Sets with interior extremal points for the Markoff inequality*

David Benko and Vilmos Totik

Abstract

In this note we verify that for a set consisting of 3 intervals it is possible that the extreme value in the Markoff inequality occurs only at interior points.

Markoff's inequality

$$\|P_n'\|_{[-1,1]} \le n^2 \|P_n\|_{[-1,1]}$$

is one of the most fundamental inequalities for the derivatives of polynomials. It is sharp in the sense that for $x_0 = \pm 1$ and for the Chebyshev polynomial $P_n = T_n$ we have

$$|T_n'(x_0)| = n^2 \|T_n\|_{[-1,1]}.$$

Let $E \subset \mathbf{R}$ be a set consisting of several intervals. Then

$$\|P_n'\|_E \le M_n \|P_n\|_E$$

holds for every polynomial P_n of degree at most n with some smallest constant M_n, which we call the n-th Markoff factor for E. It is clear that M_n is of the order n^2, and the determination of M_n is an interesting theoretical problem.

A point $x_0 \in E$ is called an extreme point for the n-th Markoff factor if there is a nonzero polynomial P_n of degree at most n such that

$$|P_n'(x_0)| = M_n \|P_n\|_E.$$

In this note we consider the problem if the extreme points should be among the endpoints of the subintervals of E.

In extending the Markoff inequality on $[-1, 1]$ to higher derivatives, a major work is to verify that the extreme points (for the higher order Markoff factors) are ± 1. Let us also note that if L is a compact subset of the interior of E, then it follows from the Bernstein inequality that

$$\|P_n'\|_L = O(n) \|P_n\|_E,$$

*Research was supported in part by the National Science Foundation, DMS–9801435 and by the Hungarian National Science Foundation for Research, T/022983

so for large n there cannot be an extreme point for the Markoff factor M_n in L. Thus, it is reasonable to expect that for general E the extreme points for the Markoff factors are among the endpoints of the intervals of E. In fact, knowing that that the extreme points are among the endpoints of the interval considerably simplifies the determination of the Markoff factors M_n.

Unfortunately, for sets consisting of several intervals the situation is more complex, for it may happen that all extreme points are inside E.

Proposition. *There is a set E consisting of three intervals such that all extreme points for M_5 lie in the interior of E.*

It may happen that the example below is not the "smallest" one regarding the number of intervals or the degree of the polynomials, but one would expect that the same phenomemon can also occurs for more intervals (depending on the configuration of intervals in question) and for higher orders.

Proof. Let us choose and fix a large positive number $M > 10$ ($M = 100$ suffices) and a small positive number γ, and set

$$T(x) = x(x^2 - M^2)(x^2 - (M + \gamma)^2).$$

T is odd, and on the interval $(M, M + \gamma)$ it is negative. As x runs through this interval, the value $T(x)$ decreases from 0 to its minimal value m, and then on it increases to 0. As $\gamma \to 0$, we have $m \to 0$, while for $\gamma = 2$ we have $m \le T(M + 1) < -1$. Thus, there is a $\gamma \in (0, 2)$ such that $m = -1$, and this is how we choose γ. Now the set

$$E = \{x \,|\, T(x) \in [-1, 1]\}$$

consists of three intervals, one-one around $-M$, 0 and M. Thus, we can write with some $0 < a < b < M < M + \gamma < c$

$$E = [-c, -b] \cup [-a, a] \cup [b, c],$$

and at the endpoints a, b, c the polynomial T takes the value 1, while at the endpoints $-a, -b, -c$ it takes the value -1. Furthermore, if the minimum of T on $[b, c]$ is attained at the point d, then $T(d) = -1$, $T(-d) = 1$. It is easy to see that for sufficiently large M ($M \ge 100$ suffices) we have $a < 2/M^4$, $c - b < 1$ (actually, as $M \to \infty$ we have $a \sim 1/M^4$, $c - b \sim \sqrt{2}/M^{3/2}$).

