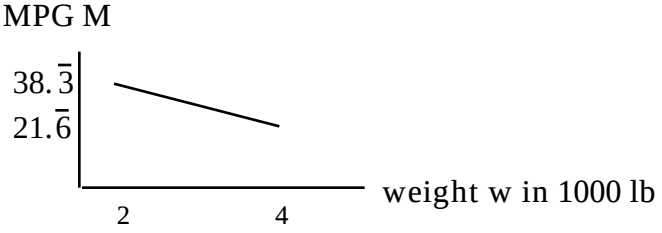
2. (a) We have two points (3360, 27) and (3960, 22). The slope is given by

$$m = \frac{(22-27)}{(3960-3360)} = \frac{-5}{600} = -\frac{1}{120}.$$

The equation of the line is now $M = -\dfrac{1}{120}w + b$. Using the first point, we have $27 = -\dfrac{1}{120} \times 3360 + b$, so $b = 27 + \dfrac{3360}{120} = 55$. We then have $M = -\dfrac{1}{120}w + 55$ for $2000 \le w \le 4000$, where $w$ is in pounds.

When $w = 2000$ lbs, the mileage is $38.\overline{3}$ mpg. For $w = 4000$ lbs, the mileage is $21.\overline{6}$ mpg. So the range is $21.\overline{6} \le M \le 38.\overline{3}$.

(b)




(c) For $w = 3600$ lbs, we obtain $M = -\dfrac{1}{120} \times 3600 + 55 = 25$ mpg.

(d) Solve $-\frac{1}{120}w + 55 = 28 \rightarrow -\frac{1}{120}w = -27 \rightarrow w = 3240$ lbs.

(e) We have $M = -\frac{1}{120}w + 55$. Now solve for $w$ as a function of $M \rightarrow$

$$M - 55 = -\frac{1}{120}w \rightarrow w = -120(M - 55) = -120M + 6600.$$

So the weight is $w = -120M + 6600$ for $21.\overline{6} \le M \le 38.\overline{3}$. The range is now $2000 \le w \le 4000$, or [2000, 4000] lbs.

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3. (a) We have two points (10, 100) and (12, 105). The slope is $\frac{105-100}{12-10} = \frac{5}{2}$. So $F = \frac{5}{2}M + b$. Then using point (10, 100), we have $100 = \frac{5}{2}(10) + b \rightarrow b = 75$.

So the fine (in dollars) is $F = \frac{5}{2}M + 75$, where $M \ge 5$ is mph over the limit.

The minimum fine is \$87.50 for $M = 5$. So the range is [87.50, ∞).

(b) Given $F = \frac{5}{2}M + 75$, solve for $M$ to obtain $M = \frac{2}{5}(F - 75)$ for $F \ge \$87.50$.

So we can say $M = \frac{2}{5}(F - 75)$, where $F \ge \$87.50$ is the amount of the fine. The range is now [5, ∞), which are the possible speeds above the limit.