Now for $x \in [b, c]$ we get by direct differentiation

$$|T'(x)| \le 5(c - b)c(2c)^2 \le 20 \cdot c^3 \le 40M^3,$$

and since

$$T'(x) = M^2(M + \gamma)^2 - 3(M^2 + (M + \gamma)^2)x^2 + 5x^4,$$

we have

$$T'(0) = M^2(M+\gamma)^2 > M^4,$$

while $T'(x)$ is positive and smaller than this number for all $x \in [-a, a]$, $x \neq 0$. From these we can infer that $|T'(x)|$ attains its maximum on E at $x = 0$ and nowhere else. Therefore,

$$M_5 > \max\{|T'(\pm a)|, |T'(\pm b)|, |T'(\pm c)|\}.$$

Hence, if we can prove that for any polynomial P of degree at most 5 and any endpoint q of E we have

$$|P'(q)| \leq |T'(q)| \|P\|_E, \tag{1}$$

then it follows that no endpoint is an extremal point for M_5, and the proposition follows. By symmetry we need to prove this only for $q = a, b, c$.

Consider first the case $q = c$. Let $x_1 = -c$, $x_2 = -d$, $x_3 = -a$, $x_4 = b$, $x_5 = d$ and $x_6 = c$. We shall interpolate at these points, and so let

$$l_i(x) = \frac{\prod_{j\neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

be the basic polynomials of Lagrange interpolation. Since for $j \neq i$ the polynomial l_i changes sign at x_j, and this sign change is from $+$ to $-$ if $j > i$ and $j - i$ is odd or $j < i$ and $i - j$ is even and it is from $-$ to $+$ in the opposite cases, we get immediately that $l_5'(c) < 0$, $l_4'(c) > 0$, $l_3'(c) < 0$, $l_2'(c) > 0$, $l_1'(c) < 0$, furthermore it is clear that $l_6'(c) > 0$. Note that these values are exactly of the same sign what T has at the associated points x_j, i.e. sign $l_j'(c) = T'(x_j)$. Thus, since

$$T'(x) = \sum_{j=1}^{6} T(x_j) l_j'(x),$$

we obtain

$$T'(c) = \sum_{j=1}^{6} T(x_j) l_j'(c) = \sum_{j=1}^{6} |l_j'(c)|,$$

and since for any polynomial P of degree at most 5 we have

$$P'(x) = \sum_{j=1}^{6} P(x_j) l_j'(x),$$

it follows that

$$|P'(c)| \leq \left| \sum_{j=1}^{6} P(x_j) l_j'(x) \right| \leq \|P\|_E \sum_{j=1}^{6} |l_j'(c)| = \|P\|_E T'(c),$$

which proves (1).

Next let $q = b$. Then we choose $x_1 = -d$, $x_2 = -a$, $x_3 = a$, $x_4 = b$, $x_5 = d$ and $x_6 = c$. In this case $l_6'(b) < 0$, $l_5'(b) > 0$, $l_3'(b) < 0$, $l_2'(b) > 0$, $l_1'(b) < 0$. As for $l_4'(b) = l_4'(x_4)$, it is equal to

$$\sum_{j \neq 4} \frac{1}{x_4 - x_j},$$

and this is clearly negative (the two terms with $j = 5$ and $j = 6$ are much bigger in absolute value than the other terms). Thus, in this case we have sign $l_j'(b) = -T(x_j)$, and the previous argument can be repeated:

$$|P'(b)| \leq \|P\|_E \sum_j |l_j'(b)| = \|P\|_E \sum_{j=1}^{5} (-T'(x_j) l_j'(b)) = \|P\|_E |T'(b)|.$$

Finally, let $q = a$. Then we can choose the same nodes that we have just used, i.e. $x_1 = -d$, $x_2 = -a$, $x_3 = a$, $x_4 = b$, $x_5 = d$ and $x_6 = c$. In this case $l_6'(a) > 0$, $l_5'(a) < 0$, $l_4'(a) > 0$, $l_2'(a) < 0$, $l_1'(a) > 0$; furthermore

$$l_3'(a) = l_3'(x_3) = \sum_{j \neq 3} \frac{1}{x_3 - x_j}$$

is positive, since the term $1/(x_3 - x_2) = 1/2a$ dominates the other ones. Thus, we have again sign $l_j'(b) = T(x_j)$ for all j, and (1) follows exactly as before.

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Department of Mathematics, University of South Florida, Tampa, FL 33620
E-mail: dbenko@chuma.cas.usf.edu

Department of Mathematics, University of South Florida, Tampa, FL 33620 and Bolyai Institute, University of Szeged, Aradi v. tere 1, 6720, Szeged, Hungary
E-mail: totik@math.usf.edu